1
Vectors

We enumerate definitions and important properties of vectors in this chapter. A vector $\mathbf{A}$ has magnitude $A = |\mathbf{A}|$ and a direction $\mathbf{A}/A$. Two vectors $\mathbf{A}$ and $\mathbf{B}$ can be summed: $\mathbf{A} + \mathbf{B}$. A scalar product of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \cdot \mathbf{B}$. The magnitude of $\mathbf{A}$ is equal to $\sqrt{\mathbf{A} \cdot \mathbf{A}} = |\mathbf{A}| \equiv A$. A vector product of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \times \mathbf{B}$, which is noncommutative: $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$.

1.1
Definition and Important Properties

1.1.1
Definitions

A *vector* $\mathbf{A}$ is a quantity specified by a *magnitude*, denoted by $|\mathbf{A}| \equiv A$ *and* a *direction* in space $\mathbf{A}/A$. A vector will be denoted by a letter in bold face in the text. The vector $\mathbf{A}$ may be represented geometrically by an arrow of length A pointing in the prescribed direction.

Addition. The sum $\mathbf{A} + \mathbf{B}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined geometrically by drawing vector $\mathbf{A}$ originating from the tip of vector $\mathbf{B}$ as shown in Figure 1.1a. The same result is obtained if we draw the vector $\mathbf{B}$ from the tip of the vector $\mathbf{A}$ as shown in Figure 1.1b. This is expressed mathematically by

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.1}$$

which expresses the *commutative* rule for addition.

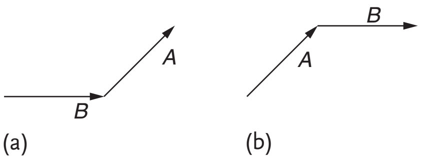

Fig. 1.1 The sum $\mathbf{A} + \mathbf{B}$, represented by (a) is equal to the sum $\mathbf{B} + \mathbf{A}$, represented by (b).

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Vectors also satisfy the *associative* rule:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{1.2}$$

The quantity represented by an ordinary (positive or negative) number is called a *scalar*, to distinguish it from a vector.

1.2
Product of a Scalar and a Vector

The product of a vector $\mathbf{A}$ and a positive scalar c is a vector, denoted by $c\mathbf{A}$, whose magnitude is equal to $c\,|\mathbf{A}|$ and whose direction is the same as that of $\mathbf{A}$. If c is negative, then $c\mathbf{A}$, by definition, is a vector of magnitude $|c|\,|\mathbf{A}|$ pointing in the direction opposite to $\mathbf{A}$. The following rules of computation hold:

$$|c\mathbf{A}| = |c|\,|\mathbf{A}| \tag{1.3}$$

$$(cd)\mathbf{A} = c(d\mathbf{A}) \tag{1.4}$$

$$\mathbf{A}c = c\mathbf{A} \tag{1.5}$$

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B} \tag{1.6}$$

$$(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A} \tag{1.7}$$

Equation (1.5) means that the same product is obtained irrespective of the order of c and $\mathbf{A}$. We say that the product $c\mathbf{A}$ is *commutative*. The properties represented by (1.6)–(1.7) are called *distributive*.

1.3
Position Vector

The position of an arbitrary point P in space with respect to a given origin 0 may be specified by the position vector $\mathbf{r}$ drawn from 0 to P. If x, y, z are the Cartesian coordinates of the point P, then we can express the vector $\mathbf{r}$ by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1.8}$$

where $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are vectors of unit length pointing along the positive x-, y-, and z-axes. See Figure 1.2. For the fixed Cartesian unit vectors, the position vector $\mathbf{r}$ is specified by a set of three real numbers, (x, y, z). We represent this by

$$\mathbf{r} = (x, y, z) \tag{1.9}$$

The distance r of point P from the origin, is given by

$$r \equiv |\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2} \tag{1.10}$$

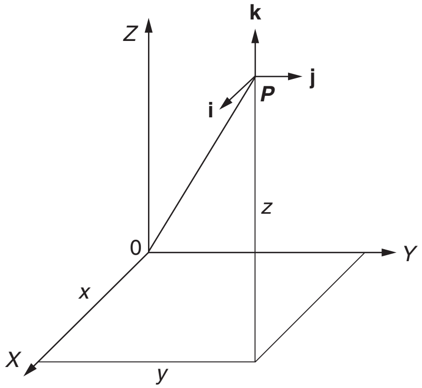

Fig. 1.2 The Cartesian coordinates (x, y, z). The orthonormal vectors **i**, **j** and **k** point in the directions of increasing x, y and z, respectively.

When point P coincides with the origin 0, we have, by definition, the *zero vector* or *null vector*, which is denoted by $\mathbf{0}$, and can be represented by $(0, 0, 0)$. The null vector has zero magnitude and no definite direction.

1.4 Scalar Product

The *dot product*, also called the *scalar product*, $\mathbf{A} \cdot \mathbf{B}$, of two vectors $\mathbf{A}$ and $\mathbf{B}$, is by definition a number equal to the product of their magnitudes times the cosine of the angle θ between them.

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos\theta \ , \quad 0 \leqslant \theta \leqslant \pi \tag{1.11}$$

From this definition, the following properties can be derived:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} \\ \mathbf{A} \cdot (c\mathbf{B}) &= (c\mathbf{A}) \cdot \mathbf{B} = c(\mathbf{A} \cdot \mathbf{B}) \\ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{C}) \end{aligned} \tag{1.12}$$

The last two equations show that the dot product is a *linear operation*. That is, given a vector $\mathbf{B}$, the dot product with a vector $\mathbf{A}$ generates a scalar $\mathbf{A} \cdot \mathbf{B}$, which is a linear function of $\mathbf{B}$. For example, if $\mathbf{B}$ is multiplied by 2, the scalar product $\mathbf{A} \cdot \mathbf{B}$ is also doubled.

The set of *Cartesian unit vectors* $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ satisfy the *orthonormality* relations:

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0 \tag{1.13}$$

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \tag{1.14}$$

The property (1.13) follows from the fact that the angles between any pair of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are 90°, and that $\cos 90° = 0$. We will say that the vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are *orthogonal* to each other. The normalization property (1.14) holds because each of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ has unit length.

An arbitrary vector $\mathbf{A}$ can be decomposed as follows:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{1.15}$$

where A_x, A_y and A_z are the projections of the vector $\mathbf{A}$ along the positive x-, y-, and z-axes, respectively, and are given numerically by

$$A_x = \mathbf{i} \cdot \mathbf{A}\,, \quad A_y = \mathbf{j} \cdot \mathbf{A}\,, \quad A_z = \mathbf{k} \cdot \mathbf{A} \tag{1.16}$$

Given the Cartesian unit vectors, the vector $\mathbf{A}$ can be represented by the set of the projections (A_x, A_y, A_z) called Cartesian components:

$$\mathbf{A} = (A_x, A_y, A_z) \tag{1.17}$$

Using the Cartesian decomposition (1.15), we obtain

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\
&= A_x B_x \mathbf{i} \cdot \mathbf{i} + A_x B_y \mathbf{i} \cdot \mathbf{j} + A_x B_z \mathbf{i} \cdot \mathbf{k} \\
&\quad + A_y B_x \mathbf{j} \cdot \mathbf{i} + A_y B_y \mathbf{j} \cdot \mathbf{j} + A_y B_z \mathbf{j} \cdot \mathbf{k} \\
&\quad + A_z B_x \mathbf{k} \cdot \mathbf{i} + A_z B_y \mathbf{k} \cdot \mathbf{j} + A_z B_z \mathbf{k} \cdot \mathbf{k} \\
&= A_x B_x + A_y B_y + A_z B_z
\end{aligned}$$

or

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.18}$$

By setting $\mathbf{A} = \mathbf{B}$ here, we obtain

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2 \geqslant 0 \tag{1.19}$$

The magnitude of the vector, $|\mathbf{A}|$, can be expressed by the square root of this quantity:

$$|\mathbf{A}| = (\mathbf{A} \cdot \mathbf{A})^{1/2} = (A_x^2 + A_y^2 + A_z^2)^{1/2} \tag{1.20}$$

We note that the properties of any vector $\mathbf{A}$ can be visualized analogously to the position vector $\mathbf{r}$ except for the difference in the physical dimension.

1.5
Vector Product

The *vector product*, $\mathbf{A} \times \mathbf{B}$, of two vectors $\mathbf{A}$ and $\mathbf{B}$ is by definition a vector having a magnitude equal to the area of the parallelogram with $\mathbf{A}$ and $\mathbf{B}$ as sides, and pointing in a direction perpendicular to the plane comprising $\mathbf{A}$ and $\mathbf{B}$. The direction of

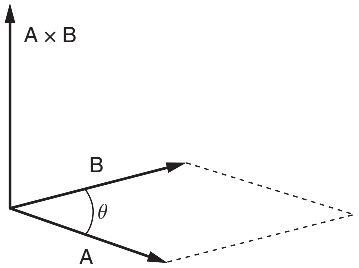

Fig. 1.3

$\mathbf{A} \times \mathbf{B}$ is, by convention, that direction in which a *right hand screw* would advance when turned from **A** to **B**, as indicated in Figure 1.3.

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta\,, \quad 0 \leqq \theta \leqq \pi \tag{1.21}$$

The vector product is a linear operation:

$$\mathbf{A} \times (c\mathbf{B}) = c\mathbf{A} \times \mathbf{B}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1.22}$$

The following properties are observed:

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \tag{1.23}$$

$$\mathbf{A} \times \mathbf{B} = 0 \qquad \text{if } \mathbf{A} \parallel \mathbf{B} \tag{1.24}$$

$$\mathbf{A} \times \mathbf{A} = 0 \tag{1.25}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.26}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}\,, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}\,, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \tag{1.27}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \tag{1.28}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{1.29}$$

The last relation (1.29) may be verified by writing out the Cartesian components of both sides explicitly.

Problem 1.5.1

By writing out the Cartesian components of both sides, show that

1. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$,
2. $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$

Problem 1.5.2

Show that

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

1.6 Differentiation

When a vector **A** depends on the time t, the derivative of **A** with respect to t, $d\mathbf{A}/dt$, is defined by

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t} \tag{1.30}$$

The following rules are observed for scalar and vector products:

$$\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} \tag{1.31}$$

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt} \tag{1.32}$$

Note that the operational rules as well as the definition are similar to the those of a scalar function.

A function $F(\mathbf{r})$ of the position $\mathbf{r} = (x, y, z)$ is called a *point function* or similarly a *field*. The space derivatives are discussed in Chapter 6.

1.7 Spherical Coordinates

For problems with special symmetries, it is convenient to use non-Cartesian coordinates. In particular, if the system under consideration has spherical symmetry, we may then use spherical coordinates (r, θ, φ), shown in Figure 1.4. These coordinates are related to the Cartesian coordinates by

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\varphi \end{aligned} \tag{1.33}$$

A system of orthogonal unit (orthonormal) vectors $(\mathbf{l}, \mathbf{m}, \mathbf{n})$ in the directions of increasing θ, φ, and r, respectively, is also shown in Figure 1.4. These unit vectors

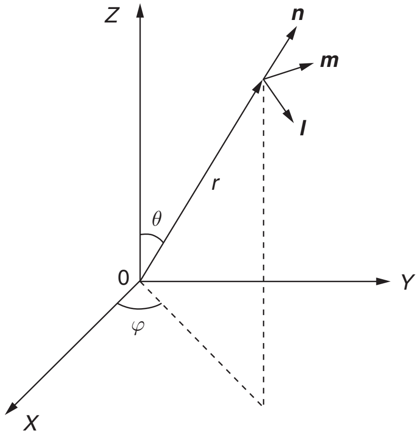

Fig. 1.4 The spherical polar coordinates (r, θ, φ). The orthonormal vectors **n**, **l**, and **m** point in the direction of increasing r, θ, and φ, respectively.

are related to the Cartesian unit vectors (**i**, **j**, **k**) as follows:

$$\mathbf{l} = -\mathbf{k}\sin\theta + \mathbf{i}\cos\theta\cos\varphi + \mathbf{j}\cos\theta\sin\varphi$$
$$\mathbf{m} = -\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi$$
$$\mathbf{n} = \mathbf{k}\cos\theta + \mathbf{i}\sin\theta\cos\varphi + \mathbf{j}\sin\theta\sin\varphi \tag{1.34}$$

An arbitrary vector **A** can be decomposed as follows:

$$\mathbf{A} = A_r\mathbf{n} + A_\theta\mathbf{l} + A_\varphi\mathbf{m} \tag{1.35}$$

where A_r, A_θ, A_φ are the components of **A** along **n**, **l**, and **m**, respectively. Further, they are given by

$$A_r = \mathbf{n}\cdot\mathbf{A}\,, \quad A_\theta = \mathbf{l}\cdot\mathbf{A}\,, \quad A_\varphi = \mathbf{m}\cdot\mathbf{A} \tag{1.36}$$

Problem 1.7.1

Two vectors point in directions (θ_1, φ_1) and (θ_2, φ_2). The angle between the two vectors is denoted by ψ. Show that

$$\cos\psi = \sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2) + \cos\theta_1\cos\theta_2$$

1.8 Cylindrical Coordinates

For problems with axial symmetry, *cylindrical coordinates* (ρ, ϕ, z), as shown in Figure 1.5 are used. These cylindrical polar coordinates are related to the Cartesian

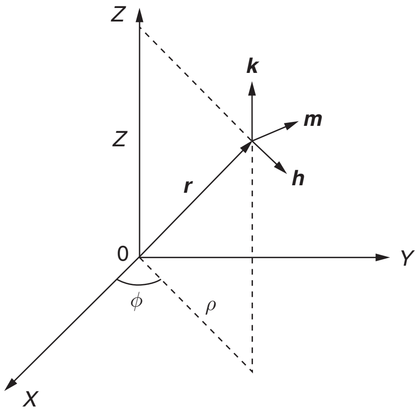

Fig. 1.5 The cylindrical polar coordinates (ρ, ϕ, z). The orthonormal vectors **h**, **m**, and **k** point in the direction of increasing ρ, ϕ, and z, respectively.

coordinates by

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned} \tag{1.37}$$

The set of orthonormal vectors **h**, **m**, **k** in the direction of increasing ρ, ϕ, and z are shown in Figure 1.5. They are related to the Cartesian unit vectors (**i**, **j**, **k**) as follows:

$$\begin{aligned} \mathbf{h} &= \mathbf{i} \cos \phi + \mathbf{j} \sin \phi \\ \mathbf{m} &= -\mathbf{i} \sin \phi + \mathbf{j} \cos \phi \\ \mathbf{k} &= \mathbf{k} \end{aligned} \tag{1.38}$$

An arbitrary vector **A** can be decomposed in the following form:

$$\mathbf{A} = A_\rho \mathbf{h} + A_\phi \mathbf{m} + A_z \mathbf{k} \tag{1.39}$$

where A_ρ, A_ϕ, A_z are the components of **A** along **h**, **m**, **k**, given by

$$A_\rho = \mathbf{h} \cdot \mathbf{A}\,, \quad A_\phi = \mathbf{m} \cdot \mathbf{A}\,, \quad A_z = \mathbf{k} \cdot \mathbf{A} \tag{1.40}$$

We note that an arbitrary vector **A** can be decomposed as follows:

$$\begin{aligned} \mathbf{A} &= \mathbf{i}(\mathbf{i} \cdot \mathbf{A}) + \mathbf{j}(\mathbf{j} \cdot \mathbf{A}) + \mathbf{k}(\mathbf{k} \cdot \mathbf{A}) \\ &= \mathbf{n}(\mathbf{n} \cdot \mathbf{A}) + \mathbf{l}(\mathbf{l} \cdot \mathbf{A}) + \mathbf{m}(\mathbf{m} \cdot \mathbf{A}) \\ &= \mathbf{h}(\mathbf{h} \cdot \mathbf{A}) + \mathbf{m}(\mathbf{m} \cdot \mathbf{A}) + \mathbf{k}(\mathbf{k} \cdot \mathbf{A}) \end{aligned} \tag{1.41}$$

where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, $(\mathbf{n}, \mathbf{l}, \mathbf{m})$ and $(\mathbf{h}, \mathbf{m}, \mathbf{k})$ are orthonormal vectors in Cartesian, spherical and cylindrical coordinates, respectively. We note that the three equations (1.41) can be written as

$$\begin{aligned} \mathbf{A} &= \mathbf{e}_1(\mathbf{e}_1 \cdot \mathbf{A}) + \mathbf{e}_2(\mathbf{e}_2 \cdot \mathbf{A}) + \mathbf{e}_3(\mathbf{e}_3 \cdot \mathbf{A}) \\ &= \sum_{j=1}^{3} \mathbf{e}_j (\mathbf{e}_j \cdot \mathbf{A}) \end{aligned} \tag{1.42}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a set of orthonormal vectors satisfying

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.43}$$

The symbol δ_{ij} is called Kronecker's delta.