1
Introducing Quantum Fields

1.1
The Classical String

We obtain a quantum field by quantizing a classical field, of which the simplest example is the classical string. To be on firm mathematical grounds, we define the latter as the long-wavelength limit of a discrete chain. Consider $N + 2$ masses described by the classical Lagrangian

$$L(q, \dot{q}) = \sum_{j=0}^{N+1} \left[\frac{m}{2} \dot{q}_j^2 - \frac{\kappa}{2} (q_j - q_{j+1})^2 \right] \tag{1.1}$$

where m is the mass, and κ a force constant. The coordinate $q_j(t)$ represents the lateral displacement of the j-th mass along a one-dimensional chain. We impose fixed-endpoint boundary conditions, by setting

$$q_0(t) = q_{N+1}(t) = 0 \tag{1.2}$$

The equations of motion for the N remaining movable masses are then

$$m\ddot{q}_j - \kappa \left(q_{j+1} - 2q_j + q_{j-1} \right) = 0 \quad (j = 1, \ldots, N) \tag{1.3}$$

The normal modes have the form

$$q_j(t) = \cos(\omega t) \sin(j p) \tag{1.4}$$

To satisfy the boundary conditions, choose p to have one of the N possible values

$$p_n = \frac{\pi n}{N+1} \quad (n = 1, \ldots, N) \tag{1.5}$$

Substituting this into the equations of motion, we obtain N independent normal frequencies ω_n:

$$\omega_n^2 = \omega_0^2 \sin^2 \left(\frac{\pi}{2} \frac{n}{N+1} \right) \quad (n = 1 \cdots N) \tag{1.6}$$

Quantum Field Theory from Operators to Path Integrals. K. Huang

ISBN: 978-3-527-40846-7

where

$$\omega_0 = 2\sqrt{\frac{\kappa}{m}} \tag{1.7}$$

This is a cutoff frequency, for the modes with $n > N$ merely repeat the lower ones. For $N = 4$, for example, the independent modes correspond to $n = 1, 2, 3, 4$. The case $n = 5$ is trivial, being equivalent to $n = 0$. The case $n = 6$ is the same as that for $n = 4$, since $\omega_6 = \omega_4$, and $\sin(j\, p_6) = -\sin(j\, p_4)$.

When N is large, and we are not interested in the behavior near the endpoints, it is convenient to use periodic boundary conditions

$$q_{j+N}(t) = q_j(t) \tag{1.8}$$

In this case the normal modes are

$$q_j(t) = e^{i(j\,p - \omega t)} \tag{1.9}$$

For N even, the boundary conditions can be satisfied by putting

$$p_n = \frac{2\pi n}{N} \quad (n = 0, \pm 1, \ldots, \pm N/2) \tag{1.10}$$

The corresponding normal frequencies are

$$\omega_n^2 = \omega_0^2 \sin^2\left(\frac{\pi n}{N}\right) \tag{1.11}$$

Compared to the fixed-end case, the spacing between normal frequencies is now doubled; but each frequency is twofold degenerate, and the number of normal modes remains the same. A comparison of the two cases for $N = 8$ is shown in Figure 1.1.

The equilibrium distance a between masses does not explicitly appear in the Lagrangian; it merely supplies a length scale for physical distances. For example, it appears in the definition of the distance of a mass from an end of the chain:

$$x \equiv j\, a \quad (j = 1, \ldots, N) \tag{1.12}$$

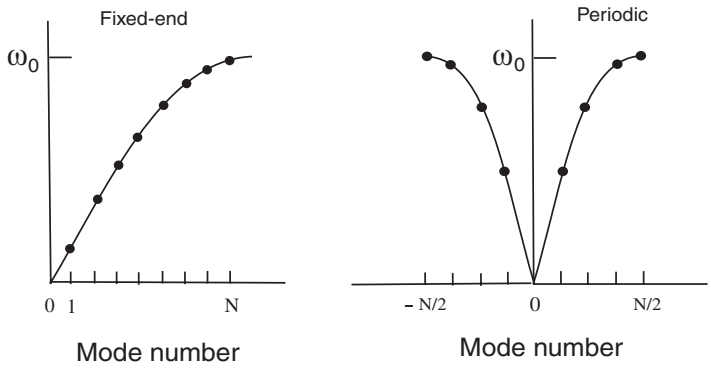

Figure 1.1 Normal modes of the classical chain for fixed-end and periodic boundary conditions.

The total length of the chain is then defined as

$$R = Na \tag{1.13}$$

In the continuum limit

$$a \to 0 \quad N \to \infty \quad (R = Na \text{ fixed}) \tag{1.14}$$

the discrete chain approaches a continuous string, and the coordinate approaches a classical field defined by

$$q(x,t) \equiv q_j(t) \tag{1.15}$$

The Lagrangian in the continuum limit can be obtained by making the replacements

$$\begin{aligned} (q_{n+1} - q_j)^2 &\to a^2 \left[\frac{\partial q(x,t)}{\partial x}\right]^2 \\ \sum_j &\to \frac{1}{a}\int_0^R dx \end{aligned} \tag{1.16}$$

Assuming that the mass density ρ and string tension σ approach finite limits

$$\rho = m/a \tag{1.17}$$

$$\sigma = \kappa a \tag{1.18}$$

we obtain the limit Lagrangian

$$L_{\text{cont}} = \frac{1}{2}\int_0^R dx \left[\rho\left(\frac{\partial q(x,t)}{\partial t}\right)^2 - \sigma\left(\frac{\partial q(x,t)}{\partial x}\right)^2\right] \tag{1.19}$$

This leads to the equation of motion

$$\frac{\partial^2 q(x,t)}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 q(x,t)}{\partial x^2} = 0 \tag{1.20}$$

which is a wave equation, with propagation velocity

$$c = \sqrt{\sigma/\rho} \tag{1.21}$$

The general solutions are the real and imaginary parts of

$$q(x,t) = e^{i(kx-\omega t)} \tag{1.22}$$

with a linear dispersion law

$$\omega = ck \tag{1.23}$$

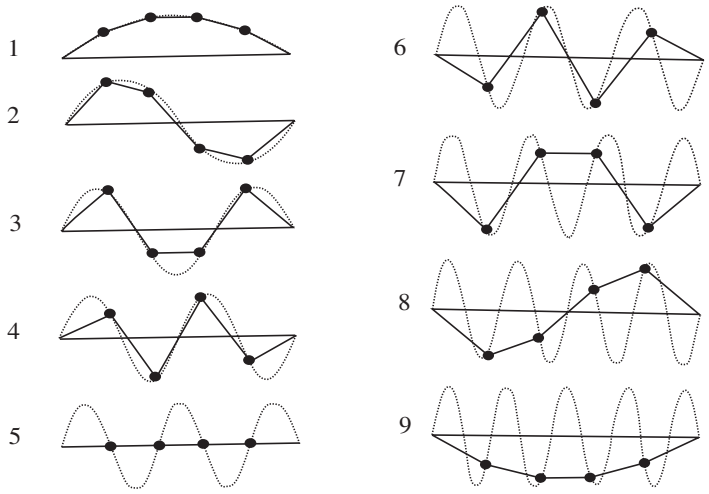

Figure 1.2 Normal modes of a discrete chain of 4 masses, compared with those of a continuous string. The former repeat themselves after the first 4 modes. (After J. C. Slater and N. H. Frank, *Mechanics*, McGraw-Hill, New York, 1947).

For fixed-end boundary conditions

$$q(0,t) = q(R,t) = 0 \tag{1.24}$$

the normal modes of the continuous string are

$$q_n(x,t) = \cos(\omega_n t)\sin(k_n x) \tag{1.25}$$

with $\omega_n = ck_n$, and

$$k_n = \frac{\pi n}{R} \quad (n = 0, 1, 2, \cdots) \tag{1.26}$$

The normal frequencies ω_n are the same as those for the discrete chain for $n/N \ll 1$, as given in (1.6). However, the number of modes of the continuum string is infinite, and only the first N modes have correspondence with those of the discrete string. This is illustrated in Figure 1.2 for $N = 4$. Thus, there is a cutoff frequency

$$\omega_c \equiv \omega_N = \frac{\pi c}{a} \tag{1.27}$$

This is of the same order, but not same as the maximum frequency defined earlier $\omega_0 = 2c/a$, for ω_c is based on a linear dispersion law. The continuum model is an accurate representation of the discrete chain only for $\omega \ll \omega_c$.

For periodic boundary conditions

$$q(0,t) = q(R,t) \tag{1.28}$$

the allowed wave numbers are

$$k_n = \frac{2\pi n}{R} \quad (n = 0, \pm 1, \pm 2, \cdots) \tag{1.29}$$

We obtain the cutoff frequency ω_c by setting $n = N/2$.

1.2 Renormalization

The high-frequency cutoff is a theoretical necessity. Without it, the specific heat of the string will diverge, since each normal mode contributes an amount kT. The value of the cutoff cannot be determined from the long-wavelength effective theory, because only the combination $c = a\omega_c/\pi$ occurs. Absorbing the cutoff into measurable parameters, as done in (1.17), is called *renormalization*. A theory for which this can be done is said to be *renormalizable*.

Non-renormalizable systems exhibit behavior that is sensitive to atomic motion. Such behavior would appear to be random on a macroscopic scale, as in the propagation of cracks in materials, and the nucleation of raindrops.

1.3 The Quantum String

We now quantize the classical chain, to obtain a quantum field in the continuum limit. The Hamiltonian of the classical discrete chain is given by

$$H(p,q) = \sum_{j=1}^{N} \left[\frac{p_j^2}{2m} + \frac{\kappa}{2}(q_j - q_{j+1})^2 \right] \tag{1.30}$$

where $p_j = m\dot{q}_j$. The system can be quantized by replacing p_j and q_j by hermitian operators satisfying the commutation relations[1)]

$$\left[p_j, q_k\right] = -i\delta_{jk} \tag{1.31}$$

We imposing periodic boundary conditions, and expand these operators in Fourier series:

$$\begin{aligned} q_j &= \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2} Q_n e^{i2\pi nj/N} \\ p_j &= \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2} P_n e^{i2\pi nj/N} \end{aligned} \tag{1.32}$$

where P_n and Q_n are operators satisfying

$$\begin{aligned} \left[P_n^\dagger, Q_m\right] &= -i\delta_{nm} \\ P_n^\dagger &= P_{-n} \\ Q_n^\dagger &= Q_{-n} \end{aligned} \tag{1.33}$$

1) We use units in which $\hbar = c = 1$, where $\hbar$ is the reduced Planck's constant, and c ist the velocity of light.

The system is reduced to a sum of independent harmonic oscillators:

$$H = \sum_{n=-N/2}^{N/2} \left[\frac{1}{2m} P_n^\dagger P_n + \frac{1}{2} m\omega_n^2 Q_n^\dagger Q_n \right]$$

$$\omega_n^2 = \frac{4\kappa}{m} \sin^2\left(\frac{\pi n}{N}\right) \tag{1.34}$$

The eigenvalues are labeled by a set of occupation numbers $\{\alpha_n\}$:

$$E_\alpha = \sum_{n=-N/2}^{N/2} \omega_n(\alpha_n + 1/2) \tag{1.35}$$

where $\alpha_n = 0, 1, 2, \cdots$. The frequency ω_n is taken to be the positive root of ω_n^2, since H is positive-definite.

In the continuum limit (1.14) the Hamiltonian becomes

$$H_{\text{cont}} = \int_0^R dx \left[\frac{1}{2\rho} p^2(x,t) + \frac{\sigma}{2} \left(\frac{\partial q(x,t)}{\partial x} \right)^2 \right] \tag{1.36}$$

where, with $x = j\,a$,

$$p(x,t) = \frac{p_j(t)}{a} = \rho \frac{\partial q(x,t)}{\partial t} \tag{1.37}$$

The quantum field $q(x,t)$ and its canonical conjugate $p(x,t)$ satisfy the equal-time commutation relation

$$\left[p(x,t), q(x',t) \right] = -i\delta(x - x') \tag{1.38}$$

Just as in the classical case, we have to introduce a cutoff frequency ω_c. General properties of the quantum field will be discussed more fully in Chapter 2.

1.4 Second Quantization

Another way to obtain a quantum field is to consider a collection of identical particles in quantum mechanics. In this case, the quantum field is an equivalent description of the system. Identical particles are defined by a Hamiltonian that is (a) invariant under a permutation of the particle coordinates, and (b) has the same form for any number of particles. The quantized-field description is called "second quantization" for historical reasons, but quantization was actually done only once.

Let $\mathcal{H}_N$ the Hilbert space of a system of N identical non-relativistic particles. The union of all $\mathcal{H}_N$ is called the Fock space:

$$\mathcal{F} = \bigcup_{N=0}^{\infty} \mathcal{H}_N \tag{1.39}$$

The subspace with $N = 0$ contains the vacuum state as its only member. We assume that N is the eigenvalues of a "number operator" N_{op}, which commutes with the Hamiltonian. It is natural to introduce operators on Fock space that connect subspaces of different N. An elementary operator of this kind creates or annihilates one particle at a point in space. Such an operator is a quantum field operator, since it is a spatial function. This is why a quantum-mechanical many-particle system automatically gives rise to a quantum field.

For definiteness, consider N non-relativistic particles in 3 spatial dimensions, with coordinates $\{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$. The Hamiltonian is

$$H = -\frac{1}{2m} \sum_{i=1}^{N} \nabla_i^2 + V(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{1.40}$$

where ∇_i^2 is the Laplacian with respect to $\mathbf{r}_i$, and where V is a symmetric function of its arguments. The eigenfunctions Ψ_n are defined by

$$H\Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_N) = E_n \Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{1.41}$$

For Bose or Fermi statistics, Ψ_n is respectively symmetric or antisymmetric under an interchange of any two coordinates $\mathbf{r}_i$ and $\mathbf{r}_j$. The particles are called bosons or fermions respectively.

We now describe the equivalent quantum field theory, and justify it later. Let $\psi(\mathbf{r})$ be the Schrödinger-picture operator that annihilates one particle at $\mathbf{r}$. Its hermitian conjugate $\psi^\dagger(\mathbf{r})$ will create one particle at $\mathbf{r}$. They are defined through the commutation relations

$$\begin{aligned} \left[\psi(\mathbf{r}), \psi^\dagger(\mathbf{r}')\right]_\pm &= \delta^3(\mathbf{r} - \mathbf{r}') \\ \left[\psi(\mathbf{r}), \psi(\mathbf{r}')\right]_\pm &= 0 \end{aligned} \tag{1.42}$$

where $[A, B]_\pm = AB \pm BA$, with the plus sign corresponding to bosons and the minus sign to fermions. The Fock- space Hamiltonian will be defined in such a manner that it reduces to (1.40) in the N-particle subspace.

A general N-particle Hamiltonian has the structure

$$H = \sum_i f(\mathbf{r}_i) + \sum_{i<j} g(\mathbf{r}_i, \mathbf{r}_j) + \sum_{i<j<k} h(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \cdots \tag{1.43}$$

where the functions g, h, etc. are symmetric functions of their arguments. The first term is a "one-particle operator", a sum of operators of the form $f(\mathbf{r})$, which act on one particle only. The second term is a "two-particle operator", a sum of operators

of the form $g(\mathbf{r}_1, \mathbf{r}_2)$, over all distinct pairs. Generally, an "n-particle operator" is a sum of operators that depend only on a set of n coordinates. To construct the Hamiltonian on Fock space, we associate an n-particle operator with an operator on Fock space, with the following correspondences:

$$\begin{aligned}
\sum_i f(\mathbf{r}_i) &\to \int d^3 r \psi^\dagger(\mathbf{r}) f(\mathbf{r}) \psi(\mathbf{r}) \\
\sum_{i<j} g(\mathbf{r}_i, \mathbf{r}_j) &\to \frac{1}{2} \int d^3 r_1 d^3 r_2 \psi_1^\dagger \psi_2^\dagger g_{12} \psi_2 \psi_1 \\
\sum_{i<j<k} h(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) &\to \frac{1}{3!} \int d^3 r_1 d^3 r_2 d^3 r_3 \psi_1^\dagger \psi_2^\dagger \psi_3^\dagger h_{123} \psi_3 \psi_2 \psi_1 \\
&\vdots
\end{aligned} \tag{1.44}$$

where for brevity we have written $\psi_1 = \psi(\mathbf{r}_1)$, $g_{12} = g(\mathbf{r}_1, \mathbf{r}_2)$, etc.

As an example, suppose the potential in (1.40) is a sum of two-body potentials:

$$V(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \sum_{i<j} v(\mathbf{r}_i, \mathbf{r}_j) \tag{1.45}$$

Then the corresponding Fock-space Hamiltonian, also denoted H, takes the form

$$\begin{aligned}
H = &-\frac{1}{2m} \int d^3 r \psi^\dagger(\mathbf{r}) \nabla^2 \psi(\mathbf{r}) \\
&+ \frac{1}{2} \int d^3 r_1 d^3 r_2 \psi^\dagger(\mathbf{r}_1) \psi^\dagger(\mathbf{r}_2) v(\mathbf{r}_1, \mathbf{r}_2) \psi(\mathbf{r}_2) \psi(\mathbf{r}_1)
\end{aligned} \tag{1.46}$$

The particle number is the eigenvalue of the number operator, defined as

$$N_{\text{op}} = \int d^3 r \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \tag{1.47}$$

By using (1.42), we can verify the relations

$$\begin{aligned}
\left[N_{\text{op}}, H\right] &= 0 \\
\left[\psi(\mathbf{r}), N_{\text{op}}\right] &= \psi(\mathbf{r}) \\
\left[\psi^\dagger(\mathbf{r}), N_{\text{op}}\right] &= -\psi^\dagger(\mathbf{r})
\end{aligned} \tag{1.48}$$

These imply that the action of $\psi(\mathbf{r})$ on a eigenstate of N_{op} is to decrease its eigenvalue by 1, while that of $\psi^\dagger(\mathbf{r})$ is to increase it by 1. Thus $\psi(\mathbf{r})$ is an annihilation operator, while $\psi^\dagger(\mathbf{r})$ is a creation operator. The vacuum state $|0\rangle$ is defined as the eigenstate of N_{op} with eigenvalue zero. It is annihilated by all annihilation operators:

$$\psi(\mathbf{r})|0\rangle = 0 \tag{1.49}$$

By applying $\psi^\dagger(\mathbf{r})$ to the vacuum state repeatedly, it is easy to show that the eigenvalues of N_{op} are nonnegative integers.

To demonstrate that the quantum field is equivalent to the many-particle system, consider a complete set of states $|E, N\rangle$ of the quantum field, which are simultaneous eigenstate of H and N_{op}:

$$H|E, N\rangle = E|E, N\rangle$$
$$N_{\text{op}}|E, N\rangle = N|E, N\rangle$$

We define the N-particle wave function $\Psi_E(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ corresponding to $|E, N\rangle$ by

$$\Psi_E(\mathbf{r}_1, \ldots, \mathbf{r}_N) \equiv \frac{1}{\sqrt{N!}}\langle 0|\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)|E, N\rangle \tag{1.50}$$

which the correct symmetry with respect to particle permutation. It tell us that the probability amplitude for finding N particle at the positions $\mathbf{r}_1, \ldots, \mathbf{r}_N$ can be found by annihilating the particles at the respective locations from the state $|E, N\rangle$, and evaluating the overlap between the resulting state and the vacuum state. We leave it as an exercise to show that this wave function satisfies the N-particle Schrödinger equation (1.41). (See Problem 1.3.)

1.5 Creation and Annihilation Operators

The field operator $\psi(\mathbf{r})$ annihilates a particle at $\mathbf{r}$. That is, it annihilates a particle whose wave function is a δ-function. Since the latter can be written as a linear superposition of a complete set of wave functions, we can express $\psi(\mathbf{r})$ as a linear superposition of operators that annihilate particles with specific types of wave functions. Suppose $u_k(\mathbf{r})$ is a member of a complete orthonormal set of single-particle wave functions:

$$\int d^3r\, u_k^*(\mathbf{r})u_{k'}(\mathbf{r}) = \delta_{kk'}$$
$$\sum_k u(\mathbf{r})u^*(\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}')$$

An example of such a set is plane-waves:

$$u_k(\mathbf{r}) = \frac{1}{\sqrt{\Omega}}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.51}$$

We can expand the field operators with respect to such a basis:

$$\psi(\mathbf{r}) = \sum_k u_k(\mathbf{r})a_k$$
$$\psi^\dagger(\mathbf{r}) = \sum_k u_k^*(\mathbf{r})a_k^\dagger$$

The coefficient a_k and $a_k^\dagger$ are operators that satisfy the commutation relations

$$\left[a_k, {a_{k'}}^\dagger\right]_\pm = \delta_{kk'}$$
$$[a_k, a_{k'}]_\pm = 0 \tag{1.52}$$

where the + sign is for bosons, and the − sign for fermions. These relations follow from (1.42) and the orthonormality of the functions $u_k(\mathbf{r})$.

It follows from the commutation relations that, for each k, the eigenvalues of $a_k^\dagger a_k$ are integers n_k, called the occupation number of the single-particle state k:

$$\begin{aligned} a^\dagger a|n\rangle &= n|n\rangle \\ \langle n|m\rangle &= \delta_{nm} \end{aligned} \tag{1.53}$$

where we have omitted the label k for brevity. The allowed values of the occupation number are given by

$$n = \begin{cases} 0, 1, 2, \ldots, \infty & \text{(Bose statistics)} \\ 0, 1 & \text{(Fermi statistics)} \end{cases}$$

The actions of a and $a^\dagger$ have the following results:

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle \\ a^\dagger|n\rangle &= \sqrt{1 \pm n}|n+1\rangle \end{aligned} \tag{1.54}$$

where the $\pm$ sign corresponds respectively to Bose (+) and Fermi (−) statistics. which show that a annihilates a particle in the state with wave function $u(\mathbf{r})$, and $a^\dagger$ creates such a particle. We leave it as an exercise to derive these basic results. (See Problem 1.2.)

The state $|0\rangle$ corresponding to $n = 0$ is the vacuum state, which satisfies

$$a|0\rangle = 0 \tag{1.55}$$

We assume that it is normalizable:

$$\langle 0|0\rangle = 1 \tag{1.56}$$

Obviously all other states can be obtained by creating particles from the vacuum:

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |n\rangle \tag{1.57}$$

We can simultaneously diagonalize ${a_k}^\dagger a_k$ for all k. The eigenstates are then labeled by a set of occupation numbers $\{n_0, n_1, \ldots\}$, and they constitute a basis for the Fock space. The total number of particles present is $N = \sum_k n_k$. We have

$$\begin{aligned} a_k^\dagger a_k \, |n_0, \ldots, n_k, \ldots\rangle &= n_k |n_0, \ldots, n_k, \ldots\rangle \\ a_k |n_0, \ldots, n_k, \ldots\rangle &= (-1)^s \sqrt{n_k} |n_0, \ldots, n_k - 1, \ldots\rangle \\ a_k^\dagger |n_0, \ldots, n_k, \ldots\rangle &= (-1)^s \sqrt{1 \pm n_k} |n_0, \ldots, n_k + 1, \ldots\rangle \end{aligned}$$

where

$$s = \begin{cases} 0 & \text{(Bose statistics)} \\ \sum_{p<k} n_p & \text{(Fermi statistics)} \end{cases} \tag{1.58}$$

That is, $s = \pm 1$ for fermions, depending on whether the number of fermions with quantum numbers less than k is even or odd, the meaning of "less than" being

set by an arbitrary but fixed ordering. This phase factor arises from the fact that fermion creation operators anticommute: $a_k^\dagger a_p^\dagger = -a_p^\dagger a_k^\dagger$.

A complete set of states can be constructed by creating particles from the vacuum:

$$\begin{aligned} |k\rangle &= a_k^\dagger |0\rangle \\ |k,p\rangle &= a_k^\dagger a_p^\dagger |0\rangle \\ &\vdots \end{aligned} \tag{1.59}$$

These states are not normalized to unity. When there are many particle present, it is more convenient to label the state with occupation numbers $\{n_k\}$, where n_k is the number of particles with single-particle quantum number k:

$$|n_0, \ldots, n_k, \ldots\rangle = C \prod_k \left[a_k{}^\dagger\right]^{n_k} |0\rangle \tag{1.60}$$

These states can be normalized to unity by choosing

$$C = \left[\prod_k n_k!\right]^{-1/2} \tag{1.61}$$

1.6 Bose and Fermi Statistics

The term "statistics" refers to the rule for counting the degeneracy of an energy level of a many-particle system. In 3 dimensional space, it depends on the symmetry of the wave function under a permutation of the particle coordinates. Technically speaking, the different possible symmetries correspond to the different irreducible representations of the permutation group.

The completely symmetric and the completely antisymmetric representations correspond respectively to Bose and Fermi statistics. They are the only possible ones in a two-particle system; but for more than two particles other possibilities exist, in which the wave function is symmetric with respect to permutations among one subset S of coordinates, and antisymmetric for the complementary set. Called "parastatistics", such representations correspond to the Young's tableaux with more than one row, or more than one column. Since the particle are identical, there is more than one way to choose the subset S. Consequently, such "para" representations must be multidimensional. That is, the carrier space for such a representation must be spanned by states having the same energy eigenvalue, and they mix under a permutation of the coordinates. Therefore, the energy levels of particles obeying parastatistics must have intrinsic degeneracies, which cannot be removed by any interaction that treat the particles as identical.

The Bose and Fermi statistics can be set apart from the parastatistics by virtue of the following properties:

- Under particle permutation, the symmetry character of wave functions is independent of the number of particles present.
- Energy eigenfunctions do not mix under particle permutation.

Parastatistics does occur in atomic physics, but only in the context of "incomplete" permutations, which interchange the positions of atomic electrons but not their spins. With respect to permutations of both position and spin, electrons obey Fermi statistics, as we know. No known examples of parastatistics have been found in nature. Perhaps the simple properties itemized above are essential for consistency on some level.

Although we live in a 3D world, some interesting physical systems are effectively 2D. These include the electron sheets that exhibit the quantum Hall effect, the copper-oxide planes in a high-temperature superconductor, and thin films of superfluid helium on various substrates. In a 2D system, the variety of statistics is far richer, because the exchange of two particle in a plane is not a unique process: we may rotate the particles about a center through angle $n\pi$, where n is any odd integer, and the paths corresponding to different n are not necessarily equivalent. Consequently, the symmetry group relevant to particle exchange is not the permutation group, but the much larger *braid group*. This circumstance allows for fractional spin and statistics; but we shall not discuss this, except for a brief discussion on fractional spin in Chapter 19.

Problems

Problem 1.1

Consider an actual string made of atoms spaced $a = 10^{-8}$ cm apart. Suppose the length of the string is 1 m, and it is kept at such a tension that the fundamental frequency is 100 cycles/sec. Find the cutoff frequency, and show that it lies in the infrared region of the spectrum. (This gives the Debye temperature.)

Problem 1.2

a) The basic commutation relation for boson annihilation and creation operator is

$$\left[a, a^\dagger\right] = 1 \quad [a, a] = 0$$

where $[A, B] = AB - BA$. From this definition show that eigenstates $|n\rangle$ of $a^\dagger a$ have the properties

$$\begin{aligned} a^\dagger a|n\rangle &= n|n\rangle \qquad (n = 0, 1, 2, \cdots) \\ a|n\rangle &= \sqrt{n}|n\rangle \\ a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \end{aligned}$$

b) Fermion annihilation and creation operators are defined by an anticommutation relation

$$\{a, a^\dagger\} = 1 \quad \{a, a\} = 0$$

where $\{A, B\} = AB + BA$. Show

$$a^\dagger a|n\rangle = n|n\rangle \qquad (n = 0, 1)$$

$$a|n\rangle = \sqrt{n}|n\rangle$$

$$a^\dagger|n\rangle = \sqrt{1-n}|n+1\rangle$$

Problem 1.3

Consider the N-particle wave function defined in (1.50)

$$\Psi_E(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \frac{1}{\sqrt{N!}}\langle 0|\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)|E, N\rangle$$

where $|E, N\rangle$ is an N-particle energy eigenstate with respect to the Hamiltonian H given in (1.46).

a) Show that it is normalized to unity:

$$\int d^3r_1 \cdots d^3r_N |\Psi_E(\mathbf{r}_1, \ldots, \mathbf{r}_N)|^2 = 1$$

b) Show

$$E\Psi_E(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \frac{1}{\sqrt{N!}}\langle 0|\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)H|E, N\rangle$$

c) Show that the wave function satisfies the N-particle Schrödinger equation

$$\left[-\sum_{i=1}^{N}\frac{1}{2m}\nabla_i^2 + \sum_{i<j} v(\mathbf{r}_i, \mathbf{r}_j)\right]\Psi_E(\mathbf{r}_1, \ldots, \mathbf{r}_N) = E\Psi_E(\mathbf{r}_1, \ldots, \mathbf{r}_N)$$

by going to the result in (b), and commute H all the way to the left, where it gives zero operating on the vacuum.

All the results stated hold for both Bose and Fermi statistics [1].

Problem 1.4

A non-relativistic boson or fermion field $\psi(\mathbf{x})$ is governed by the Hamiltonian

$$H = -\frac{1}{2m}\int d\mathbf{x}\psi^\dagger(\mathbf{x})\nabla^2\psi(\mathbf{x}) + \frac{1}{2}\int d\mathbf{x}d\mathbf{y}\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})v(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x})$$

The system is enclosed in a large cubical box of volume Ω ($\Omega \to \infty$), with periodic boundary conditions. Expand the field in terms of annihilation operators $a_\mathbf{k}$ for free-particle states of momentum $\mathbf{k}$, and show that

$$H = \sum_\mathbf{k}\frac{\mathbf{k}^2}{2m}a_\mathbf{k}^\dagger a_\mathbf{k} + \frac{1}{2\Omega}\sum_{\mathbf{p},\mathbf{q},\mathbf{k}}\tilde{v}(\mathbf{k})a_{\mathbf{p}+\mathbf{k}}^\dagger a_{\mathbf{q}-\mathbf{k}}^\dagger a_\mathbf{p} a_\mathbf{q}$$

where $\tilde{v}(\mathbf{k}) = \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}}v(\mathbf{r})$.

Problem 1.5

Consider a system of N non-relativistic electrons and N positive ions with Coulomb interactions, enclosed in a periodic box of volume Ω. The Hamiltonian is given by

$$H = \sum_{i=1}^N\frac{\mathbf{p}_i^2}{2m} + \sum_{i<j}\frac{e^2}{|\mathbf{r}_i-\mathbf{r}_j|} - \sum_{i,j=1}^N\frac{e^2}{|\mathbf{r}_i-\mathbf{R}_j|} + \sum_{i<j}\frac{e^2}{|\mathbf{R}_i-\mathbf{R}_j|} + \sum_{i=1}^N\frac{\mathbf{P}_i^2}{2M}$$

The ions are heavy. Hence consider $\mathbf{R}_i$ to be fixed numbers, neglect $\mathbf{P}_i$, and drop the last two terms.

a) Label single-electron states momentum $\mathbf{k}$ and spin s, designated collectively as $\alpha = \{\mathbf{k}, s\}$. The corresponding wave function is

$$u_\alpha(\mathbf{r}) = \frac{1}{\sqrt{\Omega}}e^{i\mathbf{k}\cdot\mathbf{r}}\zeta_s$$

$$\zeta_+ = \begin{pmatrix}1\\0\end{pmatrix}\zeta_- = \begin{pmatrix}0\\1\end{pmatrix}$$

b) To go to the quantized-field representation, replace one- and two-particle operators by the rules

$$\sum_{i=1}^N K(\mathbf{r}_i) \to \sum_{\alpha,\beta}\langle\alpha|K|\beta\rangle a_\alpha^\dagger a_\beta$$

$$\sum_{i<j}v(\mathbf{r}_i-\mathbf{r}_j) \to \frac{1}{2}\sum_{\alpha\beta\gamma\lambda}(a_\alpha a_\beta)^\dagger(a_\gamma a_\lambda)\langle\alpha\beta|v|\gamma\lambda\rangle$$

c) Define Fourier transforms:

$$\langle \mathbf{k}s|\frac{e^2}{|\mathbf{r}-\mathbf{R}|}|\mathbf{k}'s'\rangle \equiv \frac{\delta_{ss'}}{\Omega}\int d^3r\, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\frac{e^2}{|\mathbf{r}-\mathbf{R}|}$$
$$= \frac{\delta_{ss'}}{\Omega}\frac{4\pi e^2}{|\mathbf{r}-\mathbf{R}|}e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}}.$$
$$\langle \alpha\beta|v|\gamma\lambda\rangle \equiv \frac{\delta_{s_1s_3}\delta_{s_2s_4}}{\Omega}\int d^3x\, d^3y\,\frac{e^2}{|\mathbf{x}-\mathbf{y}|}e^{i[(\mathbf{k}_3-\mathbf{k}_1)\cdot\mathbf{x}+(\mathbf{k}_4-\mathbf{k}_2)\cdot\mathbf{y}]}$$
$$= \frac{\delta_{s_1s_3}\delta_{s_2s_4}}{\Omega}\delta_K(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\frac{4\pi e^2}{|\mathbf{k}_3-\mathbf{k}_1|}$$

where δ_K is the Kronecker delta.

d) Obtain the Hamiltonian in quantized-field form:

$$H = \sum_{\mathbf{k}s}\frac{\hbar^2k^2}{2m}a^\dagger_{\mathbf{k}s}a_{\mathbf{k}s} + \frac{2\pi e^2}{\Omega}\sum_{\mathbf{p},\mathbf{q},\mathbf{k}}\sum_{s,s'}\frac{1}{k^2}(a_{\mathbf{p}+\mathbf{k},s}a_{\mathbf{q}-\mathbf{k},s'})^\dagger a_{\mathbf{p},s}a_{\mathbf{q},s'}$$
$$-\frac{4\pi e^2}{\Omega}\sum_{\mathbf{p},\mathbf{k}}\sum_s\frac{1}{k^2}a^\dagger_{\mathbf{p},s}a_{\mathbf{k}+\mathbf{p},s}\sum_{i=1}^{N}e^{i\mathbf{k}\cdot\mathbf{R}_i}$$

e) Show that the second term gives, for small $\mathbf{k}$,

$$\frac{2\pi e^2}{\Omega\mathbf{k}^2}\sum_{\mathbf{p},\mathbf{q}}\sum_{s,s'}(a_{\mathbf{p}s}a_{\mathbf{q}s'})^\dagger a_{\mathbf{p}s}a_{\mathbf{q}s'} = \frac{2\pi e^2N(N-1)}{\Omega\mathbf{k}^2}$$

which is divergent at $\mathbf{k}=0$. Show that the divergent term proportional to N^2 is canceled by the $\mathbf{k}=0$ limit of the third term.

The $O(N)$ term above remains divergent. The source of this divergence is the periodic boundary conditions, by which the set of coordinates $\mathbf{r}_1,\ldots,\mathbf{r}_N$ is being repeated an infinite number of times in space. Consequently, the Coulomb energy of an electron diverges, due to long-range interactions with an infinite number of distant copies.

References

1 Huang, K. (1987) *Statistical Mechanics* 2nd ed., Wiley, New York, Appendix.