

Contents

List of Contributors XXI

Preface XXV

Part I: Probability 1

1	Stochastic Processes	3
	<i>James R. Cruise, Ostap O. Hryniv, and Andrew R. Wade</i>	
1.1	Introduction	3
1.2	Generating Functions and Integral Transforms	4
1.2.1	Generating Functions	4
1.2.2	Example: Branching Processes	7
1.2.3	Other Transforms	9
1.2.3.1	Moment Generating Functions	9
1.2.3.2	Laplace Transforms	10
1.2.3.3	Characteristic Functions	10
1.3	Markov Chains in Discrete Time	10
1.3.1	What is a Markov Chain?	10
1.3.2	Some Examples	11
1.3.3	Stationary Distribution	12
1.3.4	The Strong Markov Property	12
1.3.5	The One-Step Method	13
1.3.6	Further Computational Methods	14
1.3.7	Long-term Behavior; Irreducibility; Periodicity	15
1.3.8	Recurrence and Transience	16
1.3.9	Remarks on General State Spaces	17
1.3.10	Example: Bak–Sneppen and Related Models	17
1.4	Random Walks	18
1.4.1	Simple Symmetric Random Walk	18
1.4.2	Pólya's Recurrence Theorem	19
1.4.3	One-dimensional Case; Reflection Principle	19
1.4.4	Large Deviations and Maxima of Random Walks	21
1.5	Markov Chains in Continuous Time	22

1.5.1	Markov Property, Transition Function, and Chapman–Kolmogorov Relation	22
1.5.2	Infinitesimal Rates and Q-matrices	23
1.5.3	Kolmogorov Differential Equations	24
1.5.4	Exponential Holding-Time Construction; “Gillespie’s Algorithm”	25
1.5.5	Resolvent Computations	26
1.5.6	Example: A Model of Deposition, Diffusion, and Adsorption	27
1.5.6.1	$N = 1$	28
1.5.6.2	$N = 3$	28
1.6	Gibbs and Markov Random Fields	29
1.6.1	Gibbs Random Field	29
1.6.2	Markov Random Field	30
1.6.3	Connection Between Gibbs and Markov Random Fields	31
1.6.4	Simulation Using Markov Chain Monte Carlo	31
1.7	Percolation	31
1.8	Further Reading	33
1.A	Appendix: Some Results from Probability Theory	33
1.A.1	Set Theory Notation	33
1.A.2	Probability Spaces	34
1.A.3	Conditional Probability and Independence of Events	35
1.A.4	Random Variables and Expectation	35
1.A.5	Conditional Expectation	36
	References	37
2	Monte-Carlo Methods	39
	<i>Kurt Binder</i>	
2.1	Introduction and Overview	39
2.2	Random-Number Generation	40
2.2.1	General Introduction	40
2.2.2	Properties That a Random-Number Generator (RNG) Should Have	40
2.2.3	Comments about a Few Frequently Used Generators	41
2.3	Simple Sampling of Probability Distributions Using Random Numbers	42
2.3.1	Numerical Estimation of Known Probability Distributions	42
2.3.2	“Importance Sampling” versus “Simple Sampling”	42
2.3.3	Monte-Carlo as a Method of Integration	43
2.3.4	Infinite Integration Space	43
2.3.5	Random Selection of Lattice Sites	44
2.3.6	The Self-Avoiding Walk Problem	44
2.3.7	Simple Sampling versus Biased Sampling: the Example of SAWs Continued	45
2.4	Survey of Applications to Simulation of Transport Processes	46
2.4.1	The “Shielding Problem”	46
2.4.2	Diffusion-Limited Aggregation (DLA)	46
2.5	Monte-Carlo Methods in Statistical Thermodynamics: Importance Sampling	47

2.5.1	The General Idea of the Metropolis Importance-Sampling Method	47
2.5.2	Comments on the Formulation of a Monte-Carlo Algorithm	48
2.5.3	The Dynamic Interpretation of the Monte-Carlo Method	50
2.5.4	Monte-Carlo Study of the Dynamics of Fluctuations Near Equilibrium and of the Approach toward Equilibrium	51
2.5.5	The Choice of Statistical Ensembles	52
2.6	Accuracy Problems: Finite-Size Problems, Dynamic Correlation of Errors, Boundary Conditions	52
2.6.1	Finite-Size–Induced Rounding and Shifting of Phase Transitions	52
2.6.2	Different Boundary Conditions: Simulation of Surfaces and Interfaces	54
2.6.3	Estimation of Statistical Errors	55
2.7	Sampling of Free Energies and Free Energy Barriers	56
2.7.1	Bulk Free Energies	56
2.7.2	Interfacial Free Energies	57
2.7.3	Transition Path Sampling	57
2.8	Quantum Monte-Carlo Techniques	57
2.8.1	General Remarks	57
2.8.2	Path-Integral Monte-Carlo Methods	58
2.8.3	A Classical Application: the Momentum Distribution of Fluid ^4He	59
2.8.4	A Few Qualitative Comments on Fermion Problems	59
2.9	Lattice Gauge Theory	61
2.9.1	Some Basic Ideas of Lattice Gauge Theory	61
2.9.2	A Famous Application	62
2.10	Selected Applications in Classical Statistical Mechanics of Condensed Matter	62
2.10.1	Metallurgy and Materials Science	63
2.10.2	Polymer Science	63
2.10.3	Surface Physics	67
2.11	Concluding Remarks	67
	Glossary	68
	References	69
	Further Reading	71
3	Stochastic Differential Equations	73
	<i>Gabriel J. Lord</i>	
3.1	Introduction	73
3.2	Brownian Motion / Wiener Process	75
3.2.1	White and Colored Noise	77
3.2.2	Approximation of a Brownian Motion	80
3.3	Stochastic Integrals	80
3.3.1	Itô Integral	82
3.4	Itô SDEs	84
3.4.1	Itô Formula and Exact Solutions	86
3.5	Stratonovich Integral and SDEs	90
3.6	SDEs and Numerical Methods	92

3.6.1	Numerical Approximation of Itô SDEs	93
3.6.2	Numerical Approximation of Stratonovich SDEs	95
3.6.3	Multilevel Monte Carlo	97
3.7	SDEs and PDEs	98
3.7.1	Fokker–Planck Equation	99
3.7.2	Backward Fokker–Planck Equation	102
3.7.2.1	Sketch of Derivation.	102
3.7.2.2	Boundary Conditions	103
3.7.3	Filtering	103
	Further Reading	104
	Glossary	104
	References	105

Part II: Discrete Mathematics, Geometry, Topology 109

4	Graph and Network Theory	111
	<i>Ernesto Estrada</i>	
4.1	Introduction	111
4.2	The Language of Graphs and Networks	112
4.2.1	Graph Operators	113
4.2.2	General Graph Concepts	114
4.2.3	Types of Graphs	115
4.3	Graphs in Condensed Matter Physics	115
4.3.1	Tight-Binding Models	115
4.3.1.1	Nullity and Zero-Energy States	117
4.3.2	Hubbard Model	118
4.4	Graphs in Statistical Physics	120
4.5	Feynman Graphs	124
4.5.1	Symanzik Polynomials and Spanning Trees	125
4.5.2	Symanzik Polynomials and the Laplacian Matrix	126
4.5.3	Symanzik Polynomials and Edge Deletion/Contraction	128
4.6	Graphs and Electrical Networks	129
4.7	Graphs and Vibrations	130
4.7.1	Graph Vibrational Hamiltonians	131
4.7.2	Network of Classical Oscillators	131
4.7.3	Network of Quantum Oscillators	133
4.8	Random Graphs	134
4.9	Introducing Complex Networks	137
4.10	Small-World Networks	138
4.11	Degree Distributions	139
4.11.1	“Scale-Free” Networks	141
4.12	Network Motifs	142
4.13	Centrality Measures	143
4.14	Statistical Mechanics of Networks	146
4.14.1	Communicability in Networks	147

4.15	Communities in Networks	148
4.16	Dynamical Processes on Networks	150
4.16.1	Consensus	150
4.16.2	Synchronization in Networks	151
4.16.3	Epidemics on Networks	153
	Glossary	154
	References	155
	Further Reading	157
5	Group Theory	159
	<i>Robert Gilmore</i>	
5.1	Introduction	159
5.2	Precursors to Group Theory	160
5.2.1	Classical Geometry	161
5.2.2	Dimensional Analysis	161
5.2.3	Scaling	162
5.2.4	Dynamical Similarity	163
5.3	Groups: Definitions	164
5.3.1	Group Axioms	165
5.3.2	Isomorphisms and Homomorphisms	166
5.4	Examples of Discrete Groups	166
5.4.1	Finite Groups	166
5.4.1.1	The Two-Element Group Z_2	166
5.4.1.2	Group of Equilateral Triangle C_{3v}	167
5.4.1.3	Cyclic Groups C_n	168
5.4.1.4	Permutation Groups S_n	168
5.4.1.5	Generators and Relations	169
5.4.2	Infinite Discrete Groups	169
5.4.2.1	Translation Groups: One Dimension	169
5.4.2.2	Translation Groups: Two Dimensions	170
5.4.2.3	Space Groups	170
5.5	Examples of Matrix Groups	170
5.5.1	Translation Groups	170
5.5.2	Heisenberg Group H_3	171
5.5.3	Rotation Group $SO(3)$	171
5.5.4	Lorentz Group $SO(3, 1)$	172
5.6	Lie Groups	173
5.7	Lie Algebras	175
5.7.1	Structure Constants	175
5.7.2	Constructing Lie Algebras by Linearization	175
5.7.3	Constructing Lie Groups by Exponentiation	177
5.7.4	Cartan Metric	178
5.7.5	Operator Realizations of Lie Algebras	179
5.7.6	Disentangling Results	180
5.8	Riemannian Symmetric Spaces	181

5.9	Applications in Classical Physics	182
5.9.1	Principle of Relativity	182
5.9.2	Making Mechanics and Electrodynamics Compatible	182
5.9.3	Gravitation	184
5.9.4	Reflections	185
5.10	Linear Representations	185
5.10.1	Maps to Matrices	185
5.10.2	Group Element–Matrix Element Duality	186
5.10.3	Classes and Characters	187
5.10.4	Fourier Analysis on Groups	187
5.10.4.1	Remark on Terminology	188
5.10.5	Irreps of $SU(2)$	188
5.10.6	Crystal Field Theory	190
5.11	Symmetry Groups	190
5.12	Dynamical Groups	194
5.12.1	Conformal Symmetry	194
5.12.2	Atomic Shell Structure	195
5.12.3	Nuclear Shell Structure	195
5.12.4	Dynamical Models	198
5.13	Gauge Theory	199
5.14	Group Theory and Special Functions	202
5.14.1	Summary of Some Properties	202
5.14.2	Relation with Lie Groups	202
5.14.3	Spherical Harmonics and $SO(3)$	203
5.14.4	Differential and Recursion Relations	204
5.14.5	Differential Equation	205
5.14.6	Addition Theorems	206
5.14.7	Generating Functions	206
5.15	Summary	207
	Glossary	208
	References	210
6	Algebraic Topology	211
	<i>Vanessa Robins</i>	
6.1	Introduction	211
6.2	Homotopy Theory	212
6.2.1	Homotopy of Paths	212
6.2.2	The Fundamental Group	213
6.2.3	Homotopy of Spaces	215
6.2.4	Examples	215
6.2.5	Covering Spaces	216
6.2.6	Extensions and Applications	217
6.3	Homology	218
6.3.1	Simplicial Complexes	219
6.3.2	Simplicial Homology Groups	219

6.3.3	Basic Properties of Homology Groups	221
6.3.4	Homological Algebra	222
6.3.5	Other Homology Theories	224
6.4	Cohomology	224
6.4.1	De Rham Cohomology	226
6.5	Morse Theory	226
6.5.1	Basic Results	226
6.5.2	Extensions and Applications	228
6.5.3	Forman's Discrete Morse Theory	229
6.6	Computational Topology	230
6.6.1	The Fundamental Group of a Simplicial Complex	230
6.6.2	Smith Normal form for Homology	231
6.6.3	Persistent Homology	232
6.6.4	Cell Complexes from Data	234
	Further Reading	236
	References	236
7	Special Functions	239
	<i>Chris Athorne</i>	
7.1	Introduction	239
7.2	Discrete Symmetry	241
7.2.1	Symmetries	241
7.2.2	Coxeter Groups	243
7.2.3	Symmetric Functions	244
7.2.4	Invariants of Coxeter Groups	246
7.2.5	Fuchsian Equations	247
7.3	Continuous Symmetry	250
7.3.1	Lie Groups and Lie Algebras	250
7.3.2	Representations	251
7.3.3	The Laplace Operator	253
7.3.4	Spherical Harmonics	255
7.3.5	Separation of Variables	256
7.3.6	Bessel Functions	256
7.3.7	Addition Laws	258
7.3.8	The Hypergeometric Equation	258
7.3.9	Orthogonality	260
7.3.10	Orthogonal Polynomials	260
7.4	Factorization	261
7.4.1	The Bessel Equation	262
7.4.2	Hermite	262
7.4.3	Legendre	263
7.4.4	"Factorization" of PDEs	265
7.4.5	Dunkl Operators	267
7.5	Special Functions Without Symmetry	268
7.5.1	Airy Functions	268

7.5.1.1	Stokes Phenomenon	269
7.5.2	Liouville Theory	269
7.5.3	Differential Galois Theory	271
7.6	Nonlinear Special Functions	272
7.6.1	Weierstraß Elliptic Functions	272
7.6.1.1	Lamé Equations	277
7.6.2	Jacobian Elliptic Functions	277
7.6.3	Theta Functions	278
7.6.4	Painlevé Transcendents	280
7.7	Discrete Special Functions	283
7.7.1	∂ - δ Theory	283
7.7.2	Quantum Groups	283
7.7.3	Difference Operators	284
7.7.4	q -Hermite Polynomials	285
7.7.5	Discrete Painlevé Equations	286
	References	286
8	Computer Algebra	291
	<i>James H. Davenport</i>	
8.1	Introduction	291
8.2	Computer Algebra Systems	292
8.3	“Elementary” Algorithms	292
8.3.1	Representation of Polynomials	292
8.3.2	Greatest Common Divisors	294
8.3.2.1	Intermediate Expression Swell	294
8.3.2.2	Sparsity	295
8.3.3	Square-free Decomposition	295
8.3.4	Extended Euclidean Algorithm	296
8.4	Advanced Algorithms	296
8.4.1	Modular Algorithms – Integer	296
8.4.2	Modular Algorithms – Polynomial	297
8.4.3	The Challenge of Factorization	298
8.4.4	p -adic Algorithms – Integer	300
8.4.5	p -adic Algorithms – Polynomial	301
8.5	Solving Polynomial Systems	301
8.5.1	Solving One Polynomial	301
8.5.2	Real Roots	303
8.5.3	Linear Systems	304
8.5.4	Multivariate Systems	304
8.5.5	Gröbner Bases	305
8.5.6	Regular Chains	308
8.6	Integration	308
8.6.1	Rational Functions	308
8.6.2	More Complicated Functions	310
8.6.3	Linear Ordinary Differential Equations	311

8.7	Interpreting Formulae as Functions	312
8.7.1	Fundamental Theorem of Calculus Revisited	312
8.7.2	Simplification of Functions	313
8.7.3	Real Problems	314
8.8	Conclusion	315
	References	315
9	Differentiable Manifolds	319
	<i>Marcelo Epstein</i>	
9.1	Introduction	319
9.2	Topological Spaces	319
9.2.1	Definition	319
9.2.2	Continuity	320
9.2.3	Further Topological Notions	320
9.3	Topological Manifolds	320
9.3.1	Motivation	320
9.3.2	Definition	321
9.3.3	Coordinate Charts	321
9.3.4	Maps and Their Representations	321
9.3.5	A Physical Application	322
9.3.6	Topological Manifolds with Boundary	323
9.4	Differentiable Manifolds	323
9.4.1	Motivation	323
9.4.2	Definition	323
9.4.3	Differentiable Maps	324
9.4.4	Tangent Vectors	324
9.4.5	Brief Review of Vector Spaces	325
9.4.5.1	Definition	325
9.4.5.2	Linear Independence and Dimension	325
9.4.5.3	The Dual Space	326
9.4.6	Tangent and Cotangent Spaces	326
9.4.7	The Tangent and Cotangent Bundles	327
9.4.8	A Physical Interpretation	328
9.4.9	The Differential of a Map	328
9.5	Vector Fields and the Lie Bracket	330
9.5.1	Vector Fields	330
9.5.2	The Lie Bracket	330
9.5.3	A Physical Interpretation: Continuous Dislocations	331
9.5.4	Pushforwards	334
9.6	Review of Tensor Algebra	334
9.6.1	Linear Operators and the Tensor Product	334
9.6.2	Symmetry and Skew Symmetry	337
9.6.3	The Algebra of Tensors on a Vector Space	337
9.6.4	Exterior Algebra	339
9.7	Forms and General Tensor Fields	341

- 9.7.1 1-Forms 341
- 9.7.2 Pullbacks 341
- 9.7.3 Tensor Bundles 342
- 9.7.4 The Exterior Derivative 343
- 9.8 Symplectic Geometry 345
- 9.8.1 Symplectic Vector Spaces 345
- 9.8.2 Symplectic Manifolds 345
- 9.8.3 Hamiltonian Systems 346
- 9.9 The Lie Derivative 347
- 9.9.1 The Flow of a Vector Field 347
- 9.9.2 One-parameter Groups of Transformations Generated by Flows 348
- 9.9.3 The Lie Derivative 348
- 9.9.3.1 The Lie Derivative of a Scalar 349
- 9.9.3.2 The Lie Derivative of a Vector Field 350
- 9.9.3.3 The Lie Derivative of a 1-form 350
- 9.9.3.4 The Lie Derivative of Arbitrary Tensor Fields 350
- 9.9.3.5 The Lie Derivative in Components 351
- Further Reading 351

10 Topics in Differential Geometry 353

Marcelo Epstein

- 10.1 Integration 353
- 10.1.1 Integration of n -Forms in \mathbb{R}^n 353
- 10.1.2 Integration of Forms on Oriented Manifolds 354
- 10.1.3 Stokes' Theorem 355
- 10.2 Fluxes in Continuum Physics 355
- 10.2.1 Extensive-Property Densities 355
- 10.2.2 Balance Laws, Flux Densities, and Sources 356
- 10.2.3 Flux Forms and Cauchy's Formula 356
- 10.2.4 Differential Expression of the Balance Law 357
- 10.3 Lie Groups 358
- 10.3.1 Definition 358
- 10.3.2 Group Actions 358
- 10.3.3 One-Parameter Subgroups 360
- 10.3.4 Left- and Right-Invariant Vector Fields on a Lie Group 361
- 10.4 Fiber Bundles 361
- 10.4.1 Introduction 361
- 10.4.2 Definition 361
- 10.4.3 Simultaneity in Classical Mechanics 363
- 10.4.4 Adapted Coordinate Systems 363
- 10.4.5 The Bundle of Linear Frames 363
- 10.4.6 Bodies with Microstructure 364
- 10.4.7 Principal Bundles 364
- 10.4.8 Associated Bundles 365
- 10.5 Connections 367

10.5.1	Introduction	367
10.5.2	Ehresmann Connection	368
10.5.3	Parallel Transport along a Curve	368
10.5.4	Connections in Principal Bundles	369
10.5.5	Distributions and the Theorem of Frobenius	370
10.5.6	Curvature	371
10.5.7	Cartan's Structural Equation	372
10.5.8	Bianchi Identities	372
10.5.9	Linear Connections	372
10.5.10	The Canonical 1-Form	373
10.5.11	The Christoffel Symbols	374
10.5.12	Parallel Transport and the Covariant Derivative	374
10.5.13	Curvature and Torsion	375
10.6	Riemannian Manifolds	377
10.6.1	Inner-Product Spaces	377
10.6.2	Riemannian Manifolds	379
10.6.3	Riemannian Connections	380
	Further Reading	380

Part III: Analysis 383

11	Dynamical Systems	385
	<i>David A.W. Barton</i>	
11.1	Introduction	385
11.1.1	Definition of a Dynamical System	385
11.1.2	Invariant Sets	386
11.2	Equilibria	386
11.2.1	Definition and Calculation	386
11.2.2	Stability	387
11.2.3	Linearization	387
11.2.4	Lyapunov Functions	388
11.2.5	Topological Equivalence	389
11.2.6	Manifolds	390
11.2.7	Local Bifurcations	391
11.2.8	Saddle-Node Bifurcation	392
11.2.9	Hopf Bifurcation	393
11.2.10	Pitchfork Bifurcation	396
11.2.11	Center Manifolds	398
11.3	Limit Cycles	399
11.3.1	Definition and Calculation	399
11.3.1.1	Harmonic Balance Method	399
11.3.1.2	Numerical Shooting	400
11.3.1.3	Collocation	401
11.3.2	Linearization	402
11.3.3	Topological Equivalence	403

- 11.3.4 Manifolds 403
- 11.3.5 Local Bifurcations 404
- 11.3.6 Period-Doubling Bifurcation 404
- 11.4 Numerical Continuation 405
- 11.4.1 Natural Parameter Continuation 405
- 11.4.2 Pseudo-Arc-Length Continuation 407
- 11.4.3 Continuation of Bifurcations 407
- References 408

12 Perturbation Methods 411

James Murdock

- 12.1 Introduction 411
- 12.2 Basic Concepts 412
 - 12.2.1 Perturbation Methods versus Numerical Methods 412
 - 12.2.2 Perturbation Parameters 413
 - 12.2.3 Perturbation Series 415
 - 12.2.4 Uniformity 416
- 12.3 Nonlinear Oscillations and Dynamical Systems 418
 - 12.3.1 Rest Points and Regular Perturbations 418
 - 12.3.2 Simple Nonlinear Oscillators and Lindstedt's Method 419
 - 12.3.3 Averaging Method for Single-Frequency Systems 422
 - 12.3.4 Multifrequency Systems and Hamiltonian Systems 424
 - 12.3.5 Multiple-Scale Method 426
 - 12.3.6 Normal Forms 427
 - 12.3.7 Perturbation of Invariant Manifolds; Melnikov Functions 429
- 12.4 Initial and Boundary Layers 429
 - 12.4.1 Multiple-Scale Method for Initial Layer Problems 429
 - 12.4.2 Matching for Initial Layer Problems 431
 - 12.4.3 Slow-Fast Systems 432
 - 12.4.4 Boundary Layer Problems 433
 - 12.4.5 WKB Method 434
 - 12.4.6 Fluid Flow 434
- 12.5 The "Renormalization Group" Method 435
 - 12.5.1 Initial and Boundary Layer Problems 436
 - 12.5.2 Nonlinear Oscillations 439
 - 12.5.3 WKB Problems 441
- 12.6 Perturbations of Matrices and Spectra 442
 - Glossary 444
 - References 446
 - Further Reading 447

13 Functional Analysis 449

Pavel Exner

- 13.1 Banach Space and Operators on Them 449
 - 13.1.1 Vector and Normed Spaces 449

13.1.2	Operators on Banach Spaces	450
13.1.3	Spectra of Closed Operators	452
13.2	Hilbert Spaces	452
13.2.1	Hilbert-Space Geometry	452
13.2.2	Direct Sums and Tensor Products	454
13.3	Bounded Operators on Hilbert Spaces	454
13.3.1	Hermitean Operators	454
13.3.2	Unitary Operators	455
13.3.3	Compact Operators	456
13.3.4	Schatten Classes	456
13.4	Unbounded Operators	457
13.4.1	Operator Adjoint and Closure	457
13.4.2	Normal and Self-Adjoint Operators	458
13.4.3	Tensor Products of Operators	460
13.4.4	Self-Adjoint Extensions	460
13.5	Spectral Theory of Self-Adjoint Operators	462
13.5.1	Functional Calculus	462
13.5.2	Spectral Theorem	463
13.5.3	More about Spectral Properties	465
13.5.4	Groups of Unitary Operators	467
13.6	Some Applications in Quantum Mechanics	468
13.6.1	Schrödinger Operators	468
13.6.2	Scattering Theory	470
	Glossary	472
	References	473
14	Numerical Analysis	475
	<i>Lyonell Boulton</i>	
14.1	Introduction	475
14.2	Algebraic Equations	476
14.2.1	Nonlinear Scalar Equations	476
14.2.2	Nonlinear Systems	477
14.2.3	Numerical Minimization	479
14.3	Finite-Dimensional Linear Systems	480
14.3.1	Direct Methods and Matrix Factorization	480
14.3.2	Iteration Methods for Linear Problems	482
14.3.3	Computing Eigenvalues of Finite Matrices	485
14.4	Approximation of Continuous Data	487
14.4.1	Lagrange Interpolation	487
14.4.2	The Interpolation Error	487
14.4.3	Hermite Interpolation	488
14.4.4	Piecewise Polynomial Interpolation	489
14.5	Initial Value Problems	491
14.5.1	One-Step Methods	491
14.5.2	Multistep Methods	492

14.5.3	Runge–Kutta Methods	494
14.5.4	Stability and Global Stability	495
14.6	Spectral Problems	496
14.6.1	The Infinite-Dimensional min–max Principle	497
14.6.2	Systems Confined to a Box	498
14.6.3	The Case of Unconfined Systems	499
	Further Reading	499
	References	500
15	Mathematical Transformations	503
	<i>Des McGhee, Rainer Picard, Sascha Trostorff, and Marcus Waurick</i>	
15.1	What are Transformations and Why are They Useful?	503
15.2	The Fourier Series Transformations	506
15.2.1	The Abstract Fourier Series	506
15.2.2	The Classical Fourier Series	507
15.2.3	The Fourier Series Transformation in $L^2(S_{\mathbb{C}}(0, 1))$	509
15.3	The z -Transformation	509
15.4	The Fourier–Laplace Transformation	510
15.4.1	Convolutions as Functions of ∂_ρ	512
15.4.1.1	Functions of ∂_ρ	512
15.4.1.2	Convolutions	513
15.4.2	The Fourier–Plancherel Transformation	515
15.5	The Fourier–Laplace Transformation and Distributions	515
15.5.1	Impulse Response	517
15.5.2	Shannon’s Sampling Theorem	517
15.6	The Fourier-Sine and Fourier-Cosine Transformations	518
15.7	The Hartley Transformations H_{\pm}	519
15.8	The Mellin Transformation	520
15.9	Higher-Dimensional Transformations	521
15.10	Some Other Important Transformations	522
15.10.1	The Hadamard Transformation	522
15.10.2	The Hankel Transformation	523
15.10.3	The Radon Transformation	523
	References	525
16	Partial Differential Equations	527
	<i>Des McGhee, Rainer Picard, Sascha Trostorff, and Marcus Waurick</i>	
16.1	What are Partial Differential Equations?	527
16.2	Partial Differential Equations in \mathbb{R}^{n+1} , $n \in \mathbb{N}$, with Constant Coefficients	529
16.2.1	Evolutionarity	530
16.2.2	An Outline of Distribution Theory	531
16.2.3	Integral Transformation Methods as a Solution Tool	533
16.3	Partial Differential Equations of Mathematical Physics	537
16.4	Initial-Boundary Value Problems of Mathematical Physics	540

16.4.1	Maxwell's Equations	542
16.4.2	Viscoelastic Solids	543
16.4.2.1	The Kelvin–Voigt Model	543
16.4.2.2	The Poynting–Thomson Model (The Linear Standard Model)	544
16.5	Coupled Systems	544
16.5.1	Thermoelasticity	545
16.5.2	Piezoelectromagnetism	546
16.5.3	The Extended Maxwell System and its Uses	547
	References	548
17	Calculus of Variations	551
	<i>Tomáš Roubíček</i>	
17.1	Introduction	551
17.2	Abstract Variational Problems	551
17.2.1	Smooth (Differentiable) Case	552
17.2.2	Nonsmooth Case	554
17.2.3	Constrained Problems	555
17.2.4	Evolutionary Problems	556
17.2.4.1	Variational Principles	556
17.2.4.2	Evolution Variational Inequalities	559
17.2.4.3	Recursive Variational Problems Arising by Discretization in Time	560
17.3	Variational Problems on Specific Function Spaces	562
17.3.1	Sobolev Spaces	562
17.3.2	Steady-State Problems	563
17.3.2.1	Second Order Systems of Equations	564
17.3.2.2	Fourth Order Systems	568
17.3.2.3	Variational Inequalities	570
17.3.3	Some Examples	571
17.3.3.1	Nonlinear Heat-Transfer Problem	571
17.3.3.2	Elasticity at Large Strains	572
17.3.3.3	Small-Strain Elasticity, Lamé System, Signorini Contact	573
17.3.3.4	Sphere-Valued Harmonic Maps	574
17.3.3.5	Saddle-Point-Type Problems	575
17.3.4	Evolutionary Problems	575
17.4	Miscellaneous	576
17.4.1	Numerical Approximation	576
17.4.2	Extension of Variational Problems	577
17.4.3	Γ -Convergence	581
	Glossary	582
	Further Reading	584
	References	586
	Index	589

