1 Heat transfer

1.1 Overview

1.1.1 Heat

A first description of what heat is comes from thermodynamics. That discipline describes how systems and their environment interchange energy. Anything can be a system: a material, a building assembly, a building, part of a HVAC system, even a whole city. Energy transmitted as 'work' is purposeful and organized, whereas as 'heat' it is diffuse and chaotic. A second description resides in particle physics, where the statistically distributed kinetic energy of atoms and free electrons stands for heat. In any case, heat is the least noble, most diffuse form of energy to which each nobler form degrades; consider the second law of thermodynamics.

1.1.2 Temperature

The temperature reflects the quality of the heat. Higher values reflect the increased kinetic energy of atoms and free electrons, resulting in higher exergy and the potential to convert more heat via a cyclic process into work. Lower temperatures and therefore less kinetic energy of atoms and free electrons result in less exergy. Higher temperatures require warming up, lower temperatures cooling down of a system. Like any potential, temperature is a scalar, which, as heat, cannot be measured directly. It is sensed and because many material properties depend on it, indirectly quantifiable. A mercury thermometer uses the volumetric expansion of mercury when heated and the contraction when cooled. In a Pt100 thermometer, the electrical resistance of platinum wire changes with temperature. Temperature logging with thermocouples uses the varying contact potential between metals.

The SI system advances two temperature scales, one empiric in degrees Celsius, °C, with the symbol θ, and one thermodynamic in degrees kelvin, K, with the symbol T. Zero °C coincides with the triple point of water, and 100 °C with the boiling point of water at 1 atmosphere pressure. Zero K instead stands for the point of absolute zero, and 273.15 K coincides with the triple point of water. Temperature differences are given in K, temperatures in °C or K, with the following relationship between the two:

$$T = \theta + 273.15$$

Instead of degrees Celsius, the US generally uses degrees Fahrenheit (°F):

$$°\mathrm{F} = 32 + 9/5\ °\mathrm{C}$$

1.1.3 Sensible and latent heat

Sensible heat transfer, whether by conduction, convection or radiation, requires temperature differences. Conduction refers to the heat exchanged when vibrating

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atoms collide and free electrons move. Transmission between solids at different temperatures in ideal contact and among points at different temperature in the same solid is conduction-based. Conduction also intervenes in gases and liquids, whether or not they are in contact with solids. According to the second law of thermodynamics, conduction always goes in the direction of lower temperatures. A medium is required, but conduction induces no movement.

Convection instead is the result of macroscopic motion in liquids and gases wherein temperatures differences exist, included contact with colder or warmer solids. Whether it is an external force, a difference in density or both that fosters movement, will define the type of convection: forced, natural or mixed. Also convection needs a medium.

Radiation, finally, concerns the heat transferred due to the emission and absorption of electromagnetic waves by surfaces. At temperatures above 0 K, every surface emits. If two or more are at different temperatures, the result is heat exchange. Radiation does not need a medium, while the governing laws differ strongly from those describing conduction and convection.

Latent heat directly links to changes of state. Its release or absorption requires no temperature differences but affects the whole thermal picture. To give an example, water evaporating absorbs a quantity of sensible heat equal to the heat of evaporation, so acts as a heat sink. When the water vapour formed moves to a colder spot where it condenses again, that heat of evaporation is re-emitted, so forms a heat source. These sources and sinks not only impact the temperatures in materials and assemblies, but also the latent heat transferred.

1.1.4 Why are heat and temperature so compelling?

Why heat? The main motivation is that energy use in buildings matters. In fact, thermal comfort indoors requires an operative temperature at the desired level, which in cold and temperate climates means heating. Related fossil fuel burning still represents a substantial share of the overall end energy consumed and CO_2 emitted, such that energy efficiency in buildings became mandatory, among others by minimizing the heat traversing the enclosure.

Why temperature? There are many reasons. Inside surface temperatures close to the air temperature indoors improve thermal comfort, whereas low values increase mould and surface condensation risks. Excessive summer temperatures indoors affect usability, while higher temperature differences across envelope assemblies foster air and moisture movement, thermal stress and crack risk. Too many swings from above to below freezing point may damage frost-sensitive porous materials that stay wet. High temperature gradients favour combined moisture and dissolved salt displacement, while high temperatures accelerate the chemical breakdown of synthetics. Whether these effects remain controllable depends on how building assemblies are designed and built.

1.1.5 Some definitions

Amount of heat, symbol Q, units J	Quantifies the energy exchanged as heat. As energy is a scalar, so is the amount of heat.
Heat flow, symbol Φ, units J/s = W	Heat migrating per unit of time. Heat flow is a measure for 'power', thus a scalar.
Heat flux, symbol q, units W/m^2	Heat migrating per unit of time across a unit surface normal to the flow. Flux is a vector with the same direction as the surface vector. The components in Cartesian coordinates are q_x, q_y, q_z, in polar coordinates q_R, q_ϕ, q_Θ.

Solving a heat transfer problem means determining a scalar temperature (T) and vector heat flux (q) field. The calculation thus requires a scalar and a vector equation.

1.2 Conduction

1.2.1 Conservation of energy

A first relationship between heat flux (q) and temperature (T) follows from the conservation of energy axiom. If the system is an infinitely small material volume dV and everything that is around it the environment, then, knowing that conduction does not displace mass, the energy balance becomes:

$$\mathrm{d}\Phi + \mathrm{d}\Psi = \mathrm{d}U + \mathrm{d}W \tag{1.1}$$

with dΦ the resulting heat flow between system and environment, dΨ the heat dissipated uniformly in the system, dU the change in the system's internal energy and dW the work exchanged with the environment, all three per unit of time. Dissipation can include heat produced by an exothermic chemical reaction, heat absorbed thanks to an endothermic chemical reaction, the Joule effect due to an electric current passing through, latent heat released or absorbed, and so on. The work exchanged equals:

$$\mathrm{d}W = P\mathrm{d}(\mathrm{d}V) = P\mathrm{d}^2V$$

with P being the pressure in Pa. The balance states that the heat exchanged (= dΦ), released or absorbed modifies the internal energy of the material volume while generating an exchange of work with the environment. If isobaric, the balance can be rewritten as:

$$\mathrm{d}(U + P\mathrm{d}V) = \mathrm{d}Q + \mathrm{d}E$$

with $U + P\mathrm{d}V$ being the enthalpy H. Writing the resulting heat flow, the enthalpy change and the heat dissipated as:

$$\mathrm{d}\Phi = -\mathrm{div}(q)\mathrm{d}V \qquad \mathrm{d}H = \left|\frac{\partial(\rho\, c_p T)}{\partial t}\right|\mathrm{d}V \qquad \mathrm{d}\Psi = \Phi'\mathrm{d}V$$

where c_p is the specific heat capacity at constant pressure of the material (J/(kg.K)), ρ is its density (kg/m^3) and Φ' is the dissipated heat per unit of time and volume, positive for a source, negative for a sink, which makes the conservation law:

$$\left(\mathrm{div}(q) + \Phi' + \frac{\partial\left(\rho c_p T\right)}{\partial t}\right)\mathrm{d}V = 0 \tag{1.2}$$

For solids and liquids the specific heat capacity depends little on the change of state. One value, symbol c, can be used, with the product ρc equal to the volumetric specific heat capacity. For gases, the value varies according to the change of state, giving as the relationship between the specific heat capacity at constant pressure (c_p) and constant volume (c_v):

$$c_p = c_v + R$$

with R the specific gas constant (in Pa.m^3/(kg.K)). Because conservation of energy holds for any infinitely small material volume, the relationship between heat flux (q) and temperature (T) finally becomes:

$$\mathrm{div}\,(q) = -\frac{\partial(\rho c T)}{\partial t} - \Phi' \tag{1.3}$$

1.2.2 The conduction laws

1.2.2.1 First law

The first law is the empirical conduction equation introduced by the French physicist Fourier in 1822 (Figure 1.1):

$$q = -\lambda\,\mathrm{grad}\,T = -\lambda\,\mathrm{grad}\,\theta \tag{1.4}$$

Fig. 1.1 The French physicist Fourier

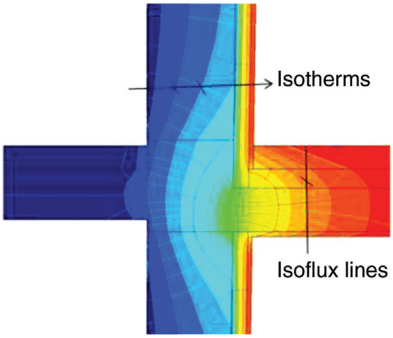


Fig. 1.2 Lines of equal temperature (isotherms) and equal heat flow rates (isoflux lines)

This is a vector relation, which states that the conductive heat flux anywhere in a solid, liquid or gas varies proportionally to the temperature gradient there. The multiplier λ is the thermal conductivity, units W/(m.K). The minus sign indicates that flux and temperature gradient, positive from colder to warmer, oppose each other. Thermodynamics in fact teaches that, if not forced externally, heat always moves in the direction of lower temperatures. Otherwise entropy would decrease without energy input from the environment, which is impossible.

The following observation supports the law. With the surfaces of equal temperature, called isotherms, in a material traced and the heat fluxes visualized, it can be observed that these develop perpendicular to the isotherms and increase where these are near to one another (Figure 1.2).

At the same time the fluxes keep proportionality with the thermal conductivity, which for reasons of simplicity is assumed to be scalar and constant, even though for building and insulating materials such assumptions do simplify reality. The value in fact depends on temperature, moisture content, thickness and sometimes age. In anisotropic materials, it is even a tensor. Fortunately, for first-order calculations, ‘scalar and constant’ often suffices.

In right-angled Cartesian coordinates [*x*,*y*,*z*], the heat fluxes along the axes become:

$$q_x = \boldsymbol{q_x u_x} = -\lambda \frac{\partial T}{\partial x} \qquad q_y = \boldsymbol{q_y u_y} = -\lambda \frac{\partial T}{\partial y} \qquad q_z = \boldsymbol{q_z u_z} = -\lambda \frac{\partial T}{\partial z}$$

Often the temperature in °C replaces K. The heat flow across a surface area dA with direction n then equals:

$$\mathrm{d}\Phi_n = \boldsymbol{q\mathrm{d}A_n} = -\lambda \frac{\partial \theta}{\partial n} \mathrm{d}A_n u_n^2 = -\lambda \frac{\partial \theta}{\partial n} \mathrm{d}A_n$$

Along the three axes, one gets:

$$\mathrm{d}\Phi_x = -\lambda \frac{\partial \theta}{\partial x} \mathrm{d}A_x \qquad \mathrm{d}\Phi_y = -\lambda \frac{\partial \theta}{\partial y} \mathrm{d}A_y \qquad d\Phi_z = -\lambda \frac{\partial \theta}{\partial z} dA_z$$

1.2.2.2 Second law

To arrive at a second law, the conduction equation is implemented in the conservation of energy expression:

$$\mathrm{div}(\lambda\,\mathrm{grad}\,T) = \frac{\partial(\rho c T)}{\partial t} - \Phi' \tag{1.5}$$

The result is a scalar equation that allows the calculation of temperature fields. In the case that the thermal conductivity and the volumetric specific heat capacity are constant, the expression simplifies to what is known as Fourier's second law:

$$\nabla^2 T = \left(\frac{\rho c}{\lambda}\right)\frac{\partial T}{\partial t} - \frac{\Phi'}{\lambda} \tag{1.6}$$

with ∇^2 being the Laplace operator, in Cartesian coordinates equal to:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.7}$$

Further discussions focus on solving both laws for a series of building-related cases.

1.2.3 Steady state

Steady state indicates that the temperatures and heat fluxes remain time-independent. For that, constant boundary conditions, constant material properties and constant energy dissipation are needed. When all are invariable, then:

$$\partial T/\partial t = 0$$

With the temperature in °C, Fourier's second law so simplifies to:

$$\nabla^2 \theta = -\Phi'/\lambda \tag{1.8}$$

1.2.3.1 One-dimensional flat assemblies

In one dimension with temperature changes normal to the surface, the equation further reduces to:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = -\frac{\Phi'}{\lambda} \tag{1.9}$$

Without dissipation, it becomes:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = 0$$

with, as a solution:

$$\theta = \mathrm{C}_1 x + \mathrm{C}_2 \tag{1.10}$$

where C_1 and C_2 are the integration constants, fixed by the boundary conditions. That simple equation governs conduction across flat assemblies with the end faces at constant but different temperatures. Buildings include numerous flat assemblies such as low-slope roofs, sloped roof pitches, outer walls, floors, partition walls, glass surfaces, and so on. In cross-section, these can be single-layer or composite.

Single-layer

If single-layer, the thermal conductivity of the material and the layer thickness (d) has to be known. The boundary conditions are: $x = 0$: $\theta = \theta_{s1}$; $x = d$: $\theta = \theta_{s2}$ with θ_{s1} and θ_{s2} being the different temperatures at the end faces, whereby θ_{s1} is assumed to be colder than θ_{s2}. The integration constants then become

$$C_2 = \theta_{s1} \qquad C_1 = (\theta_{s2} - \theta_{s1})/d$$

which changes the temperatures in the layer into:

$$\theta = \frac{\theta_{s2} - \theta_{s1}}{d} x + \theta_{s1} \tag{1.11}$$

Or, at steady state, the temperatures in a flat, single-layer assembly with neither heat source nor sink will vary linearly between the values at both end faces (Figure 1.3).

Related heat flux equals:

$$q = -\lambda \,\mathrm{grad}\, \theta = -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x} = -\lambda \left(\frac{\theta_{s1} - \theta_{s2}}{d} \right) \tag{1.12}$$

In absolute terms, this equation becomes:

$$q = \lambda \left(\frac{\theta_{s2} - \theta_{s1}}{d} \right) \tag{1.13}$$

showing that the flux is proportional to the thermal conductivity of the material and the temperature difference between both end faces and inversely proportional to the layer thickness. For given thickness and temperature difference, a lower thermal

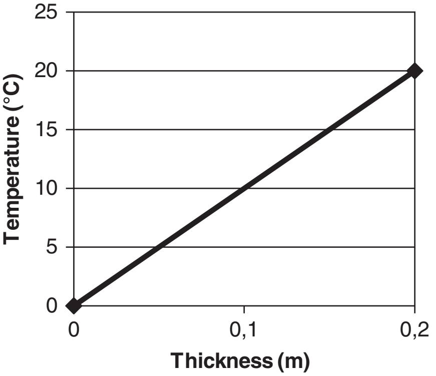


Fig. 1.3 Temperatures in a single-layer assembly

conductivity reduces the flux, meaning less heat being lost or gained. Materials with very low thermal conductivity are therefore called insulation materials. Rearranging gives:

$$q = \frac{\Delta\theta}{d/\lambda} \tag{1.14}$$

with d/λ the thermal resistance of the flat single-layer assembly, symbol R, units m^2.K/W. The higher that number, the lower the heat flux for a given temperature difference between both end faces, or, the better the assembly insulates. Higher values require larger thicknesses or, for a given thickness, better insulating materials. The inverse of the thermal resistance is the thermal conductance, symbol P, units W/(m^2.K), a quantity telling how much heat per unit of time and surface passes across the assembly for a 1 °C or 1 K temperature difference between both end faces.

Composite

Composite assemblies consist of two or more plane-parallel layers. In buildings, most flat parts are composite, an example being the filled cavity wall of Figure 1.4, showing the inside leaf, cavity fill and veneer wall.

In steady state, without heat sources or sinks, the flux must be the same in each layer. Otherwise, thermal storage or discharge will make the regime transient. Suppose the temperature of end face 1 is θ_{s1}, and for end face 2, θ_{s2} ($\theta_{s1} < \theta_{s2}$). If the thermal conductivities and thicknesses of all layers are known and possible contact resistances between layers can be neglected, which fits for non-metallic materials, the heat flux q being constant means:

$$\text{Layer 1} \quad q = \lambda_1 \frac{\theta_1 - \theta_{s1}}{d_1}$$

$$\text{Layer 2} \quad q = \lambda_2 \frac{\theta_2 - \theta_1}{d_2} \ldots$$

Fig. 1.4 Filled cavity wall

$$\text{Layer } n-1 \quad q = \lambda_{n-1}\frac{\theta_{n-1}-\theta_{n-2}}{d_{n-1}}$$

$$\text{Layer } n \quad q = \lambda_{n}\frac{\theta_{s2}-\theta_{n-1}}{d_{n}}$$

with θ_1, θ_2, . . . , θ_{n-1} the unknown interface temperatures. Rearrangement and summing gives:

$$\begin{array}{ll}
q\dfrac{d_1}{\lambda_1} & = \theta_1 - \theta_{s1} \\
+q\dfrac{d_2}{\lambda_2} & = \theta_2 - \theta_1 \\
+\ldots & \\
+q\dfrac{d_n}{\lambda_n} & = \theta_{s2} - \theta_{n-1} \\
\hline
q\displaystyle\sum_{i=1}^{n}\left(\frac{d_i}{\lambda_i}\right) & = \theta_{s2} - \theta_{s1}
\end{array}$$

or:

$$q = \frac{\theta_{s2}-\theta_{s1}}{\sum_{i=1}^{n}(d_i/\lambda_i)} \tag{1.15}$$

The sum $\sum(d_i/\lambda_i)$, symbol R_T, units $m^2\,K/W$, is called the total thermal resistance of the assembly, and the ratio d_i/λ_i, symbol R_i, same units as R_T, the thermal resistance of layer *i*. The higher the total thermal resistance, the lower the steady-state heat flux and the better the assembly insulates. A high thermal resistance requires a sufficiently thick insulation layer. Since the energy crises of the 1970s, thermal insulation of high performance became a prime measure to lower the net heating demand and CO_2 release where heating systems burn fossil fuels.

How the assembly is designed fixes its total thermal resistance. As a whole, the commutation property applies and the layer sequence does not matter. Inside insulation should perform as well as outside insulation. From a whole building physics perspective, that conclusion is somewhat deceptive: the same thermal resistance, yes, but both diverge in overall performance, a fact not highlighted in the steady state. The analogy with the current (I) in an electrical circuit subjected to a voltage difference ΔV when the electrical resistances (R_{ei}) stay in series is instructive:

$$I = \Delta V / \sum R_{ei}$$

Clearly, temperature replaces voltage, heat flux the current, and thermal resistance the electrical resistance. This allows the conversion of heat conduction problems into an electrical analogy.

With both end face temperatures known, all interface temperatures follow from rearranging the heat flux equation per layer:

$$\begin{aligned}
\theta_1 &= \theta_{s1} + q\frac{d_1}{\lambda_1} = \theta_{s1} + (\theta_{s2} - \theta_{s1})\frac{R_1}{R_T} \\
\theta_2 &= \theta_1 + q\frac{d_2}{\lambda_2} = \theta_1 + q\,R_2 = \theta_{s1} + (\theta_{s2} - \theta_{s1})\frac{(R_1 + R_2)}{R_\mathrm{T}}\ldots \\
\theta_{n-1} &= \theta_{n-2} + q\frac{d_{n-1}}{\lambda_{n-1}} = \theta_{s1} + (\theta_{s2} - \theta_{s1})\frac{\sum\limits_{i=1}^{n-1} R_i}{R_\mathrm{T}}
\end{aligned}$$

Writing $\theta_i = \theta_{i-1} + qd_i/\lambda_i$ as $(\theta_i - \theta_{i-1})/d_i = q/\lambda_i$ underlines that the temperature gradient in a layer is inversely proportional to its thermal conductivity. Hence, gradients are large in insulation and small in conductive layers. Each layer equation can yet be rewritten as:

$$\theta_x = \theta_{s1} + q\,R_{s1}^x \tag{1.16}$$

where R_{s1}^x is the thermal resistance between end face s1 and interface x in the assembly. When the calculation starts at end face s2, the equation becomes:

$$\theta_x = \theta_{s2} - q\,R_{s2}^x \tag{1.17}$$

In a $[R,\theta]$ plane with the thermal resistance R as abscissa and the temperature θ as ordinate, both represent a straight line linking both end face temperatures $(0, \theta_{s1})$ and (R, θ_{s2}) with the heat flux as slope. Effectively a composite assembly responds as if it is single-layered. To construct the temperature line in the thickness graph, first draw the assembly in the $[R,\theta]$ plane with the layers as thick as their thermal resistance. Mark the temperature θ_{s1} on end face s1, temperature θ_{s2} on end face s2, and trace the straight line in between. The thickness-related course then follows from transposing all intersections with the successive interfaces into the thickness graph and then linking them per layer as shown in Figure 1.5. Of course, the correct layer sequence must be kept.

For single-layer as well as composite assemblies, the product of the heat flux with the surface area (A) gives the heat flow:

$$\Phi = q\,A \tag{1.18}$$

Special cases

Three special cases demand consideration. The first concerns a single-layer assembly where the thermal conductivity changes with temperature or moisture content (w), so along the ordinate x (Figure 1.6).

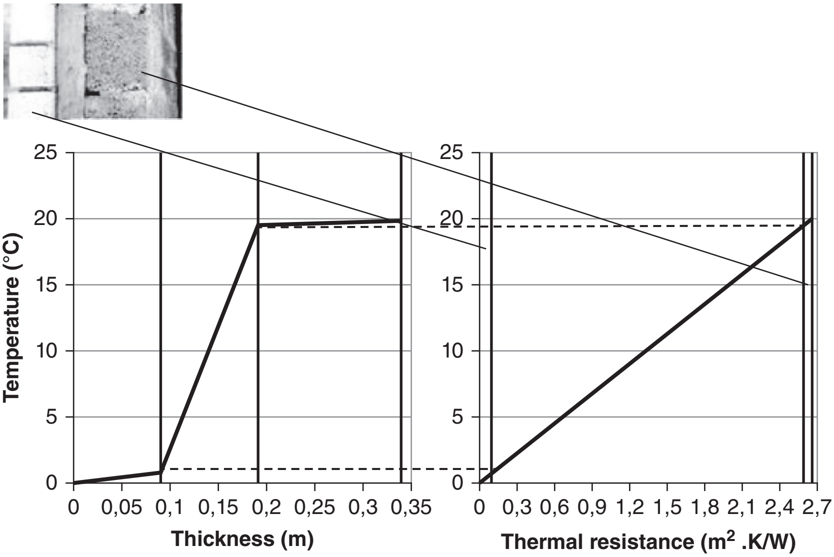


Fig. 1.5 Temperatures in a composite assembly, graphic construction

Should the thermal conductivity vary linearly with temperature ($\lambda = \lambda_o + a\theta$), then the heat flux will follow from ($x = 0$, $\theta = \theta_{s1}$; $x = d$, $\theta = \theta_{s2}$):

$$\int_{\theta_{s1}}^{\theta_{s2}} (\lambda_0 + a\theta)\mathrm{d}\theta = \int_0^d q\,\mathrm{d}x$$

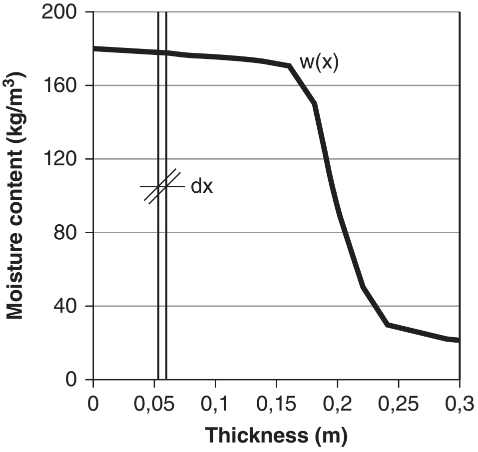


Fig. 1.6 Moisture profile with thermal conductivity as a function of *x*

giving as a solution:

$$\lambda_0(\theta_{s2}-\theta_{s1})+\frac{a\left(\theta_{s2}^2-\theta_{s1}^2\right)}{2}=q\,d \qquad \text{or} \qquad q=\lambda(\theta_m)\frac{\theta_{s2}-\theta_{s1}}{d} \tag{1.19}$$

with $\lambda(\theta_m)$ the thermal conductivity at the average temperature in the layer. The thermal resistance is $d/\lambda(\theta_m)$, while the temperature curve changes into a parabola:

$$a\theta^2/2+\lambda_0\theta=q\,x+C$$

For an assembly loaded with a varying moisture content $w(x)$, so that $\lambda = F(w(x)) = f(x)$, the heat flux becomes ($x=0$: $\theta=\theta_{s1}$; $x=d$: $\theta=\theta_{s2}$):

$$q\int_0^d \frac{dx}{\lambda(x)}=\int_{\theta_{s1}}^{\theta_{s2}} d\theta$$

In case the moisture distribution is such that the thermal conductivity increases proportionally to x ($\lambda=\lambda_o+ax$), the integrals give:

$$q=\frac{\theta_{s2}-\theta_{s1}}{\dfrac{1}{a}\ln\left(\dfrac{\lambda_0+a\,d}{\lambda_0}\right)} \tag{1.20}$$

Again, the denominator stands for the thermal resistance. The temperature course now becomes:

$$\theta_x=\theta_{s1}+q\left[\frac{1}{a}\ln\left(\frac{\lambda_0+a\,x}{\lambda_0}\right)\right] \tag{1.21}$$

The second special case concerns a single-layer assembly that dissipates or absorbs Φ joules of heat per unit of time and volume. If this happens uniformly over the thickness, the steady-state balance equation changes to:

$$\frac{d^2\theta}{dx^2}=-\frac{\Phi'}{\lambda}$$

with, as boundary conditions: $x=0$, $\theta=\theta_{s1}$; $x=d$, $\theta=\theta_{s2}$ ($\theta_{s1}<\theta_{s2}$). The solution is:

$$\theta=-\frac{\Phi'}{2\lambda}x^2+C_1x+C_2 \tag{1.22}$$

a parabolic temperature curve: convex for a heat source, and concave for a heat sink (Figure 1.7).

The heat flux changes to:

$$q=-\lambda\frac{d\theta}{dx}=\Phi'\,x-C_1\lambda \tag{1.23}$$

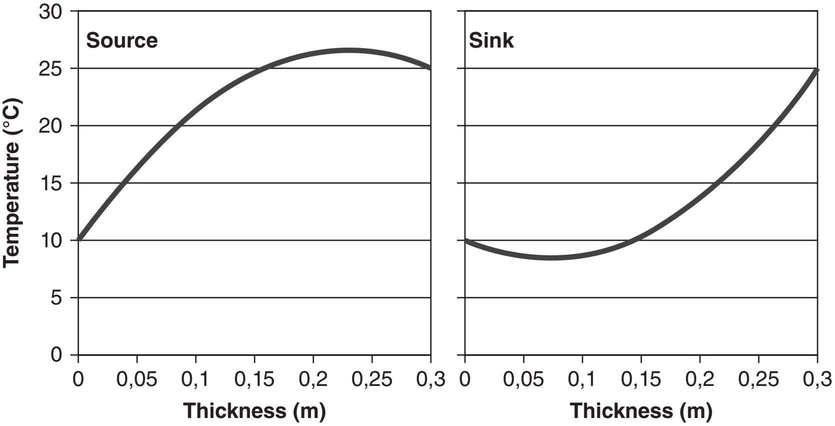


Fig. 1.7 Temperature in a uniformly spread heat source or sink in a single-layer assembly

Instead of no heat dissipated, the heat flux now changes from point to point across the assembly. The boundary conditions give as integration constants:

$$C_1 = \frac{\theta_{s2} - \theta_{s1}}{d} + \frac{\Phi d'}{2\lambda} \qquad C_2 = \theta_{s1}$$

The third special case concerns a composite assembly with a local heat source or sink, for example, due to a deposit or drying of condensate in an interface between two layers. The heat flux dissipated this way is q', while the temperature θ_{s1} at one end face passes θ_{s2} at the other end face. Then, a steady-state heat balance at the interface (x) with the source or sink gives the temperatures in the assembly. Presuming that the heat flows from both end faces to x:

$$\text{End face s1:} \quad q_{s1}^x = \frac{\theta_{s1} - \theta_x}{R_{s1}^x}$$
$$\text{End face s2:} \quad q_{s2}^x = \frac{\theta_{s2} - \theta_x}{R_{s2}^x}$$

In both equations, θ_x is the unknown temperature in x. Setting the sum of the two heat fluxes and the dissipated heat zero, with the dissipated heat negative for drying and positive for condensation, gives:

$$\theta_x = \frac{R_{s2}^x \theta_{s1} + R_{s1}^x \theta_{s2} + q' R_{s1}^x R_{s2}^x}{R_{s1}^x + R_{s2}^x} \tag{1.24}$$

Introducing that temperature in the flux equations from both end faces results in:

$$q_{s1}^x = \left(\frac{\theta_{s1} - \theta_{s2}}{R_T}\right) - \frac{q' R_{s2}^x}{R_T} \qquad q_{s2}^x = -\left[\left(\frac{\theta_{s1} - \theta_{s2}}{R_T}\right) + \frac{q' R_{s1}^x}{R_T}\right] \tag{1.25}$$

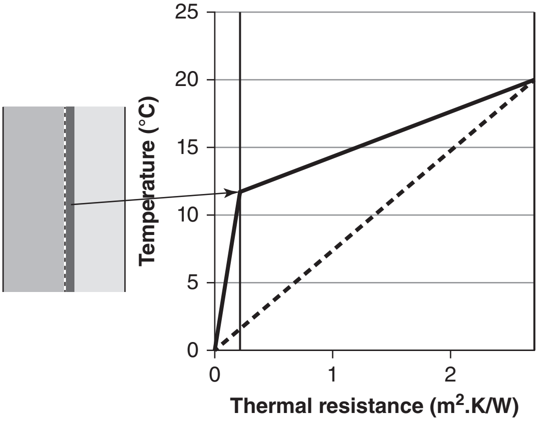


Fig. 1.8 Composite assembly, with heat source in the interface between two layers

with R_T the total thermal resistance of the assembly, equal to:

$$R_T = R_{s1}^x + R_{s2}^x$$

For a heat source the incoming flux drops and the outgoing flux increases compared with no source (Figure 1.8). For a sink, it is the inverse.

1.2.3.2 Two dimensions, cylinder symmetric

In cylinder coordinates, cylinder symmetric cases behave as one-dimensional, such as hung heating pipes. Of interest are the heat loss per metre run, the pipe temperature and the insulation efficiency. Consider a pipe with an inside radius r_1 and outside radius r_2. The temperature at the inside face is θ_{s1}, and at the outside face θ_{s2} (Figure 1.9). In steady state with no dissipation, the same heat flow must pass each cylinder concentric to the pipe's centre. With that centre as origin, the flow per metre run is:

$$\Phi = -\lambda(2\pi r)\mathrm{d}\theta/\mathrm{d}r = \mathrm{C}^{\mathrm{t}}$$

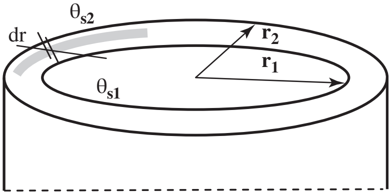


Fig. 1.9 The pipe problem

Integration gives:

$$\Phi \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r} = -2\pi\lambda \int_{\theta_{s1}}^{\theta_{s2}} \mathrm{d}\theta \qquad \text{or} \qquad \Phi = \frac{\theta_{s1} - \theta_{s2}}{\left(\dfrac{\ln(r_2/r_1)}{2\pi\lambda}\right)} \tag{1.26}$$

The denominator is the equivalent of the thermal resistance of a flat assembly. This is the thermal resistance per metre pipe, with units m.K/W.

For a composite pipe, the same reasoning as for a composite flat assembly gives the heat flow per metre run as:

$$\Phi = \frac{\theta_{s1} - \theta_{s2}}{\sum_{i=1}^{n} \left[\dfrac{\ln(r_{i+1}/r_i)}{2\pi\lambda_i}\right]} \tag{1.27}$$

The temperatures then follow from:

$$\theta_{i+1} = \theta_{s1} + \Phi \sum_{i=1}^{i} \left[\frac{\ln(r_{i+1}/r_i)}{2\pi\lambda_i}\right] \tag{1.28}$$

1.2.3.3 Two and three dimensions: thermal bridges

When looking in detail at outside walls, roofs, floors and partition walls, the assumption of 'flat' does not apply everywhere. What about lintels above windows? What about window reveals? What about junctions between two outside walls? What about the corners between two outer walls and a low-slope roof? Also, the end faces of flat assemblies are not necessarily isothermal (Figure 1.10). Studying steady-state

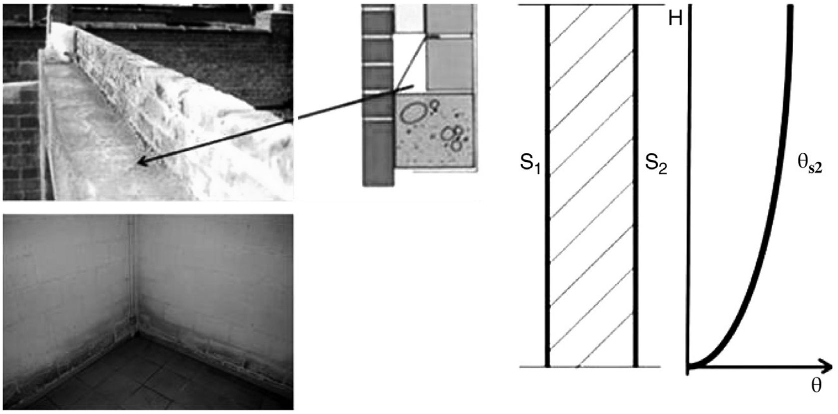


Fig. 1.10 Lintel above a window; corner between two outside walls and the floor; flat assembly with non-isothermal inside face

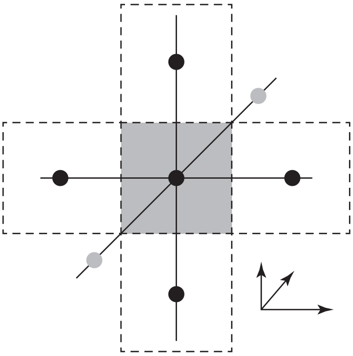


Fig. 1.11 Central and neighbouring control volumes

heat transfer now requires solving:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=\pm\frac{\Phi'}{\lambda}$$

or, without dissipation:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=0$$

For some very elementary cases, with one material, easy geometry and simple boundary conditions, this partial differential equation can be solved analytically. In the majority of building-related cases, only a numeric approach, for example using the control volume method (CVM), offers a solution. Many building details yet consist of rectangular material volumes. In CVM, such detail is meshed in cubic or beam-like control volumes and the sum of the steady-state heat flows coming from the six adjacent volumes is set to zero (Figure 1.11).

If the meshing coincides with the interfaces between layers, then all control volumes remain material-homogenous but the calculations do not give the interface temperatures. The preference therefore goes to control volumes whose centres lie on the interfaces. They are not material-homogenous but the calculations give the interface temperatures. Along the x, y and z axes, the sides of the control volumes around each centre-point equal the sum of half the distance in the negative and positive directions to the adjacent centre points. If all seven centre points are situated in the same material and the mesh width is equal along all three axes, the heat flow from the centre point in the nearest by volume $(\mathrm{l}-1,\mathrm{m},\mathrm{n})$ to the centre point in the central volume $(\mathrm{l},\mathrm{m},\mathrm{n})$ equals:

$$\Phi_{\mathrm{l-1,m,n}}^{\mathrm{l,m,n}}=\lambda\frac{\left(\theta_{\mathrm{l-1,m,n}}-\theta_{\mathrm{l,m,n}}\right)\mathrm{a}^2}{\mathrm{a}}=\mathrm{a}\lambda\left(\theta_{\mathrm{l-1,m,n}}-\theta_{\mathrm{l,m,n}}\right)$$

with a being the mesh width. The other five give:

$$\begin{aligned}
\Phi_{\mathrm{l+1,m,n}}^{\mathrm{l,m,n}} &= \lambda\frac{\left(\theta_{\mathrm{l+1,m,n}}-\theta_{\mathrm{l,m,n}}\right)\mathrm{a}^2}{\mathrm{a}}=\mathrm{a}\lambda\left(\theta_{\mathrm{l+1,m,n}}-\theta_{\mathrm{l,m,n}}\right)\\
\Phi_{\mathrm{l,m-1,n}}^{\mathrm{l,m,n}} &= \lambda\frac{\left(\theta_{\mathrm{l,m-1,n}}-\theta_{\mathrm{l,m,n}}\right)\mathrm{a}^2}{\mathrm{a}}=\mathrm{a}\lambda\left(\theta_{\mathrm{l,m-1,n}}-\theta_{\mathrm{l,m,n}}\right)\\
\Phi_{\mathrm{l,m+1,n}}^{\mathrm{l,m,n}} &= \lambda\frac{\left(\theta_{\mathrm{l,m+1,n}}-\theta_{\mathrm{l,m,n}}\right)\mathrm{a}^2}{\mathrm{a}}=\mathrm{a}\lambda\left(\theta_{\mathrm{l,m+1,n}}-\theta_{\mathrm{l,m,n}}\right)\\
\Phi_{\mathrm{l,m,n-1}}^{\mathrm{l,m,n}} &= \lambda\frac{\left(\theta_{\mathrm{l,m,n-1}}-\theta_{\mathrm{l,m,n}}\right)\mathrm{a}^2}{\mathrm{a}}=\mathrm{a}\lambda\left(\theta_{\mathrm{l,m,n-1}}-\theta_{\mathrm{l,m,n}}\right)\\
\Phi_{\mathrm{l,m,n+1}}^{\mathrm{l,m,n}} &= \lambda\frac{\left(\theta_{\mathrm{l,m,n+1}}-\theta_{\mathrm{l,m,n}}\right)\mathrm{a}^2}{\mathrm{a}}=\mathrm{a}\lambda\left(\theta_{\mathrm{l,m,n+1}}-\theta_{\mathrm{l,m,n}}\right)
\end{aligned}$$

Adding and setting to zero results in:

$$\theta_{\mathrm{l-1,m,n}}+\theta_{\mathrm{l+1,m,n}}+\theta_{\mathrm{l,m-1,n}}+\theta_{\mathrm{l,m+1,n}}+\theta_{\mathrm{l,m,n-1}}+\theta_{\mathrm{l,m,n+1}}-6\theta_{\mathrm{l,m,n}}=0$$

Unknowns in this linear equation are the temperatures in the central and six adjacent control volumes.

In two dimensions, each control volume has only four adjacent volumes, so the seven heat balances reduce to five:

$$\theta_{\mathrm{l-1,m}}+\theta_{\mathrm{l+1,m}}+\theta_{\mathrm{l,m-1}}+\theta_{\mathrm{l,m+1}}-4\theta_{\mathrm{l,m}}=0$$

If the central control volume bridges an interface between two materials, if that interface is parallel to the $[x,\ y]$ plane and if the thermal conductivities of the materials at both sides are λ_1 and λ_2, then for a mesh width a along the three axes, the heat flow from volume (l − 1,m,n) to the central volume (l,m,n) equals:

$$\Phi_{\mathrm{l-1,m,n}}^{\mathrm{l,m,n}}=\lambda_1\frac{\left(\theta_{\mathrm{l-1,m,n}}-\theta_{\mathrm{l,m,n}}\right)\mathrm{a}^2}{2\mathrm{a}}+\lambda_2\frac{\left(\theta_{\mathrm{l-1,m,n}}-\theta_{\mathrm{l,m,n}}\right)\mathrm{a}^2}{2\mathrm{a}}$$

or:

$$\Phi_{\mathrm{l-1,m,n}}^{\mathrm{l,m,n}}=\frac{\mathrm{a}(\lambda_1+\lambda_2)\left(\theta_{\mathrm{l-1,m,n}}-\theta_{\mathrm{l,m,n}}\right)}{2}$$

The other five give:

$$\begin{aligned}
\Phi_{\mathrm{l+1,m,n}}^{\mathrm{l,m,n}} &= \frac{\mathrm{a}(\lambda_1+\lambda_2)\left(\theta_{\mathrm{l+1,m,n}}-\theta_{\mathrm{l,m,n}}\right)}{2}\\
\Phi_{\mathrm{l,m-1,n}}^{\mathrm{l,m,n}} &= \frac{\mathrm{a}(\lambda_1+\lambda_2)\left(\theta_{\mathrm{l,m-1,n}}-\theta_{\mathrm{l,m,n}}\right)}{2}\\
\Phi_{\mathrm{l,m+1,n}}^{\mathrm{l,m,n}} &= \frac{\mathrm{a}(\lambda_1+\lambda_2)\left(\theta_{\mathrm{l,m+1,n}}-\theta_{\mathrm{l,m,n}}\right)}{2}\\
\Phi_{\mathrm{l,m,n-1}}^{\mathrm{l,m,n}} &= \mathrm{a}\lambda_1\left(\theta_{\mathrm{l,m,n-1}}-\theta_{\mathrm{l,m,n}}\right)\\
\Phi_{\mathrm{l,m,n+1}}^{\mathrm{l,m,n}} &= \mathrm{a}\lambda_2\left(\theta_{\mathrm{l,m,n+1}}-\theta_{\mathrm{l,m,n}}\right)
\end{aligned}$$

Adding and setting to zero results in:

$$\frac{(\lambda_1+\lambda_2)\left(\theta_{\mathrm{l-1,m,n}}+\theta_{\mathrm{l+1,m,n}}+\theta_{\mathrm{l,m-1,n}}+\theta_{\mathrm{l,m+1,n}}\right)}{2}+\lambda_2\theta_{\mathrm{l,m,n-1}}+\lambda_1\theta_{\mathrm{l,m,n+1}}$$
$$-3(\lambda_1+\lambda_2)\theta_{\mathrm{l,m,n}}=0$$

which is a linear equation with the temperatures in the central and six adjacent control volumes as the seven unknowns. Two dimensions give:

$$\frac{(\lambda_1+\lambda_2)\left(\theta_{\mathrm{l-1,m}}-\theta_{\mathrm{l+1,m}}\right)}{2}+\lambda_2\theta_{\mathrm{l,m-1}}+\lambda_1\theta_{\mathrm{l,m+1}}-2(\lambda_1+\lambda_2)\theta_{\mathrm{l,m,n}}=0$$

If the central control volume lies on the intersection between three materials with the interfaces parallel to the $[x, y]$ and $[y, z]$ planes and the thermal conductivities of the three materials are λ_1, λ_2 and λ_3, then the sum equals:

$$(\lambda_2+\lambda_3)\frac{\theta_{\mathrm{l-1,m,n}}}{2}+(\lambda_2+\lambda_3)\frac{\theta_{\mathrm{l+1,m,n}}}{2}+\lambda_3\theta_{\mathrm{l,m-1,n}}+(\lambda_1+\lambda_2)\frac{\theta_{\mathrm{l,m+1,n}}}{2}$$
$$+(\lambda_1+\lambda_2+\lambda_3)\frac{\theta_{\mathrm{l,m,n-1}}-\theta_{\mathrm{l,m,n+1}}}{4}-(3\lambda_1+3\lambda_2+6\lambda_3)\frac{\theta_{\mathrm{l,m,n}}}{2}=0$$

which is again a linear equation with seven unknowns. Two dimensions give:

$$\lambda_3\theta_{\mathrm{l-1,m}}+(\lambda_1+\lambda_2)\frac{\theta_{\mathrm{l+1,m}}}{2}+(\lambda_2+\lambda_3)\frac{\theta_{\mathrm{l,m-1}}}{2}+(\lambda_1+\lambda_3)\frac{\theta_{\mathrm{l,m+1}}}{2}-(\lambda_1+\lambda_2+\lambda_3)\theta_{\mathrm{l,m}}=0$$

All other cases are solved the same way. In three dimensions, a control volume may contain eight materials, while in two dimensions, four materials. All generate a linear equation with seven or five unknowns. For p control volumes, the result is a system of p equations with p unknown temperatures, except those figuring as boundary conditions. Solving the system gives the temperature distribution. Then, the above equations allow calculation of the heat flux components along the axes.

To generalize the algorithm, suppose P_s is the surface-linked thermal conductance between two adjacent control volumes (units W/K). If in the same material, then:

$$P_\mathrm{s}=(\lambda/d)A$$

In different materials, the named conductance consists of a serial and/or a parallel circuit of separate conductances (Figure 1.12).

$$\Phi^{\mathrm{l,m,n}}_{\mathrm{l-1,m,n}}=P^{\mathrm{l,m,n}}_{\mathrm{sl-1,m,n}}\left(\theta_{\mathrm{l-1,m,n}}-\theta_{\mathrm{l,m,n}}\right) \tag{1.29}$$

For the sum of the heat flows per control volume, the following equation, which demands the calculation of all surface-linked conductances P_s, applies:

$$\sum_{\substack{i=\mathrm{l,m,n}\\ j=\pm 1}}\left[P_{\mathrm{s},i+j}\theta_{i+j}\right]-\theta_{\mathrm{l,m,n}}\sum_{\substack{i=\mathrm{l,m,n}\\ j=\pm 1}}\mathrm{P}_{\mathrm{s},i+j}=0 \tag{1.30}$$

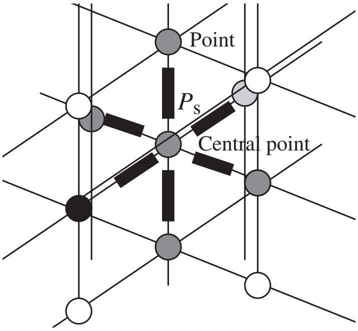


Fig. 1.12 Thermal conductances around the centre point in a meshing volume

Transferring the control volumes with known temperature to the right converts any two- or three-dimensional building detail into a system of equations:

$$[P_{\mathrm{s}}]_{p,p}[\theta]_p = \left[P_{\mathrm{s}.i,j,k}\theta_{i,j,k}\right]_p$$

where $[P_{\mathrm{s}}]_{p,p}$ is a p rows, p columns conductance matrix, $[\theta]_p$ a column matrix of the p unknown temperatures, and $[P_{\mathrm{s},i,j,k}\theta_{i,j,k}]_p$ a column matrix of the p known temperatures. The accuracy of a CVM calculation depends on the meshing applied. The finer the mesh, the closer the solution will approach the exact one. However, the consequence is more equations. Infinitely fine meshes produce the exact solution, but the price paid is an infinite number of equations. Therefore, a compromise between accuracy and processing time must been sought. Fine meshes where large temperature gradients are expected, and less fine where small temperature gradients are expected, will minimize the divergence.

Today, powerful software packages for two- and three-dimensional heat transfer exist (Trysco$^{\mathrm{R}}$, Heat$^{\mathrm{R}}$, etc.).

1.2.4 Transient

Transient means that the temperatures and heat fluxes change with time. Varying material properties, time-dependent heat dissipation and time-dependent boundary conditions are responsible for that. If only the last intervenes, Fournier's second law can be written as:

$$\nabla^2\theta = \frac{\rho c}{\lambda}\frac{\partial\theta}{\partial \mathrm{t}}$$

The end face temperatures and heat fluxes often vary either periodically or suddenly. Outdoors, the air temperature fluctuates over periods of a year, n days, one day. Heating up in turn causes a jump in temperature and heat flux (Figure 1.13).

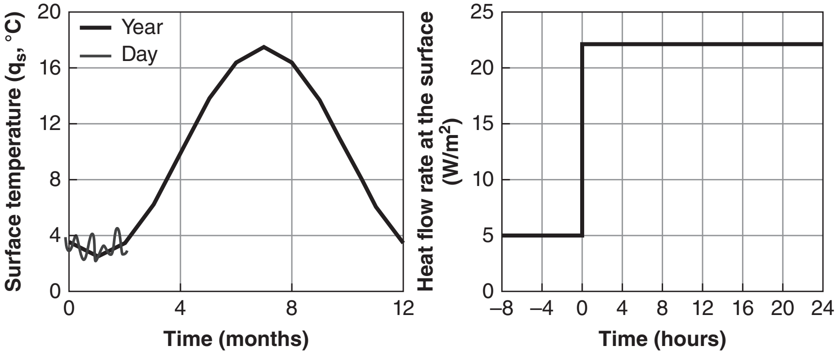


Fig. 1.13 Left: periodic change; right: non-periodic change

1.2.4.1 Periodic boundary conditions: flat assemblies

For flat layers, the second law simplifies to:

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\rho c}{\lambda}\frac{\partial\theta}{\partial t}$$

or, with $a=\lambda/\rho c$: $$a\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t}$$

The ratio a, units m^2/s, stands for the thermal diffusivity of the material. Its value indicates how easily a local temperature change spreads over any material. The higher it is, the faster this happens. A high thermal diffusivity requires a lightweight material with high thermal conductivity, or a heavyweight one with low volumetric heat capacity. None exists. Lightweights have a low thermal conductivity while the volumetric heat capacity of heavyweights is high. Many materials, except the metals, therefore have similar thermal diffusivities.

Substituting the thickness x in the equation by the thermal resistance R $(=x/\lambda)$, which means multiplying both terms with λ^2, changes the formula in:

$$\frac{\partial^2\theta}{\partial R^2}=\rho c\lambda\frac{\partial\theta}{\partial t} \tag{1.31}$$

with as heat flux:

$$q=-\frac{\partial\theta}{\partial R}$$

The temperatures at both end faces now fluctuate periodically with time. Transforming such periodic signals with base period T into a Fourier series gives for these temperatures:

$$\theta_{\mathrm{s}}=\frac{B_{\mathrm{s0}}}{2}+\sum_{n=1}^{\infty}\left[A_{\mathrm{sn}}\sin\left(\frac{2n\pi t}{T}\right)+B_{\mathrm{sn}}\cos\left(\frac{2n\pi t}{T}\right)\right]$$

with:

$$A_{\mathrm{sn}} = \frac{2}{T}\int_0^T \theta_{\mathrm{s}}(t)\sin\left(\frac{2n\pi t}{T}\right)\mathrm{d}t \qquad B_{\mathrm{sn}} = \frac{2}{T}\int_0^T \theta_{\mathrm{s}}(t)\cos\left(\frac{2n\pi t}{T}\right)\mathrm{d}t$$

The constant $B_{\mathrm{so}}/2$ represents the average value over the base period T, $A_{\mathrm{s1}}, A_{\mathrm{s2}}\ldots,$ A_{sn}, B_{s1}, $B_{\mathrm{s2}}\ldots, B_{\mathrm{sn}}$, the harmonics of the 1st, 2nd, . . . , nth order. Rewriting in a complex form, using Euler's formulas:

$$\sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i} \qquad \cos(x) = \frac{\exp(ix) + \exp(-ix)}{2}$$

where $x = 2n\pi t/T$, the series becomes:

$$\theta_{\mathrm{s}}(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\alpha_{\mathrm{sn}}\exp\left(\frac{2in\pi t}{T}\right)\right] = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[(A_{\mathrm{sn}} + iB_{\mathrm{sn}})\exp\left(\frac{2in\pi t}{T}\right)\right] \tag{1.32}$$

where the nth complex temperature α_{sn} has:

Amplitude: $\sqrt{B_{\mathrm{sn}}^2 + A_{\mathrm{sn}}^2}$
Phase shift: $\mathrm{atan}\,(-A_{\mathrm{sn}}/B_{\mathrm{sn}})$

The amplitude indicates how large half the temperature variation is, while the phase shift fixes the time delay compared with a cosine function with period $T/(2n\pi)$ (radians). Consider the monthly mean outdoor air temperatures of Figure 1.14 The thick line proxy has an equation:

$$\theta_{\mathrm{e}} = 9.45 + 7.18\cos\left(\frac{2\pi t}{365.25} - (-2.828)\right)$$

with t time in days, being 365.25 in a year included the leap years, 9.45 the mean annual temperature (θ_{em}), 7.18 the annual amplitude (θ_{e1}) and −2.828 the related phase shift (ϕ_{e1} in radians). The time-dependent term resembles a rotating vector with value 7.18 starting −2.828 radians away from the real axis. As Fourier series, this gives:

$$\theta_{\mathrm{e}} = \frac{B_{\mathrm{o}}}{2} + \theta_{\mathrm{e1}}\left[A_1\sin\left(\frac{2\pi t}{365.25}\right) + B_1\cos\left(\frac{2\pi t}{365.25}\right)\right]$$

or, with $\mathrm{B_o} = 18.9$, $A_1 = \theta_{\mathrm{e1}}\sin(\phi_{\mathrm{e1}}) = -2.21$ and $B_1 = \theta_{\mathrm{e1}}\cos(\phi_{\mathrm{e1}}) = -6.83$:

$$\theta_{\mathrm{e}} = 9.45 - 2.21\sin\left(\frac{2\pi t}{365.25}\right) - 6.83\cos\left(\frac{2\pi t}{365.25}\right)$$

The related complex value is $\alpha_{\mathrm{e1}} = -6.83 - i\,2.21$ with amplitude 7.18 °C and phase shift −2.828 radians.

Assume now that from time zero one or both end faces endure a periodic temperature and heat flux. The response will be twofold: a transient that dies slowly

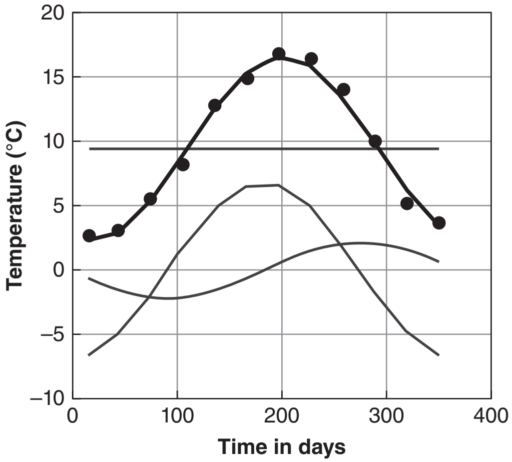


Fig. 1.14 Monthly mean temperatures. The black circles are the measured averages between 1901 and 1930. The thick line gives the result of a Fourier analysis with one harmonic. The thin lines are the annual mean, the sine term and cosine term

and a lasting periodic. As assemblies can neither compress nor extend thermal signals, the periodic must contain the same harmonics as the surface signal, but related temperature and heat flux amplitudes will dampen and gradually run behind, so building up a phase shift during the traverse. So the two will behave as complex, or the solution will be a Fourier series with the thermal resistance as an independent variable:

$$\theta(R,t)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\alpha_{\mathrm{n}}(R)\exp\left(\frac{2in\pi t}{T}\right)\right]$$

$$q(t)\quad=-\frac{\mathrm{d}\theta(R,t)}{\mathrm{d}R}=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\alpha'_{\mathrm{sn}}(R)\exp\left(\frac{2in\pi t}{T}\right)\right]$$

The prime for α'_{sn} reminds us that, mathematically, the complex heat flux is the first derivative of the complex temperature with respect to the thermal resistance:

Amplitude: $\sqrt{B'^2_{sn}+A'^2_{sn}}$
Phase shift: $\operatorname{atan}\left(-A'_{\mathrm{sn}}/B'_{\mathrm{sn}}\right)$

Single layer

Inserting the complex temperature into Fourier's second law gives:

$$\sum_{n=-\infty}^{\infty}\left\{\left[\frac{\mathrm{d}^2\alpha_n(R)}{\mathrm{d}R^2}-\frac{2\rho c\lambda in\pi}{T}\alpha_n(R)\right]\exp\left(\frac{2in\pi t}{T}\right)\right\}=0 \tag{1.33}$$

Setting to zero presumes that all coefficients of the time exponentials must be zero, or:

$$\frac{d^2\alpha_n(R)}{dR^2} - \frac{2\rho c\lambda in\pi}{T}\alpha_n(R) = 0 \tag{1.34}$$

The second-order partial differential equation so breaks into $2\infty + 1$ second-order differential equations, with the complex temperature as the dependent variable and the integer n moving from $-\infty$ past 0 to $+\infty$. Because the solutions for n positive and negative mirror each other, only infinity plus one $(\infty + 1)$ equations have to be solved. As n figures only as a parameter and all are otherwise identical, one solution suffices:

$$\alpha_n(R) = C_1\exp(\omega_n R) + C_2\exp(-\omega_n R) \qquad (0 \leq n \leq \infty) \tag{1.35}$$

where:

$$\omega_n^2 = \frac{2\rho c\lambda in\pi}{T}$$

The quantity ω is called the thermal pulsation. Using Euler's formulas then gives as complex temperatures and heat fluxes in a single-layer flat assembly:

$$\begin{aligned}\alpha_n(R) &= (C_1 - C_2)\sinh(\omega_n R) + (C_1 + C_2)\cosh(\omega_n R)\\ \alpha'_n(\mathrm{R}) &= \frac{d\alpha}{dR} = \omega_n[(C_1 - C_2)\cosh(\omega_n R) + (C_1 + C_2)\sinh(\omega_n R)]\end{aligned}$$

The integration constants $(C_1 - C_2)$ and $(C_1 + C_2)$ follow from the boundary condition that at the face $R = 0$, the complex temperature and heat flux are $\alpha_{sn}(0)$ and $\alpha'_{sn}(0)$, or:

$$\begin{aligned}\alpha_{sn}(0) &= (C_1 - C_2)0 + (C_1 + C_2) = C_1 + C_2\\ \alpha'_{sn}(0) &= \omega_n[(C_1 - C_2) + (C_1 + C_2)0] = \omega_n(C_1 - C_2)\end{aligned}$$

This converts the complex temperature and heat flux equations into:

$$\alpha_n(R) = \alpha_{sn}(0)\cosh(\omega_n R) + \alpha'_{sn}(0)\frac{\sinh(\omega_n R)}{\omega_n} \tag{1.36}$$

$$\alpha'_n(R) = \alpha_{sn}(0)\omega_n \sinh(\omega_n R) + \alpha'_{sn}(0)\cosh(\omega_n R) \tag{1.37}$$

The equations means two unknowns can be solved, whose inclusion in the Fourier series then gives the time functions. Of interest is the relationship between the complex quantities at both end faces. Suppose end face $R = 0$ is the inside. For envelopes R_T then is the outside end face, for partitions the end face at the other side. The system thus becomes $(R = R_T)$:

$$[A_{sn}(R_T)] = [W_n][A_{sn}(0)] \tag{1.38}$$

with $[A_{sn}(0)]$ the column matrix of the unknown complex quantities at one end face, and $[A_{sn}(R_T)]$ the column matrix of the known complex quantities at the other end face. The system matrix $[W_n]$ for the nth harmonic solely depends on the thickness, the material properties, the base period and which harmonic is considered. As an alternative for the thermal resistance, it contains much more information.

For $n = 0$, $\alpha_{\text{so}}/2$ and $\alpha'_{\text{so}}/2$ are the average temperature and heat flux on the inside end face over the period considered (T), with as thermal pulsation:

$$\omega_{\text{o}}^2(n = 0) = 2i\rho c\lambda 0\pi/T = 0$$

This turns the equations for the complex temperatures and heat fluxes into:

$$\alpha_{\text{so}}(R_{\text{T}}) = \alpha_{\text{so}}(0) + \alpha'_{\text{so}}(0)\frac{0}{0} \qquad \alpha'_{\text{so}}(R_{\text{T}}) = \alpha_{\text{so}}(0)0 + \alpha'_{\text{so}}(0) = \alpha'_{\text{so}}(0)$$

The incoming and outgoing average fluxes look the same. As this holds for any thermal resistance between 0 and R_{T}, thus at any spot in the single layer, for $n = 0$ the heat flux is a constant as in steady state. The ratio 0/0 in the temperature equation is solved using de l'Hopitâl's rule:

$$\lim_{\omega_0 \to 0}\left[\sinh(\omega_{\text{o}}R_{\text{T}})/\omega_{\text{o}}\right] = \lim_{\omega_0 \to 0}\left[R_{\text{T}}\cosh(\omega_{\text{o}}R_{\text{T}})\right] = R_{\text{T}}$$

which allows us to write:

$$\alpha_{\text{so}}(R_{\text{T}}) = \alpha_{\text{so}}(0) + \alpha'_{\text{so}}(0)R_{\text{T}}.$$

Anywhere in the single layer the same result comes out. Or, in the $[R,\theta]$ plane, the average temperatures lie on a straight line with the slope equal to the heat flux, again as in steady-state conditions. Both results extend the steady state from invariable to average over a sufficiently long time span.

Considering the harmonics, suppose that at the inside surface temperature remains constant. Then there no complex temperatures intervene, which gives at the other end face:

$$\alpha_{\text{sn}}(R_{\text{T}}) = \alpha'_{\text{sn}}(0)\frac{\sinh(\omega_n R_{\text{T}})}{\omega_n} \quad \text{or} \quad \frac{\alpha_{\text{sn}}(R_{\text{T}})}{\alpha'_{\text{sn}}(0)} = \frac{\sinh(\omega_n R_{\text{T}})}{\omega_n} \tag{1.39}$$

The function on the right reflects the ratio between the complex temperature at the outside face or the end face at the other side, and the complex heat flux at the inside face. In steady state, this ratio defines the thermal resistance. The name here is dynamic thermal resistance for the nth harmonic, symbol D_{q}^n, units m^2.K/W, with as amplitude the size and as argument (ϕ_{q}^n) the time shift between the complex temperature at the one and the complex heat flux at the other end face:

$$\phi_{\text{q}}^n = \arg\left[\sinh(\omega_n R)/\omega_n\right]$$

The assumption made looks purely theoretical. However, in reality the case is applicable to buildings with constant indoor temperature. For an infinitely long period the amplitude and phase shift become:

$$\left[D_{\text{q}}^n\right] = \lim_{n \to 0}\left[\frac{\sinh(0)}{0}\right] = \lim_{n \to 0}\left[R_{\text{T}}\cosh(0)\right] = R_{\text{T}} \qquad \phi_{\text{q}}^n = \lim_{n \to 0}\left[\arg\frac{\sinh(0)}{0}\right] = \infty$$

Or, the dynamic thermal resistance then equals the steady-state value. For a really fast pulsation, period nil, the amplitude and phase shift change to:

$$\left[D_{\mathrm{q}}^{n}\right] = \lim_{n\to\infty}\left[\frac{\sinh(\infty)}{\infty}\right] = \lim_{n\to\infty}\left[R_{\mathrm{T}}\cosh(\infty)\right] = \infty \qquad \phi_{\mathrm{q}}^{n} = \lim_{n\to\infty}\left[\arg\frac{\sinh(\infty)}{\infty}\right] = 0$$

Ever-faster fluctuations so push the dynamic thermal resistance to infinity, meaning the assembly dampens the signal completely. The dynamic value thus is always higher than the steady state one. Or, imposing a large thermal resistance suffices to get a high dynamic one.

Next, assume the heat flux at the inside face does not change. So with no complex heat fluxes there, the complex temperature at the other end face becomes:

$$\alpha_{\mathrm{sn}}(R_{\mathrm{T}}) = \alpha_{\mathrm{sn}}(0)\cosh(\omega_n R_{\mathrm{T}}) \qquad \text{or} \qquad \frac{\alpha_{\mathrm{sn}}(R_{\mathrm{T}})}{\alpha_{\mathrm{sn}}(0)} = \cosh(\omega_n R_{\mathrm{T}}) \tag{1.40}$$

The function on the right gives the ratio between the complex temperature at the outside face or the face at the other side and the complex temperature at the inside face. This is termed the temperature damping for the *n*th harmonic, symbol D_θ^n, no units. The amplitude again gives the size, and the argument (ϕ_θ^n) the time shift:

$$\Phi_\theta^n = \arg[\cosh(\omega_n R)]$$

The case anyhow looks fictitious. The property has no equivalent in steady state. Yet, in practice, it reflects the ability of an assembly to moderate the indoor impact of temperature changes outdoors. With few heat gains indoors and restricted ventilation, an enclosure with high temperature damping will keep the indoor temperature stable, a fact turning the number into a performance metric. For an infinitely long period the amplitude and phase shift become:

$$\left[D_\theta^n\right] = \lim_{n\to 0}[\cosh(0)] = 1 \qquad \phi_\theta^n = \lim_{n\to 0}\left[\arg[\cosh(0)]\right] = \infty$$

Then temperature damping nears 1. In fact, in steady state, without heat gains or losses, the temperatures on both end faces must be equal. For a zero period the amplitude and phase shift become:

$$\left[D_\theta^n\right] = \lim_{n\to\infty}[\cosh(\infty)] = \infty \qquad \phi_\theta^n = \lim_{n\to\infty}\left[\arg[\cosh\infty]\right] = 0$$

For ever faster fluctuations, temperature damping thus approaches infinity.

Finally, the temperature on the outside or the other face being constant converts the complex temperature on the inside face into:

$$0 = D_\theta^n \alpha_{\mathrm{sn}}(0) + D_{\mathrm{q}}^n \alpha'_{\mathrm{sn}}(0) \qquad \text{or} \qquad \frac{\alpha'_{\mathrm{sn}}(0)}{\alpha_{\mathrm{sn}}(0)} = -\frac{D_\theta^n}{D_{\mathrm{q}}^n} = -\omega_n \operatorname{cotgh}(\omega_n R) \tag{1.41}$$

with D_θ^n the temperature damping and D_{q}^n the dynamic thermal resistance. Their ratio, equal to the complex heat flux on the inside face divided by its complex temperature, is named the 'admittance' for the nth harmonic, symbol Ad^n, units W/(m^2.K), with as amplitude the value of that ratio, and as argument (ϕ^n_{Ad}) the time shift between the two:

$$\phi^n_{Ad} = \phi^n_\theta - \phi^n_q$$

The admittance also looks like an impossible quantity. However, it relates to how easily single-layer assemblies pick up heat from within when the inside surface temperature fluctuates. The higher the amplitude, the more effective the heat storage. The statement 'capacitive' indicates façade walls or inside partitions with high admittance. As the thermal pulsation ω_n also writes:

$$\omega_\mathrm{n} = \sqrt{\mathrm{i}}\,\sqrt{\rho \mathrm{c} \lambda}\,\sqrt{2\mathrm{n}\pi/\mathrm{T}}$$

a high admittance requires a large square root of the product of volumetric heat capacity and thermal conductivity, a value called the contact coefficient or effusivity of a material, symbol b, units J/(m^2.s$^{-1/2}$.K). The larger the contact coefficient, the more active that material as a heat storage medium. Heavy materials without an insulating finish inside have high effusivities, and thus large admittances. The amplitude and time shift for a period nearing infinity are:

$$[Ad^n] = \lim_{n \to 0}\left(\frac{D^n_\theta}{D^n_\mathrm{q}}\right) = \frac{1}{R_\mathrm{T}} \qquad \phi^n_\mathrm{As} = \lim_{n \to 0}\left[\arg\left(-\frac{D^n_\theta}{D^n_\mathrm{q}}\right)\right] = \infty$$

For really slow fluctuations, the admittance thus approaches the thermal conductance (P). In steady state the heat flux entering or leaving an assembly equals $P\Delta\theta_\mathrm{s}$. For a very fast pulsation, the amplitude and phase shift equal:

$$[Ad^n] = \lim_{n \to \infty}\left(\frac{D^n_\theta}{D^n_\mathrm{q}}\right) = \infty \qquad \phi^n_\mathrm{As} = \lim_{n \to \infty}\left[\arg\left(\frac{D^n_\theta}{D^n_\mathrm{q}}\right)\right] = 0$$

Or, ever-faster fluctuations move the admittance to infinity.

With the dynamic thermal resistance and temperature damping known, the complex system matrix of a single-layer assembly becomes:

$$[W_n] = \begin{bmatrix} D^n_\theta & D^n_\mathrm{q} \\ \omega_n^2 D^n_\mathrm{q} & D^n_\theta \end{bmatrix}$$

Moving to real numbers requires first that the thermal pulsation is reformulated:

$$\omega_n = \sqrt{i}\,b\sqrt{\frac{2n\pi}{T}} = \frac{(1+i)\,b\sqrt{\dfrac{2n\pi}{T}}}{\sqrt{2}} = (1+i)\,b\sqrt{\frac{n\pi}{T}}$$

whereby the conversion of $\sqrt{i}$ is based on $(1+i)^2 = 2i$ or $\sqrt{i} = (1+i)/\sqrt{2}$. With the thermal pulsation redressed, the product $\omega_n R$ becomes:

$$\omega_n R = \frac{(1+i)b\,x\sqrt{\dfrac{n\pi}{T}}}{\lambda} = (1+i)\,x\sqrt{\frac{n\pi}{aT}}$$

with x the ordinate from zero to the full thickness of the single-layer and $a = \lambda^2/b^2$ its thermal diffusivity. For $X_n = x\sqrt{n\pi/aT}$, the thermal pulsation equals $(1+i)X_n/R$. Thus temperature damping D_θ^n is given by:

$$D_\theta^n = \cosh(\omega_n R) = \cosh[(1+i)X_n] = \cosh(X_n)\cosh(iX_n) + \sinh(X_n)\sinh(iX_n)$$

or, with $\cosh(iX_n) = \cos(X_n)$ and $\sinh(iX_n) = i\sin(X_n)$:

$$\cosh(\omega_n R) = \cosh(X_n)\cos(X_n) + i\sinh(X_n)\sin(X_n)$$

Analogously, the dynamic thermal resistance D_q^n and the admittance Ad^n become:

$$D_\mathrm{q}^n = \frac{\sinh(\omega_n R)}{\omega_n} = \frac{R}{2X_n}\left\{\begin{aligned}&[\sinh(X_n)\cos(X_n) + \cosh(X_n)\sin(X_n)]\\ &\quad + i[\cosh(X_n)\sin(X_n) - \sinh(X_n)\cos(X_n)]\end{aligned}\right\}$$

$$Ad^n = \omega_n\sinh(\omega_n R) = \frac{X_n}{R}\left\{\begin{aligned}&[\sinh(X_n)\cos(X_n) - \cosh(X_n)\sin(X_n)]\\ &\quad + i[\cosh(X_n)\sin(X_n) + \sinh(X_n)\cos(X_n)]\end{aligned}\right\}$$

As six functions G_{n1} to G_{n6} are defined:

$G_{n1} = \cosh(X_n)\cos(X_n)$ $\quad G_{n2} = \sinh(X_n)\sin(X_n)$

$G_{n3} = [\sinh(X_n)\cos(X_n) + \cosh(X_n)\sin(X_n)]/(2X_n)$ $\quad G_{n4} = [\cosh(X_n)\sin(X_n) - \sinh(X_n)\cos(X_n)]/(2X_n)$

$G_{n5} = X_n[\sinh(X_n)\cos(X_n) - \cosh(X_n)\sin(X_n)]$ $\quad G_{n6} = X_n[\cosh(X_n)\sin(X_n) + \sinh(X_n)\cos(X_n)]$

They allow the expression of the temperature damping, the dynamic thermal resistance and the other term in the system matrix as:

$$\cosh(\omega_n R) = G_{n1} + iG_{n2} \qquad \frac{\sinh(\omega_n R)}{\omega_n} = R(G_{n3} + iG_{n4}) \qquad \omega_n\sinh(\omega_n R) = \frac{G_{n5} + iG_{n6}}{R}$$

The amplitude and time shift for the three properties then become:

Amplitude	Phase shift
$[D_\mathrm{q}^n] = R_T\sqrt{G_{n3}^2 + G_{n4}^2}$	$\phi_\mathrm{q}^n = \mathrm{bgtg}(G_{n4}/G_{n3})$
$[D_\theta^n] = \sqrt{G_{n1}^2 + G_{n2}^2}$	$\phi_\mathrm{q}^n = \mathrm{bgtg}(G_{n2}/G_{n1})$
$[Ad^n] = [D_\theta^n]/[D_\mathrm{q}^n]$	$\phi_{Ad}^n = \phi_\theta^n - \phi_\mathrm{q}^n$

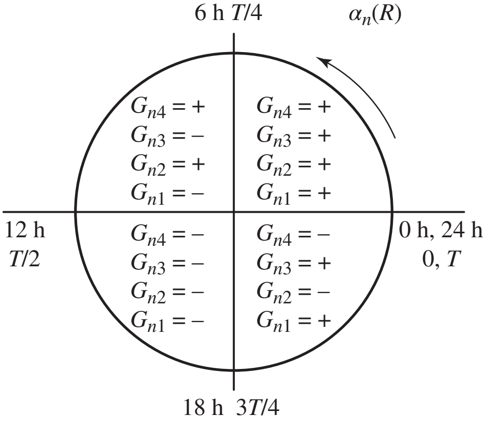


Fig. 1.15 Phase shift: sign of the G-functions

Per harmonic n, the complex 2×2 system matrix for a single-layer assembly herewith converts into a real 4×4 matrix:

$$W_n = \begin{bmatrix} G_{n1} & G_{n2} & R_T G_{n3} & R_T G_{n4} \\ -G_{n2} & G_{n1} & -R_T G_{n4} & R_T G_{n3} \\ G_{n5}/R_T & G_{n6}/R_T & G_{n1} & G_{n2} \\ -G_{n6}/R_T & G_{n5}/R_T & -G_{n2} & G_{n1} \end{bmatrix}$$

For the phase shift, Figure 1.15 shows which quarter-circle corresponds with which sign of the G-functions.

Composite
Composite assemblies figure as series-connected single-layers, each with a system matrix $[W_{n,i}]$, now called the layer matrix. The whole has as system matrix $[W_{nT}]$ (Figure 1.16):

$$[A_{sn}(R_T)] = W_{nT}[A_{sn}(0)] \tag{1.42}$$

with $[A_{sn}(R_T)]$ and $[A_{sn}(0)]$ the column matrices of the nth complex temperature and heat flux at each end face.

The relationship between the complex temperatures and heat fluxes at interface j and $j+1$ is:

$$\left[A_{n,j+1}\right] = W_{n,j}\left[A_{n,j}\right],$$

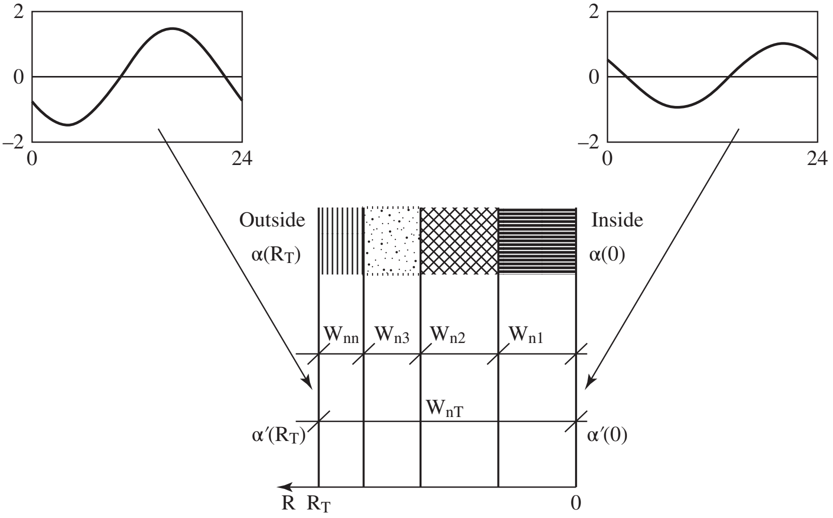


Fig. 1.16 Composite assembly, system matrix

an equation that holds per layer. Starting at the end face $R = 0$ which coincides with inside, the series becomes:

$$
\begin{aligned}
\left[A_{n,1}\right] &= W_{n,1}\left[A_{s,n}(0)\right]\\
\left[A_{n,2}\right] &= W_{n,2}\left[A_{n,1}\right]\\
&\ldots\\
\left[A_{n,m-1}\right] &= W_{n,m-1}\left[A_{n,m-2}\right]\\
\left[A_{s,n}(R_T)\right] &= W_{n,m}\left[A_{n,m-1}\right]
\end{aligned}
$$

Transposing each preceding into the next equation gives:

$$\left[A_{s,n}(R_T)\right] = W_{n,m}W_{n,m-1}\ldots W_{n,2}W_{n,1}\left[A_{s,n}(0)\right]$$

or, the system matrix of a composite assembly equals:

$$[W_{nT}] = \prod_{j=1}^{m}\left[W_{n,j}\right] \tag{1.43}$$

Multiplication starts on the inside, to end at the layer outside or at the layer on the other side. Each next-layer matrix is multiplied by the product of the preceding ones. Because the commutation property does not apply for a product of matrixes, that rule must be respected. So, in contrast to the thermal resistance, the transient response of a composite assembly changes with the layer sequence, which is very important in practice. Of course, each layer matrix keeps the same formulation and meaning as for a single-layer assembly. When two of the four complex temperatures and heat fluxes at the end faces are known, the others follow from:

$$[A_{sn}(R_T)] = [W_{nT}][A_{sn}(0)]$$

The complex temperatures and heat fluxes in the interfaces are then found by ascending or descending the layer equations in the correct sequence.

To get the complex temperatures and heat fluxes in a single-layer assembly, it is parcelled in m sub-layers with thickness Δx, layer matrix $[W_{\Delta x}]$ and system matrix $[W_n] = (W_{\Delta x})^m$. Per interface between each Δx, the same approach applies as for composite assemblies.

1.2.4.2 Any boundary conditions: flat assemblies

When the temperature or heat flux at an end face stays zero except for an infinitesimally short period dt, when it jumps to 1 (Figure 1.17), this is called a Dirac impulse.

Return to Fourier's second law now, and assume that the temperature θ_{s1} on end face s1 pulses this way. As a result, the heat flux there ($q_{s1}(t)$) and the temperature and heat flux at the other end face s2 ($\theta_{s2}(t)$, $q_{s2}(t)$) will vary. With constant material properties, the functions $q_{s1}(t)$, $\theta_{s2}(t)$ and $q_{s2}(t)$ are called the response factors and are written as:

$$I_{\theta_{s1}q_{s1}} \quad I_{\theta_{s1}\theta_{s2}} \quad I_{\theta_{s1}q_{s2}}$$

Analogously:

$$\begin{matrix} I_{q_{s1}\theta_{s1}} & I_{q_{s1}\theta_{s2}} & I_{q_{s1}q_{s2}} \\ I_{\theta_{s2}q_{s1}} & I_{\theta_{s2}\theta_{s1}} & I_{\theta_{s1}q_{s2}} \\ I_{q_{s2}q_{s1}} & I_{q_{s2}\theta_{s1}} & I_{q_{s2}\theta_{s2}} \end{matrix}$$

Depending on the layer sequence, their value changes with the flow direction, from end face s1 to s2 or vice versa. Consider a temperature pulse with value θ_{so} at end face

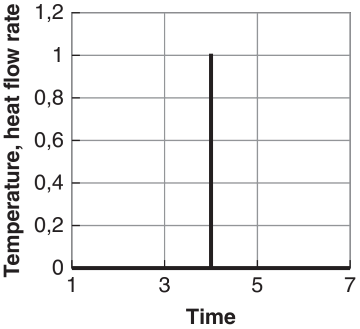
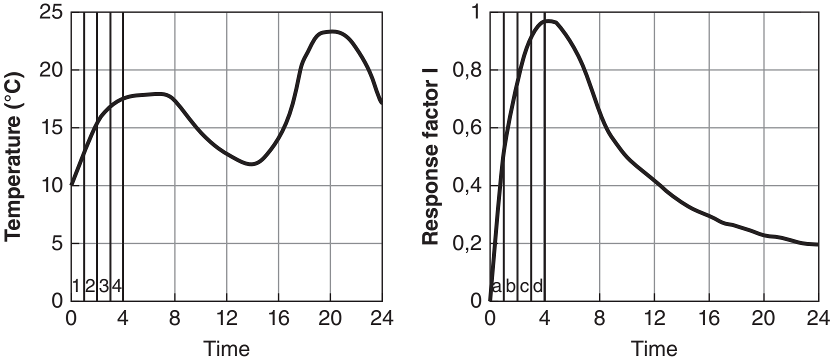


Fig. 1.17 Dirac impulse



3 Δt: 1*c+2*b+3*a
4 Δt: 1*d+2*c+3*b+4*a

Fig. 1.18 Convolution, the principle

s1. With the response factors known, the temperature and heat flux at end face s2 and the heat flux at end face s1 become:

Impulse θ_o at face s1	Impulse q_o at face s1
$q_{s1} = \theta_o I_{\theta_{s1} q_{s1}}$	$\theta_{s1} = q_o I_{q_{s1} \theta_{s1}}$
$\theta_{s2} = \theta_o I_{\theta_{s1} \theta_{s2}}$	$\theta_{s2} = q_o I_{q_{s1} \theta_{s2}}$
$q_{s2} = \theta_o I_{\theta_{s1} q_{s2}}$	$q_{s2} = q_o I_{q_{s1} q_{s2}}$

A pulse at end face s2 results in analogue relationships. Any random signal $\theta_{s1}(t)$ can now be split into a continuous series of pulses $\theta_{s1}(t)\Delta t$. For the temperature response (θ_{s2}) at end face s2 on a signal θ_{s1} at end face s1, the following applies (Figure 1.18):

$$\begin{array}{ll}
t = 0 & \theta_{s2}(0) = 0 \\
t = \Delta t & \theta_{s2}(\Delta t) = \theta_{s1}(t = 0) I_{\theta_{s1}\theta_{s2}}(t = \Delta t) \\
t = 2\Delta t & \theta_{s2}(2\Delta t) = \theta_{s1}(t = 0) I_{\theta_{s1}\theta_{s2}}(t = 2\Delta t) + \theta_{s1}(t = \Delta t) I_{\theta_{s1}\theta_{s2}}(t = \Delta t) \\
t = 3\Delta t & \begin{aligned} \theta_{s2}(3\Delta t) &= \theta_{s1}(t = 0) I_{\theta_{s1}\theta_{s2}}(t = 3\Delta t) + \theta_{s1}(t = \Delta t) I_{\theta_{s1}\theta_{s2}}(t = 2\Delta t) \\ &\quad + \theta_{s1}(t = 2\Delta t) I_{\theta_{s1}\theta_{s2}}(t = \Delta t) \end{aligned} \\
\ldots & \\
t = n\Delta t & \begin{aligned} \theta_{s2}(n\Delta t) &= \theta_{s1}(t = 0) I_{\theta_{s1}\theta_{s2}}(t = n\Delta t) + \theta_{s1}(t = \Delta t) I_{\theta_{s1}\theta_{s2}}(t = (n-1)\Delta t) \\ &\quad + \theta_{s1}(t = 2\Delta t) I_{\theta_{s1}\theta_{s2}}(t = (n-2)\Delta t) + \ldots \end{aligned}
\end{array}$$

or:

$$\theta_{s2}(n\Delta t) = \sum_{j=0}^{n-1} \theta_{s1}(j\Delta t) I_{\theta_{s1}\theta_{s2}}((n-j)\Delta t)$$

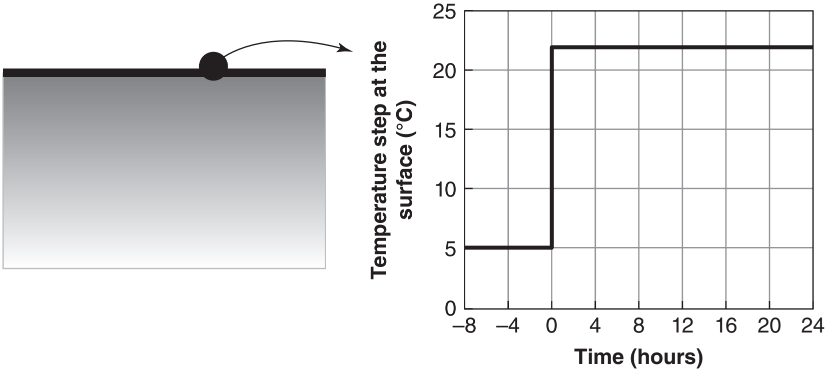


Fig. 1.19 Temperature step at the surface of a semi-infinite medium

As an integral:

$$\theta_{s2}(t) = \int_0^t \theta_{s1}(\tau)\, I_{\theta_{s1}\theta_{s2}}(t-\tau)\mathrm{d}\tau$$

This integral of the product of the signal scanned clockwise and the response factor scanned counter-clockwise is called the *convolution* integral of temperature θ_{s2} at end face s2 for the signal $\theta_{s1}(t)$ at end face s1. The same approach, be it with the correct response factors, holds for the two heat fluxes and for all situations where changing boundary conditions intervene. Response factors and convolution integrals have to be calculated numerically.

Now consider a temperature step at the surface of a semi-infinite medium. At time zero the medium is still uniformly warm, temperature θ_{s0}, but then the surface temperature suddenly jumps to a value $\theta_{s0} + \Delta\theta_{s0}$ (Figure 1.19).

A separation of variables allows the solution of Fourier's second law with $t = 0$, $0 \le x \le \infty$, $\theta = \theta_{s0}$ as initial conditions, and $t \ge 0$, $x = 0$, $\theta_s = \theta_{s0} + \Delta\theta_{s0}$ as boundary conditions. The outcome is:

$$\theta(x,t) = \theta_{s0} + \Delta\theta_{s0}\left(\frac{2}{\sqrt{\pi}} \int_{q=\frac{x}{2\sqrt{at}}}^{\infty} \exp\left(-q^2\right)\mathrm{d}q\right) \tag{1.44}$$

The term between brackets is the inverse error function, with a the thermal diffusivity of the medium. The heat flux at the surface ($x = 0$) equals:

$$q = -\lambda\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\lambda\left[\frac{2\Delta\theta_{s0}}{2\sqrt{\pi a t}}\exp\left(-\frac{x^2}{4at}\right)\right]_{x=0} = -\frac{\Delta\theta_{s0} b}{\sqrt{\pi t}} \tag{1.45}$$

with b the contact coefficient of the medium. Applying the definition of response factor to this equation gives:

$$I_{\theta_{s1}q_{s1}} = -b/\sqrt{\pi t}$$

Semi-infinite mediums do not exist, but the soil or very thick material layers are close. In any case, the heat flux equation illustrates what the contact coefficient does. A high value means a rapid uptake of heat when the surface temperature rises and a rapid release when it drops, whereas a low value stands for a slow uptake and release. Materials with high contact coefficients act as storage media. If, in passive solar buildings, walls and floors could not store the solar gains temporarily, the indoor conditions would become intolerable. Partitions should be thick enough and consist of heavy materials with a high contact coefficient.

The heat per m^2 flowing in or out of a semi-infinite medium beyond time zero equals:

$$Q = \int_0^t q\,\mathrm{d}t = \frac{2b\Delta\theta_{s0}}{\sqrt{\pi}}\sqrt{t} = A_q\sqrt{t} \tag{1.46}$$

with the quantity A_q called the heat absorption coefficient, units $J/(m^2.K.s^{½})$.

When two materials make contact and the one is at temperature θ_1, the other at temperature θ_2, the heat flux in the contact will equal ($\theta_1 > \theta_2$):

$$\begin{aligned} &\textit{Material}\ 1: \quad q_{s1} = (\theta_1 - \theta_c)b_1/\sqrt{\pi t} \\ &\textit{Material}\ 2: \quad q_{s2} = (\theta_c - \theta_2)b_2/\sqrt{\pi t} \end{aligned}$$

Since both fluxes must be identical in absolute value, the contact temperature θ_c becomes:

$$\theta_c = \frac{b_1\theta_1 + b_2\theta_2}{b_1 + b_2} \tag{1.47}$$

This instant value depends on the temperature and the contact coefficient of the two materials in contact, a reality affecting the impression when humans touch a material. A capacitative one with a high contact coefficient will feel cold or hot, while a material with low contact coefficient will feel comfortably warm. Indeed, in the first case the skin temperature will suddenly move from 32–33 °C to the material temperature, while in the second case the contact will adapt to the skin. So, touching concrete and aluminium is unpleasant, while touching wood is pleasant. To indicate that feeling, the term cold or warm material is used. The contact coefficient is an important metric when choosing floorings or chair finishes.

1.2.4.3 Two and three dimensions: thermal bridges

Where heat is transmitted two- or three-dimensionally, Fourier's second law for transient conduction applies. Analytical solutions for building details do not exist, which is why CVM is used. The transient heat capacity $\rho c\Delta V$ of each control volume intervenes. Without dissipation, the resulting heat flow now equals the change in heat content of each central control volume, or:

$$\left(\sum \Phi_{\mathrm{m}}\right) \Delta t = \rho c \, \Delta V \, \Delta \theta$$

with Φ_{m} the average heat flow from each nearest volume during the time step Δt. If set equal to a weighted average of the flow at time t (= 1) and time $t + \Delta t$ (= 2), then:

$$\Phi_{\mathrm{m}} = p \, \Phi_{t+\Delta t} + (1-p)\Phi_t \qquad (1 > p > 0)$$

The balance now writes as ($i = l,m,n$ and $j = \pm 1$):

$$p\left[\sum \left(P_{s,i+j}\theta_{i+j}\right) - \theta_{l,m,n}\sum P_{s,i+j}\right]^{(2)} + (1-p)\left[\sum \left(P_{s,i+j}\theta_{i+j}\right) - \theta_{l,m,n}\sum \mathrm{P}_{s,i+j}\right]^{(1)}$$

$$= \rho c \Delta V \frac{\theta_{l,m,n}^{(2)} - \theta_{l,m,n}^{(1)}}{\Delta t}$$

For $p = 0$, the system only contains forward difference equations, and for $p = 1$, only backward difference equations. For $p = 0.5$, which means using the arithmetic average, a choice called the Cranck-Nicholson scheme, rearrangement gives:

$$\left[\sum \left(P_{s,i+j}\theta_{i+j}\right) - \theta_{l,m,n}\left(\sum P_{s,i+j} + \frac{2\rho c \Delta V}{\Delta t}\right)\right]^{(2)}$$

$$= \left[-\sum \left(P_{s,i+j}\theta_{i+j}\right) + \theta_{l,m,n}\left(\sum P_{s,i+j} - \frac{2\rho c \Delta V}{\Delta t}\right)\right]^{(1)}$$

with the actual temperatures $^{(2)}$ unknown and the preceding ones $^{(1)}$ known. This way, each time step gives a system of as many equations as there are control volumes at unknown temperature. For given initial and boundary conditions, solving the system per time step is possible. Questions to decide upon beforehand are how to mesh (finer in materials with high thermal diffusivity) and the time step to use: in accordance with mesh density, in accordance with the step in boundary condition values, or in accordance with the information density required. Poor choices can induce instability in the solution generated.

1.3 Heat exchange at surfaces

Up to now, end face temperatures were assumed to be known. However, in most cases, it is the air temperatures and sometimes the heat flux at an end face that are the quantities figuring as known boundary conditions. Every weather station, in fact, registers the air temperature outdoors, while measuring the value indoors is much simpler than logging surface temperatures. Therefore the focus now shifts from heat transfer face-to-face to heat transfer between the ambient at both sides. But, how does heat reach a surface? Two mechanisms are responsible for that: convection between the air and the surface, and radiant exchanges with all surfaces seen. The

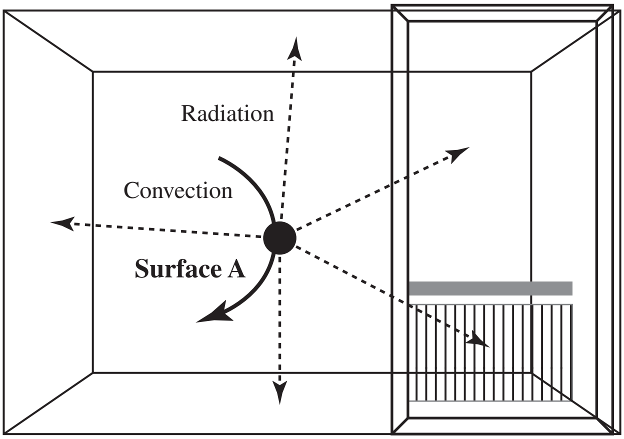


Fig. 1.20 Heat exchange at surface A by convection to and from the room air and radiation to and from all other surfaces

inside surface A in Figure 1.20, for example, radiates to and receives radiation from all the partitions, the radiator and the furniture, while exchanging convective heat with the air in the room.

1.4 Convection

1.4.1 In general

The term ‘convective’ aims at describing the heat transfer between fluids and surfaces. The heat flux and flow exchanged are then given as:

$$q_c = h_c(\theta_{fl} - \theta_s) \qquad \Phi_c = h_c(\theta_{fl} - \theta_s)A \tag{1.48}$$

where θ_{fl} is the temperature in the undisturbed fluid, θ_s the surface temperature and h_c the convective surface film coefficient, units $W/(m^2.K)$. The equations are known under the name of Newton’s law. Convection thus seems to be linearly related to the driving temperature difference $(\theta_{fl} - \theta_s)$. The equations, however, rather serve as a definition of the surface film coefficient, which accommodates the whole complexity of the heat and mass flow in a fluid touching a surface. Together, the scalar law of mass conservation and the vector law of momentum conservation fix the mass flow at the surface:

$$\begin{aligned}&Mass\,(scalar)\,div\,(\rho\mathbf{v}) = 0\\&Momentum\,(Navier\text{-}Stokes,\ vector)\frac{\mathrm{d}(\rho\mathbf{v})}{\mathrm{d}t} = \rho\mathbf{g} - \mathrm{grad}\,P + \mu\nabla^2\mathbf{v}\end{aligned}$$

In these equations, ρ is the density and μ the dynamic viscosity of the fluid, P the total pressure and $\rho\mathbf{g}$ the gravity gradient. Unknown are the velocity components v_x, v_y, v_z and the total pressure. When the mass flow turns turbulent, the turbulence equations must be added. Often used is the (k, ε) model, where k is the turbulent kinetic energy and ε its dissipation. Both give an extra scalar equation. Otherwise said, turbulent flow requires three scalar equations, while the velocity components in the x, y, and z directions turn the vector momentum equation into another three scalar equations.

For constant properties, negligible kinetic energy and negligible friction, conservation of energy in a moving fluid simplifies to:

$$a\nabla^2\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{1.49}$$

with a being the fluid's thermal diffusivity. In the Navier-Stokes equation as well as in the energy equation, d/dt is a total derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial}{\partial t} = \underbrace{\frac{\partial}{\partial x}v_x}_{(1)} + \underbrace{\frac{\partial}{\partial y}v_y}_{(2)} + \underbrace{\frac{\partial}{\partial z}v_z}_{(3)} + \frac{\partial}{\partial t}$$

where in the energy equation the terms (1), (2) and (3) multiplied with $\rho c\theta$ quantify the enthalpy that a fluid at temperature θ moving at velocity $\mathbf{v}$ with components v_x, v_y, v_z carries with it. Energy conservation so introduces temperature θ as a seventh unknown.

All information on convection follows from solving this system of seven scalar partial differential equations, which computational fluid dynamics (CFD) allows us to do numerically. Once the temperature field is known, the heat fluxes at a surface follow from Fourier's first law, because right at the contact, convection becomes conduction across a laminar boundary layer.

The following relationship thus holds for the surface film coefficient:

$$h_\mathrm{c} = -\lambda_\mathrm{fl}\frac{(\mathrm{grad}\,\theta)_\mathrm{s}}{\theta_\mathrm{fl} - \theta_\mathrm{s}} \tag{1.50}$$

where (grad $\theta)_\mathrm{s}$ is the temperature gradient in the boundary layer and θ_fl is the temperature at a particular location in the undisturbed fluid.

Solving the system analytically is possible for natural convection along a semi-infinite vertical surface and for forced convection along a semi-infinite horizontal surface, in both cases as long as the flow is laminar. With air as the fluid, velocity and temperature follow from similar equations:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial y} = \nu_\mathrm{fl}\frac{\partial^2 v_x}{\partial y^2} \qquad v_x\frac{\partial \theta}{\partial x} + v_y\frac{\partial \theta}{\partial y} = a_\mathrm{fl}\frac{\partial^2\theta}{\partial y^2}$$

Moreover, if the thermal diffusivity equals the kinematic viscosity, the equations become identical. The heat transfer and mass flow patterns then coincide, so solving gives as the surface film coefficient:

$$h_{\mathrm{c},x} = \frac{0.664\lambda_\mathrm{fl}}{x}\sqrt{\frac{v_\infty x}{\nu_{fl}}}$$

where x is the distance to the surface's free edge and v_∞ the velocity in the undisturbed fluid. Thus the surface film coefficient changes along the surface.

However, any trial to calculate the convective heat transfer along wall faces in a room, along the outside face of a building or of any other construction analytically, is doomed to fail. Even in the most simple situations, convection depends upon all parameters and properties determining the mass and heat transfer:

Fluid properties	Flow parameters
Thermal conductivity (λ_{fl})	Geometry
Density (ρ_{fl})	Surface roughness
Specific heat capacity (c_{fl})	Temperature difference ($\theta_{\text{fl}} - \theta_{\text{s}}$)
Kinematic viscosity ($\mu_{\text{fl}}/\rho_{\text{fl}}$)	Nature and direction of the flow
Volumetric expansion coefficient	Velocity components v_x, v_y, v_z

1.4.2 Typology

1.4.2.1 Driving forces

If differences in fluid density due to temperature and concentration gradients are the sole driving force, then natural convection with flow patterns reflecting the related temperature and concentration fields will develop, as is most often the case indoors. If, instead, the driving force is an imposed pressure difference, for example wind, then forced convection will be the result with a flow pattern independent of temperature and concentration. But the two types are not clearly separated. Forced convection at low velocities adds a natural component, as is the case with wind around buildings, inducing pressure differences that are generally too weak to eliminate buoyancy. Pure natural convection also rarely occurs. Opening and closing doors in a building, for example, causes some mixed convection.

1.4.2.2 Flow types

Flow can be laminar, turbulent or transitory. Diverging and converging streamlines that never cross, and wherein particle and overall flow velocity coincide, characterize laminar flow. In turbulent flow, chaotic momentum creates whirling eddies with particle velocities different from the flow velocity. Turbulent kinetic energy (k) builds up in the eddies, while turbulent dissipation (ε) continuously extinguishes eddy motion. An exact description of turbulent flow is practically impossible, even applying a fractional eddy approach. Transient flow fills the gap between laminar and turbulent. In fact, small disturbances along the flow path suffice to switch from laminar to turbulent, after which the eddies die and the flow turns laminar again. Intense turbulent mixing favours heat transfer more than laminar flow, for which only conduction perpendicular to the flow direction plays a part. Transient flow switches between the other two types. To summarize:

	Flow		
Driving force	Laminar	Transient	Turbulent
Density differences	X	X	x
Density and pressures	X	X	x
Pressures	X	X	X

1.4.3 Quantifying the convective surface film coefficient

1.4.3.1 Analytically

The few solvable cases underline that the convective surface film coefficient changes from spot to spot along a surface. When the heat flow is the subject of interest, using a surface-averaged value circumvents that complication:

$$h_c = \frac{1}{A}\int h_{cA}\,dA$$

The standard convective surface film coefficients are such averages, unless local surface temperature values are the subject of interest. Then spot-by-spot values should be used.

1.4.3.2 Numerically

Thanks to CFD, simulations of convection have improved a lot, albeit the velocity profiles in the boundary layer near surfaces still have to be assumed. Table 1.1 and Figure 1.21 list some CFD-related results.

Table 1.1 Convective surface film coefficient (h_e) along a rectangular, detached building: CFD-based correlations for wind speeds (v_w) below 15 m/s and a 10 °C surface-to-air temperature difference (Emmel *et al.*, 2007)

	Surface to wind angle, °	h_e, W/(m^2.K)
Outer walls	0	$5.15v_w^{0.31}$
	45	$3.34v_w^{0.34}$
	90	$4.78v_w^{0.71}$
	135	$4.05v_w^{0.77}$
	180	$3.54v_w^{0.76}$
Roofs	0	$5.11v_w^{0.78}$
	45	$4.6v_w^{0.79}$
	90	$3.76v_w^{0.85}$

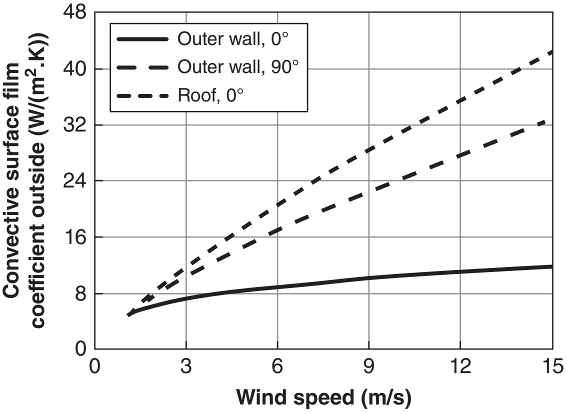


Fig. 1.21 Convective surface film coefficient outside, rectangular building, CFD-based correlations

1.4.3.3 Dimensionally

Most information about convection is still obtained from experiment and dimensional analysis. This latter technique determines which dimensionless ratios amongst the fluid properties, geometric parameters and kinematic parameters that define the movement, need the same value in experiment as in reality to allow extrapolation to real-world situations. These ratios either follow from the differential equations or from Buckingham's π-theorem, which states that if a problem depends on n single-valued physical properties involving p basic dimensions, then $n-p$ dimensionless ratios define the solution. The description of forced convection, for example, needs seven physical properties (L, λ_{fl}, ν_{fl}, ρ_{fl}, μ_{fl}, c_{fl}, h_{c}: see above) with four basic dimensions: L (representative length) t (time) M (mass) and θ (temperature). So, three dimensionless ratios (π_1, π_2, π_3) are required, expressed as $\pi_1 = f(\pi_2, \pi_3)$. These follow from rewriting the π-functions as:

$$\pi = L^{\text{a}}\,\lambda_{\text{fl}}^{\text{b}}\,\nu_{\text{fl}}^{\text{c}}\,\rho_{\text{fl}}^{\text{d}}\,\mu_{\text{fl}}^{\text{e}}\,c_{\text{fl}}^{\text{f}}\,h_{\text{c}}^{\text{g}}$$

an equation only containing the basic dimensions, or:

$$\pi = [L]^{\text{a}}\left[\frac{ML}{t^3\theta}\right]^{\text{b}}\left[\frac{L}{t}\right]^{\text{c}}\left[\frac{M}{L^3}\right]^{\text{d}}\left[\frac{M}{Lt}\right]^{\text{e}}\left[\frac{L^2}{t^2\theta}\right]^{\text{f}}\left[\frac{M}{t^3\theta}\right]^{\text{g}}$$

Because the three ratios must be dimensionless, the sum of the exponents per dimension should be zero, or:

For M			b			+	d	+	e			+	g	=	0
For L	a	+	b	+	c	−	3d	−	e	+	2f			=	0
For t		−	3b	−	c			−	e	−	2f	−	3 g	=	0
For θ		−	b							−	f	−	g	=	0

The three so follow from recalculating the π-function three times, first for g = 1, c = 0, d = 0, then for g = 0, a = 1, f = 0, and finally for g = 0, e = 1, c = 0:

Solution 1	a = 1, b = −1, e = 0, f = 0	or	$\pi_1 = \frac{h_c L}{\lambda_{fl}}$
Solution 2	b = 0, c = 1, d = 1, e = −1	or	$\pi_2 = \frac{\rho_{fl} v_{fl} L}{\mu_{fl}}$
Solution 3	a = 0, b = −1, d = 0, f = 1	or	$\pi_3 = \frac{c_{fl} \mu_{fl}}{\lambda_{fl}}$

Natural convection instead demands four dimensionless ratios, of which two combine into one. The three for forced convection are called the Reynolds, Nusselt and Prandtl numbers, and the three for natural convection are the Grasshof, Nusselt and Prandtl numbers.

Reynolds number is given by:

$$\mathrm{Re} = \mathrm{v}_{fl} L / \nu_{fl} \qquad (= \pi_2) \tag{1.51}$$

with v_{fl} being velocity, ν_{fl} the kinematic viscosity of the fluid and L a characteristic length representing the geometry, such as the hydraulic diameter for pipes, the dimension in the flow direction for walls, or a calculated length for more complex cases. The Reynolds number gives the ratio between the inertia force and viscous friction. If the number is small, friction gains and the result is laminar flow. If large, the inertia force wins and turbulent flow occurs. So the number determines the flow type: laminar for $\mathrm{Re} \leq 2000$, turbulent for $\mathrm{Re} \geq 20\ 000$, and transient in between.

The Nusselt number is calculated as:

$$\mathrm{Nu} = h_c L / \lambda_{fl} \qquad (= \pi_1) \tag{1.52}$$

where λ_{fl} is the thermal conductivity of the fluid. Multiplying the left- and right-hand sides of the convective surface film coefficient equation (see above) by the characteristic length gives:

$$h_c L / \lambda_{fl} = (\mathrm{grad}\,\theta)_s / \left[(\theta_{fl} - \theta_s)/L\right]$$

meaning that the Nusselt number represents the ratio between the temperature gradient in the fluid at the surface and the mean temperature gradient along the characteristic length. A large value indicates a significant gradient at the surface and a small gradient along the characteristic length, as is the case for high fluid velocities.

Physically, the number says that conduction governs heat transfer at a surface. Even in a turbulent flow regime, a laminar boundary layer remains whose thickness reduces with increasing fluid velocity but never becomes zero. The number underlines the importance of convection compared with conduction.

The Prandtl number is:

$$\mathrm{Pr} = \nu_{fl}/a_{fl} \qquad (= \pi_3) \tag{1.53}$$

This number combines heat and mass transfer by rating two analogous quantities: the thermal diffusivity, which determines how easily a local temperature change spreads into the fluid, and the kinematic viscosity ν, which indicates how easily a local velocity change affects the fluid.

The Grasshof number in turn is given by:

$$\mathrm{Gr} = \beta_{\mathrm{fl}} g L^3 \Delta\theta / \nu_{\mathrm{fl}}^2 \tag{1.54}$$

with β_{fl} the volumetric expansion coefficient, g the acceleration due to gravity and $\Delta\theta$ the representative temperature difference. Grasshof replaces Reynolds in the case of natural convection. Velocity then is indeed mainly the result of temperature-related differences in density (βg). As velocity influences temperature, the terms L^3 and ν_{fl}^2 replace L and ν_{fl} in the Reynolds number.

Finally, the Rayleigh number equals:

$$\mathrm{Ra} = \mathrm{Gr}\ \mathrm{Pr} \tag{1.55}$$

This number has no physical meaning. It was introduced because in many formulae for natural convection, Grasshof and Prandtl appear as a product.

All experimental, numerical and analytical expressions for the convective surface film coefficient may be written as:

Natural convection: $\mathrm{Nu} = c(\mathrm{Ra})^n$

Mixed convection: $\mathrm{Nu} = F(\mathrm{Re}/\mathrm{Gr}^{1/2}, \mathrm{Pr})$

Forced convection: $\mathrm{Nu} = F(\mathrm{Re}, \mathrm{Pr})$

where the coefficient c, the exponent n and the function $F()$ differ among geometries, the nature of the flow, and the flow direction for natural convection. A model and reality will coincide when in both cases the numbers have equal values.

1.4.4 Values for the convective surface film coefficient

1.4.4.1 Flat surfaces

In cases of natural convection, the characteristic length L (see the subscript) is the height for vertical surfaces, the side for square horizontal surfaces, and the average of length and width for rectangular horizontal surfaces. The mean temperature between wall surface and the undisturbed fluid fixes the property values. The relations are:

	Conditions	Functions
Vertical surfaces		
	$\mathrm{Ra}_L \leq 10^9$	$\mathrm{Nu}_L = 0.56\,\mathrm{Ra}_L^{1/4}$
	$\mathrm{Ra}_L > 10^9$	$\mathrm{Nu}_L = 0.025\,\mathrm{Ra}_L^{2/5}$
Horizontal surfaces		
Heat exchanged upwards	$10^5 < \mathrm{Ra}_L \leq 2\times10^7$	$\mathrm{Nu}_L = 0.56\,\mathrm{Ra}_L^{1/4}$
	$2\times10^7 < \mathrm{Ra}_L < 3\times10^{10}$	$\mathrm{Nu}_L = 0.138\,\mathrm{Ra}_L^{1/3}$
Heat exchanged downwards	$3\times10^5 < \mathrm{Ra}_L < 10^{10}$	$\mathrm{Nu}_L = 0.27\,\mathrm{Ra}_L^{1/4}$

For forced convection, the relationships with L as the characteristic length are:

	Conditions	Functions
Laminar flow	$\mathrm{Pr} > 0.1$ $\mathrm{Re}_L < 5\times10^5$	$\mathrm{Nu}_L = 0.644\,\mathrm{Re}_L^{1/2}\mathrm{Pr}^{1/3}$
Turbulent flow	$\mathrm{Pr} > 0.5$ $\mathrm{Re}_L > 5\times10^5$	$\mathrm{Nu}_L = 0.036\,\mathrm{Pr}^{1/3}\left(\mathrm{Re}_L^{4/5} - 23,200\right)$

In buildings with air-based HVAC systems, the convective surface film coefficients for mixed convection indoors are often related to the air change rate (n in air changes per hour, symbol ach), a number describing how many times an hour the air in a room is replaced by air delivered by the system. For a rectangular room (Fischer, 1995):

Configuration		Convective surface film coefficient W/(m^2.K)
Forced convection, air diffusers at the ceiling, room isothermal	Walls	$-0.199 + 0.18n^{0.8}$
	Floor	$0.159 + 0.116n^{0.8}$
	Ceiling	$-0.166 + 0.484n^{0.8}$
Forced convection, air diffusers at the walls, room isothermal	Walls	$-0.110 + 0.132n^{0.8}$
	Floor	$0.704 + 0.168n^{0.8}$
	Ceiling	$0.064 + 0.00444n^{0.8}$

In and around buildings, the fluid is air at atmospheric pressure and ambient temperature. This allows simplification of the given equations. For natural convection the temperature difference $\Delta\theta$ between the surface and the undisturbed air largely determines the surface film coefficient, a fact reflected by the following relationship:

$$h_c = a(\Delta\theta/L)^b$$

with as values for a, b and L:

	Conditions	a	b	L
Vertical surfaces				
	$10^{-4} < L^3\Delta T \le 7$	1.4	¼	Height
	$7 < L^3\Delta T \le 10^3$	1.3	⅓	1
Horizontal surfaces				
Heat flow upwards	$10^{-4} < L^3\Delta T \le 0.14$	1.3	¼	Eq. side1
	$0.14 < L^3\Delta T \le 200$	1.5	⅓	
Heat flow downwards	$2 \times 10^{-4} < L^3\Delta T \le 200$	0.6	¼	1

For forced convection outdoors, wind is the main actor, which gives as relationships:

Wind speed	Relationship	Remarks
$v \le 5$ m/s	$h_c = 5.6 + 3.9v$	For $v \le 5$ m/s, natural convection still intervenes, hence the constant 5.6
$v > 5$ m/s	$h_c = 7.2\ v^{0.78}$	

The fact that the value rises with wind speed is a direct consequence of the reduction in boundary layer thickness.

In any case, all these simple relationships only apply for air flowing along freestanding flat surfaces. Angles between two surfaces and corners between three surfaces have a disturbing effect. Moreover, if the surfaces form a room, the overall flow pattern must satisfy the continuity equation. All this makes convection so complex that, for the sake of simplicity, all standards advance constant average values:

European Normalization (EN) standard	Heat loss	Surface temperatures
Natural convection (= *inside*)		
Vertical surfaces	3.5	2.5
Horizontal surfaces:		
Heat upwards	5.5	2.5
Heat downwards	1.2	1.2
Forced convection (= *outside*)	19.0	19.0

Reference temperatures in rooms use the air temperature at a point 1.7 m above the floor's centre. Outdoors, the reference is the air temperature measured by the nearest weather station. When calculating surface temperatures using local convective surface film coefficients, the reference moves to the air temperature just outside the boundary layer.

In case of large temperature differences, a complex geometry or surfaces screened by furniture where we require a more correct calculation, the more complete formulae given in the tables above or formulae mentioned in the literature must be used.

1.4.4.2 Cavities

The word cavity refers to an air or gas layer with a small width compared with either the length or height. At the warm face the convective heat flux equals:

$$q_{\mathrm{c1}} = h_{\mathrm{c1}}(\theta_{\mathrm{s1}} - \theta_{\mathrm{c}})$$

At the cold face, it is:

$$q_{\mathrm{c2}} = h_{\mathrm{c2}}(\theta_{\mathrm{c}} - \theta_{\mathrm{s2}})$$

In both relations, θ_{c} is the gas temperature in the middle of the cavity. If the cavity remains unvented, the two must on average be equal, giving as mean flux:

$$q_{\mathrm{c}} = \frac{h_{\mathrm{c1}} h_{\mathrm{c2}}}{h_{\mathrm{c1}} + h_{\mathrm{c2}}}(\theta_{\mathrm{s1}} - \theta_{\mathrm{s2}}) \tag{1.56}$$

Replacing the two surface film coefficients by a common value h_{c} simplifies this formula to:

$$q_{\mathrm{c}} = (h_{\mathrm{c}}/2)(\theta_{\mathrm{s1}} - \theta_{\mathrm{s2}})$$

In reality, convection in a cavity is partly conduction, so the expression above is mostly rewritten as if conduction dominates, though with the thermal conductivity of the gas (λ_{fl}) multiplied by the Nusselt number:

$$q + q_{\mathrm{c}} = (\lambda_{\mathrm{fl}} \mathrm{Nu}) \Delta\theta_{\mathrm{s}}/d = \mathrm{h}_{\mathrm{c}}' \, \Delta\theta_{\mathrm{s}} \tag{1.57}$$

In this formula, d is the cavity width in m and $\Delta\theta_{\mathrm{s}}$ the temperature difference between both cavity faces in °C. In a horizontal, infinitely extending cavity, circular eddies, called Bénard cells, develop. In vertical infinitely extending cavities, some air rotation intervenes. In both cases, the Nusselt number can be written as:

$$\mathrm{Nu}_d = \max\left[1; 1 + \frac{m\,\mathrm{Ra}_d^r}{\mathrm{Ra}_d + n}\right] \qquad \left(10^2 \le \mathrm{Ra}_d \le 10^8\right) \tag{1.58}$$

with:

	m	n	r
Horizontal cavity			
Heat transfer downwards	0		
Heat transfer upwards	0.07	3200	1.33
Vertical cavity, or tilted cavity with slope above 45°	0.024	10 100	1.39
Tilted cavity with slope below 45°			
Heat transfer upwards	0.043	4100	1.36
Heat transfer downwards	0.025	13 000	1.36

For the Raleigh number, the temperature difference between the two faces figures as the reference, while the cavity width (d) acts as characteristic length. A Raleigh number below 100 means conduction with $\mathrm{Nu}_d = 1$.

Convection in finite cavities diverges strongly from infinite cavities. With d, H and L the cavity width, height and length in m, $^{\mathrm{lam}}$ a superscript for laminar, $^{\mathrm{turb}}$ a superscript for turbulent, and $^{\mathrm{transient}}$ a superscript for transient flow, the Nusselt numbers become:

		Nu_d
Vertical cavity		
Raleigh number upper limit value ($\mathrm{Ra}_{\max}$) for the applicability of Nu_d depending on the ratio H/d:		$\max\left(\mathrm{Nu}_d^{\mathrm{lam}}, \mathrm{Nu}_d^{\mathrm{turb}}, \mathrm{Nu}_d^{\mathrm{transient}}\right)$, with $\mathrm{Nu}_d^{\mathrm{lam}} = 0.242\left(\frac{\mathrm{Ra}_d d}{H}\right)^{0.273}$
H/d: 5 20 40 80 100 $\mathrm{Ra}_{\max}$: 10^8 2×106 2×105 3×104 1.2×104		$\mathrm{Nu}_d^{\mathrm{turb}} = 0.0.0605\ \mathrm{Ra}_d^{0.33}$ $\mathrm{Nu}_d^{\mathrm{transient}} = \left[1+\left(\frac{0.104\mathrm{Ra}_d^{0.293}}{1+(6310/\mathrm{Ra}_d)^{1.36}}\right)^3\right]^{0.33}$
Horizontal cavity		
Heat transfer upwards	$\mathrm{Ra}_d \leq 1708$	1
	$\mathrm{Ra}_d > 1708$	$\max\left[1, 1537 d^2\left(\frac{\Delta\theta}{L}\right)^{1/4}\right]$
Heat transfer downwards		1
Tilted cavity: see literature		

1.4.4.3 Pipes

Experimental and semi-experimental work on convection between pipes and the ambient fluid has resulted in a series of formulae. For natural convection:

Vertical pipe	$\mathrm{Ra}_d \leq 10^9$	$\mathrm{Nu}_L = 0.555\ \mathrm{Ra}_d^{1/4}$
	$\mathrm{Ra}_d > 10^9$	$\mathrm{Nu}_L = 0.021\ \mathrm{Ra}_d^{2/5}$
Horizontal pipe	$\mathrm{Ra}_d \leq 10^9$	$\mathrm{Nu}_L = 0.530\ \mathrm{Ra}_d^{1/4}$

For forced convection:

$\mathrm{Re}_d < 500$	$\mathrm{Nu}_d = 0.43 + 0.48\ \mathrm{Re}_d^{1/2}$
$\mathrm{Re}_d > 500$	$\mathrm{Nu}_L = 0.46 + 0.00\ 128\ \mathrm{Re}_d$

In all equations, the characteristic length (d) relates to the outer diameter of the pipe, while all properties link to the average (θ_{conv}) between the temperature in the undisturbed fluid (θ_{fl}) and the temperature of the pipe's outside surface (θ_{s}).

1.5 Radiation

1.5.1 In general

Thermal radiation differs fundamentally from conduction and convection. Radiation involves electromagnetic waves in the heat exchange. Any surface warmer than 0 K emits electromagnetic waves, while their absorption by surfaces agitates atoms and electrons, which is effectively heating. Electromagnetic waves span an impressive interval of wavelengths (λ), but only the 10^{-7} to 10^{-3} m range with ultraviolet (UV), visible light (L) and infrared (IR) are quoted as being thermal (Table 1.2).

Table 1.2 Categorization of electromagnetic radiation by wavelength

Wavelength	Radiation type
$\lambda \leq 10^{-6}$ µm	Cosmic radiation
$10^{-6} < \lambda \leq 10^{-4}$ µm	Gamma rays
$10^{-4} < \lambda \leq 10^{-2}$ µm	X-rays
$10^{-2} < \lambda \leq 0.38$ µm	**Ultraviolet**
$0.38 < \lambda \leq 0.76$ µm	**Visible light**
$0.76 < \lambda \leq 10^3$ µm	**Infrared**
$10^3 < \lambda$	Radio waves

Due to its electromagnetic nature, thermal radiation does not require a medium. On the contrary, only in a vacuum, where the photons move at a speed of 299 792.5 km/s, is transfer unhindered. How much will be emitted depends on the nature of a surface and its temperature, while net heat exchanges only happen among surfaces at different temperatures. Aside, the wavelength is given by the ratio between the propagation speed in m/s and the frequency in Hz.

1.5.2 Definitions

Table 1.3 outlines how thermal radiation is quantified, with the spectral values standing for ‘deduced with respect to wavelength’. A single wavelength gives monochromatic radiation, while several wavelengths together give coloured radiation.

1.5.3 Reflection, absorption and transmission

When a radiant flux (q_{uid}) emitted by a surface at temperature T touches another surface, a part is absorbed (b_{ra}), a part reflected (e_{rr}) and, if transparent, a part

Table 1.3 Variables of radiant heat transfer

Variable	Definition, units	Equations
Radiant heat Q_{R}	The heat emitted or received in the form of electromagnetic waves. Scalar, units J	
Radiant heat flow Φ_{R}	The radiant heat per unit of time. Scalar, units W	$\frac{\mathrm{d}Q_{\mathrm{R}}}{\mathrm{d}t}$
Radiant heat flux q_{R}	The radiant heat flow per unit of surface. As a surface emits radiation and receives it from all directions, the flux, units W/m^2, is a scalar. The term *irradiation*, symbol E, is used for the incoming radiation, the term *emittance*, symbol M, for the emitted radiant heat flux.	$\frac{\mathrm{d}^2Q_{\mathrm{R}}}{\mathrm{d}A\mathrm{d}t}$
Radiation intensity I	The radiant energy emitted in a specific direction. The intensity is a vector, units W/(m^2.rad) with dω the elementary angle in the direction considered.	$\frac{\mathrm{d}q_{\mathrm{R}}}{\mathrm{d}\omega}$ or $\frac{\mathrm{d}^2\Phi_{\mathrm{R}}}{\mathrm{d}A\mathrm{d}\omega}$
Luminosity L	The ratio between the radiant heat flow rate in a direction ϕ and the apparent surface, seen from that direction. The luminosity is a vector, units W/(m^2.rad). It describes how a receiving surface sees an emitting one.	$\frac{\mathrm{d}^2\Phi_{\mathrm{R}}}{\cos\phi\,\mathrm{d}A\,\mathrm{d}\omega}$

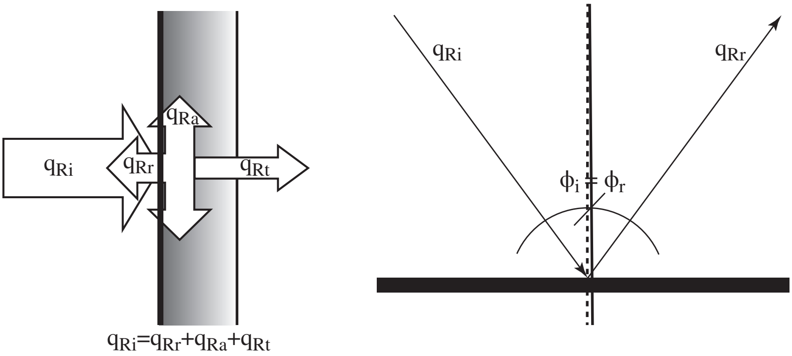


Fig. 1.22 Reflection, absorption and transmission at a surface, specular reflection

transmitted (a_{rt}):

$$\alpha = q_{\mathrm{Ra}}/q_{\mathrm{Ri}} \qquad \rho = q_{\mathrm{Rr}}/q_{\mathrm{Ri}} \qquad \tau = q_{\mathrm{Rt}}/q_{\mathrm{Ri}} \tag{1.59}$$

where α, ρ and τ are the average absorptivity, reflectivity and transmissivity at a temperature T of the receiving surface. Conservation of energy now imposes that the sum of these three must be 1, or:

$$\alpha + \rho + \tau = 1$$

This does not hold if the three are at different temperatures – never add in such a case. A difference also exists between diffuse and specular reflection. The latter obeys the laws of optics, with the incident and reflected beam in the same plane, and the angles with the normal to the surface (ϕ_{r} and ϕ_{i}) being equal (Figure 1.22). Most surfaces, however, show diffuse reflection, meaning that the reflected radiation scatters in all directions.

Reflectivity in a given direction (α) can be defined in relation to the radiation intensity incident under an angle ϕ on the surface:

$$\rho_{\phi} = I_{\mathrm{Rr}\alpha}/I_{\mathrm{Ri}\phi}$$

with $I_{\mathrm{Rr}\alpha}$ the reflected intensity in direction α. For a specularly reflecting surface, the reflectivity becomes:

$$\rho_{\phi} = (I_{\mathrm{Rr}}/I_{\mathrm{Ri}})_{\phi}$$

with ρ_{ϕ} being a function of the angle of incidence.

Most building and insulation materials are opaque for thermal radiation ($\tau = 0$). What arrives is absorbed in a thin surface layer, 10^{-6} m thick for metals and 10^{-4} m for other materials. Therefore the terms 'absorbing surface' or 'absorbing body' are

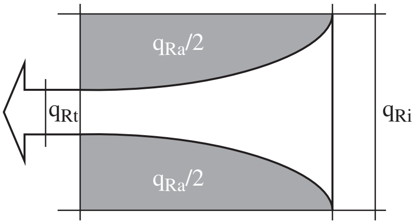


Fig. 1.23 Absorption of radiation passing through transparent materials

often used. Instead, most gases, fluids and solids such as glass and a number of synthetics are selectively transparent, although they show selective mass absorption depending on their extinction coefficient (a) (Figure 1.23):

$$\frac{dq_R}{q_R} = -a\,dx \tag{1.60}$$

For a layer with thickness d, the transmitted and absorbed radiant heat flux then equals:

$$q_{Rt} = q_{Ri}\exp(-ad) \qquad q_{Ra} = q_{Ri} - q_{Rt} = q_{Ri}[1 - \exp(-ad)]$$

with q_{Ri} the incoming flux. Absorptivity and transmissivity so become:

$$\alpha = 1 - \exp(-ad) \qquad \tau = \exp(-ad)$$

Specular reflectivity at an irradiated interface separating two media is given by:

$$\rho = \frac{I_r}{I_i} = \left[\frac{n_1\cos(\phi_i) - n_2\cos(\phi_t)}{n_1\cos(\phi_i) + n_2\cos(\phi_t)}\right]^2$$

where n_1 and n_2 are the refractive indexes of the media at either side, for example air ($n = 1$) and glass, ϕ_i is the angle of incidence in the first medium, and ϕ_t is the angle of transmittance to the second medium.

As stated, absorptivity, reflectivity, and transmissivity vary with temperature, and thus with wavelength, although the angle of incidence also matters. The impact can be impressive. Take glass, whose transmissivity for visible light is large, whereas for UV and IR it approaches zero with the absorptivity then exceeding a value of 0.9. Those differences explain the greenhouse effect. The short-wave, high-temperature solar radiation transmitted by the glass is absorbed by all surfaces indoors and re-emitted as low-temperature IR radiation, which the glass absorbs, leaving conduction as the only way to get rid of the heat. At the same time, the radiant bodies indoors slowly release the absorbed solar heat. The combination can make the indoors uncomfortably warm. Transparent synthetic materials act analogously, although some also transmit IR.

1.5.4 Radiant bodies

Ideal black surfaces absorb all incident radiation ($\alpha = 1, \rho = 0, \tau = 0, \alpha \neq f(\lambda,\phi)$). Their study is enlightening for grey bodies, which have a constant absorptivity, blank bodies, which have an absorptivity of zero, and coloured bodies, whose absorptivity depends on the temperature and the direction of the incident radiation. Although blank and grey bodies are ideal, and do not exist in reality, most real surfaces are assumed to behave as grey bodies. A distinction is made between short solar (subscript S) and long-wave ambient radiation (subscript L). Both stand for a different absorptivity and reflectivity.

1.5.4.1 Black

Of all surfaces, black bodies (subscript b) emit most radiant energy, independent of temperature. Their emissivity is 1. In fact, according to the second law of thermodynamics, in closed systems black bodies at different temperatures must evolve irrevocably towards temperature equilibrium. Because at equilibrium each of them emits and absorbs the same amount of radiation, the emissivity must equal the absorptivity, which is 1. With respect to direction, a black body obeys Lambert's law: luminosity constant. Hence, the related radiant intensity must obey (Figure 1.24):

$$I_{\mathrm{b}\phi} = L_\mathrm{b} \cos\phi \tag{1.61}$$

This equation, known as the cosine law, offers a simple relationship between emittance and luminosity. From the definitions in Table 1.3:

$$M_\mathrm{b} = L_\mathrm{b} \int_\omega \cos\phi \,\mathrm{d}\omega$$

where the integral covers the hemisphere. The angle $\mathrm{d}\omega$ is calculated assuming a hemisphere with radius r_o surrounds the surface $\mathrm{d}A$ with emittance M_b (Figure 1.25). The angle $\mathrm{d}\omega$ thus equals:

$$\mathrm{d}\omega = r_0^2 \sin\phi \,\mathrm{d}\phi \,\mathrm{d}\vartheta$$

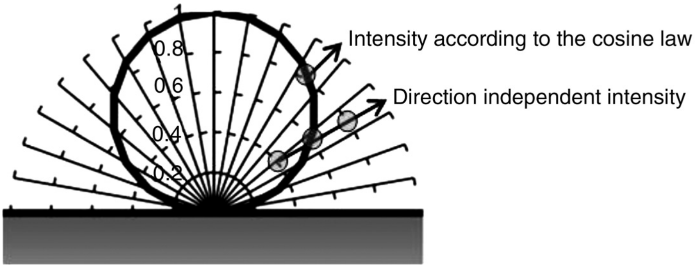


Fig. 1.24 The thick line representing the effect of the cosine law on radiation intensity

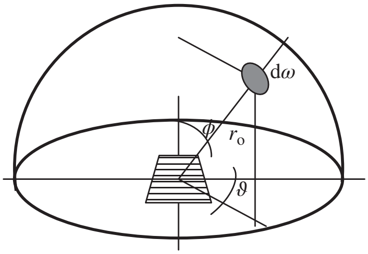


Fig. 1.25 Proving the cosine law

while on the hemisphere, the intensity drops to $I_{b\phi}/r_0^2$, or:

$$M_b = L_b \int_0^{2\pi}\int_0^{\pi/2} \frac{\cos\phi}{r_o^2} r_o^2 \sin\phi \, d\phi \, d\vartheta = \left[-\pi L_b \cos^2\phi\right]_0^{\pi/2} = \pi L_b \tag{1.62}$$

Planck's law gives the spectral density of the emittance:

$$M_{b\lambda} = \frac{2\pi c^2 h \lambda^{-5}}{\exp\left(\frac{ch}{k\lambda T}\right) - 1} \tag{1.63}$$

with constants: c as the speed of light in m/s, h as Planck's constant (6.624×10^{-34} J.s), and k as Boltzmann's constant (1.38047×10^{-23} J/K). The products $2\pi c^2 h$ and ch/k are called the radiation constants for a black body, symbols C_1 (3.7415×10^{-16} W.m^2) and C_2 (1.4388×10^{-2} m.K). Figure 1.26 shows the spectral density of the emittance for different values of the absolute temperature.

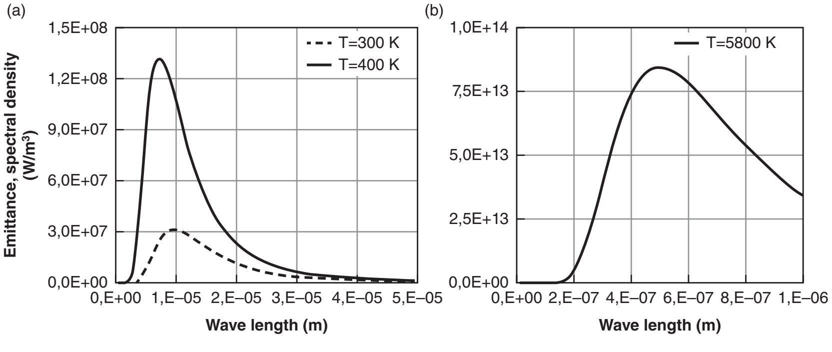


Fig. 1.26 Spectral density of the emittance (a) at ambient temperature, at a warmer temperature and (b) for the sun as a black body

The emittance, equal to the surface under the curve, increases quickly with temperature, while the maxima occur at ever-shorter wavelengths. In the $[\lambda, M_{B\lambda}]$ plane, these maxima have as a geometric locus a fifth order hyperbole, while their wavelengths obey Wien's law:

$$\lambda_M T = 2898 \qquad (\lambda_M \text{ in } \mu\text{m}) \tag{1.64}$$

At 20 °C, with $\lambda_M = 2898/293.15 = 9.9\ \mu\text{m}$, the maximum lies in the infrared interval. For the sun, with a radiant temperature of 5800 K, with $\lambda_M = 2898/5800 = 0.5\ \mu\text{m}$, it sits in the middle of the visible light interval. The emittance M_b itself follows from integrating Planck's law with respect to the wavelength:

$$M_b = \int_0^\infty M_{b\lambda}\, d\lambda = \frac{2\pi^5 k^4}{15 c^2 h^3} T^4 = \sigma T^4 \tag{1.65}$$

This equation is known as the Stefan-Boltzmann law, with σ as Stefan's constant, $5.67 \times 10^{-8}\ \text{W}/(\text{m}^2.\text{K}^4)$. This and Wien's law preceded Planck's, for which quantum mechanics had to come first. The Stefan-Boltzmann law is mostly written as:

$$M_b = C_b \left(\frac{T}{100}\right)^4 \tag{1.66}$$

with C_b the black body constant, $5.67\ \text{W}/(\text{m}^2.\text{K}^4)$, and $T/100$ the reduced radiant temperature. Luminosity and radiation intensity then become:

$$L_b = \frac{M_b}{\pi} = \frac{C_b}{\pi}\left(\frac{T}{100}\right)^4 \qquad I_{b\phi} = \frac{d^2\Phi_{Rb}}{dA\, d\omega} = L_b \cos\phi = \frac{C_b}{\pi}\left(\frac{T}{100}\right)^4 \cos\phi$$

When two black bodies 1 and 2 with surfaces A_1 and A_2 and no medium in between are positioned as shown in Figure 1.27, the elementary radiant heat flow going from 1 to 2 equals:

$$d^2\Phi_{R,1\to 2} = I_{b1} dA_1\, d\omega_1 = \frac{M_{b1}}{\pi} \cos\phi_1\, dA_1\, d\omega_1$$

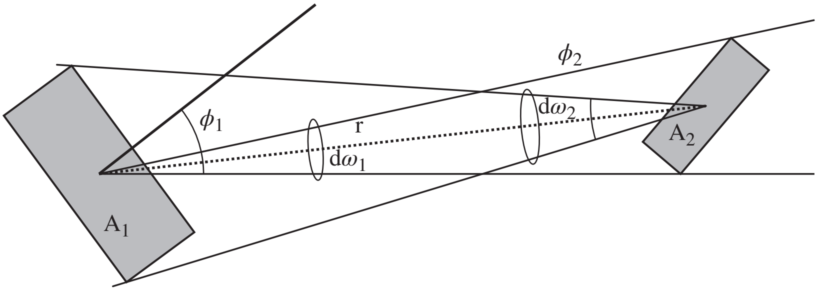


Fig. 1.27 Calculation of the view factor between the surfaces A_1 and A_2

with $d\omega_1$ the angle at which A_1 sees A_2:

$$d\omega_1 = dA_2 \cos\phi_2 / r^2$$

This redresses that flow interchanged to:

$$d^2\Phi_{R,1\to2} = \frac{M_{b1}}{\pi}\cos\phi_1\cos\phi_2\, dA_1 \frac{dA_2}{r^2}$$

From surface A_2 to surface A_1 the flow is:

$$d^2\Phi_{R,2\to1} = \frac{M_{b2}}{\pi}\cos\phi_1\cos\phi_2\, dA_1 \frac{dA_2}{r^2}$$

The resulting flow between the two becomes:

$$\text{From body 1 to body 2:}\quad d^2\Phi_{R,12} = d^2\Phi_{R,1\to2} - d^2\Phi_{R,2\to1} = \frac{(M_{b1} - M_{b2})\cos\phi_1\cos\phi_2\, dA_1\, dA_2}{\pi r^2}$$

$$\text{From body 2 to body 1:}\quad d^2\Phi_{R,21} = d^2\Phi_{R,2\to1} - d^2\Phi_{R,1\to2} = \frac{(M_{b2} - M_{b1})\cos\phi_1\cos\phi_2\, dA_1\, dA_2}{\pi r^2}$$

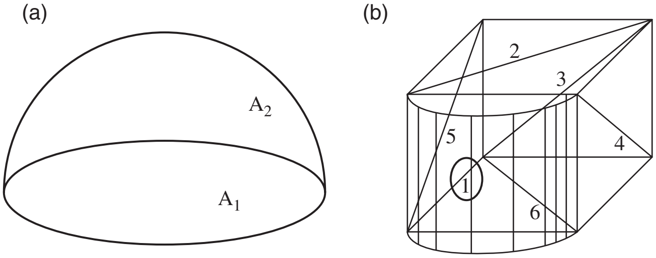
If both are finite in shape, then that flow becomes:

$$\text{From body 1 to body 2:}\quad \Phi_{R,12} = (M_{b1} - M_{b2})A_1\left[\frac{1}{\pi A_1}\int_{A_1}\int_{A_2}\frac{\cos\phi_1\cos\phi_2\, dA_2\, dA_1}{r^2}\right] \tag{1.67}$$

$$\text{From body 2 to body 1:}\quad \Phi_{R,21} = (M_{b2} - M_{b1})A_2\left[\frac{1}{\pi A_2}\int_{A_2}\int_{A_1}\frac{\cos\phi_1\cos\phi_2\, dA_1\, dA_2}{r^2}\right] \tag{1.68}$$

The term between square brackets in both formulae is called the view factor, symbol F. Other names are the angle, shape or configuration factor. If A_1 is considered as emitting, the view factor is written F_{12}. Vice versa, F_{21} is used. View factors are geometric quantities indicating what fraction of the radiant flow emitted by the one reaches the other. The size of each body, their form, the distance, and the angle at which they see each other all define the value, which equals 1 when all emitted radiation touches the other.

Concerning the view factor properties, firstly, reciprocity exists in the sense that $A_1F_1 = A_2F_2$. This relationship follows from the definition. Further, when a surface A_2 surrounds a surface A_1, the view factor from A_1 to A_2 must be 1 (Figure 1.28), a



$F_{12}=1$ $F_{11}+F_{12}+F_{13}+F_{14}+F_{15}+F_{16}=1$

Fig. 1.28 View factor between (a) surface 1 completely surrounded by surface 2 or (b) the surfaces 2 to 6. In the second case, surface 1 also radiates to itself

result that applies for each surface surrounded by $n-1$ others that together form a closed volume:

$$\sum_{j=2}^{n} F_{1j} = 1$$

Finally, two infinitely parallel surfaces, such as the bounding faces of a cavity, also have a view factor of 1.

Some simple configurations allow an analytical calculation. For a point sitting at a distance D from a rectangle with sides L_1 and L_2, the view factor calculates as the ratio between the angle at which the point sees the rectangle with surface $L_1 \times L_2 = \mathrm{A}_2$, and its whole view angle 4π:

$$F_{12} = \frac{1}{4\pi} \int_{\mathrm{A}_2} \frac{\cos\phi}{r^2} \mathrm{dA}_2$$

For a point above a corner, the formula becomes $\left(\cos\phi = D/r,\ r^2 = D^2 + x^2 + y^2\right)$:

$$F_{12} = \frac{1}{4\pi} \int_0^{L_1} \int_0^{L_2} \frac{D}{\left(D^2 + x^2 + y^2\right)^{3/2}} \mathrm{d}y\, \mathrm{d}x$$

Integration gives:

$$F_{12} = \frac{1}{8} - \frac{1}{4\pi} a \tan\left(\frac{D\sqrt{L_1^2 + L_2^2}}{L_1 L_2}\right)$$

Other positions of the point convert to the corner case by the construction of Figure 1.29. The resulting view factor is:

$$F_{12} = F_{1a} + F_{1b} + F_{1c} + F_{1d}$$

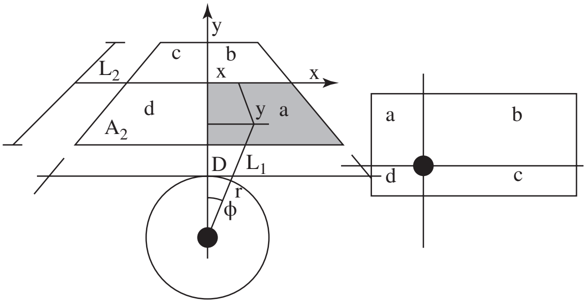


Fig. 1.29 View factor, point at distance *D* of surface A_2

Radiation between the human head and the ceiling is an example of a point-to-surface situation.

Another case concerns an infinitesimal surface dA_1 at an orthogonal distance D from a rectangle with sides L_1 and L_2 and surface A_2 in parallel. The formula for the view factor then is:

$$F_{12} = \frac{1}{dA_1} \int_{dA_1} \int_{A_2} \frac{\cos\phi_1 \cos\phi_2 \, dA_2 \, dA_1}{\pi r^2}$$

The way dA_1 sees each infinitesimal surface dA_2 on A_2 is independent of its position, or:

$$F_{12} = \frac{1}{\pi} \int_{A_2 \text{ seen by } dA_1} \frac{\cos\phi_1 \cos\phi_2 \, dA_2}{r^2}$$

Suppose now that surface dA_1 lies at a distance D above the corner (0, 0) of a rectangle A_2 (Figure 1.30). The view factor then is:

$$F_{12} = \frac{1}{\pi} \int_0^{L_1} \int_0^{L_2} \frac{\cos\phi_1 \cos\phi_2}{r^2} \, dx \, dy$$

an equation that can be simplified to $\left(\cos\phi_1 = \cos\phi_2 = D/r,\ r^2 = D^2 + x^2 + y^2\right)$:

$$F_{12} = \frac{1}{\pi} \int_0^{L_1} \int_0^{L_2} \frac{D^2 dy \, dx}{\left(D^2 + x^2 + y^2\right)^2}$$

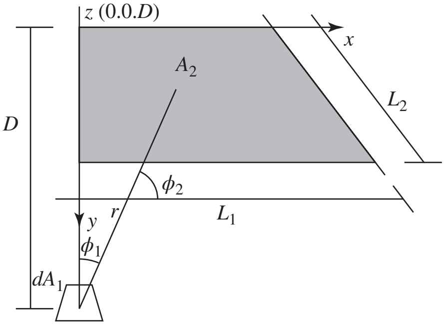


Fig. 1.30 View factor, infinitesimal surface dA_1 parallel to surface A_2 at an orthogonal distance D from the corner (0, 0)

with as a solution:

$$F_{12} = \frac{1}{2\pi}\left[\frac{L_1}{\sqrt{D^2+L_1^2}}\operatorname{atan}\left(\frac{L_2}{\sqrt{D^2+L_1^2}}\right)+\frac{L_2}{\sqrt{D^2+L_2^2}}\operatorname{atan}\left(\frac{L_1}{\sqrt{D^2+L_2^2}}\right)\right]$$

Other configurations have to be solved numerically. Consider an infinitesimally small perpendicular surface dA_1 at a distance D from a rectangular surface A_2 that has sides L_1 and L_2. dA_1 does not see that part of A_2 that lies behind the intersection of the plane containing dA_1 with A_2. The numerical formula for the view factor thus becomes:

$$F_{12} = \frac{D\Delta x\Delta y}{2\pi}\left[\sum_{x=\Delta x/2 \text{ to } L_1-\Delta x/2 \text{ step } \Delta x}\ \sum_{y=\Delta y/2 \text{ to } L_2-\Delta y/2 \text{ step } \Delta y}\frac{\sqrt{x^2+y^2}}{\left(x^2+y^2+D^2\right)^2}\right]$$

To give a practical example, the surface configurations in beam-shaped rooms include three pairs of two identical parallel walls, while all are perpendicular to each other and have common edges and corners. An analytical calculation of all view factors is not doable but related numerics are easily programmed on a spreadsheet.

With the view factor included, the radiant heat flows and fluxes are written as:

$$\begin{aligned}\phi_{R,12} &= A_1F_{12}(M_{b1}-M_{b2}) \quad q_{R,12} = F_{12}(M_{b1}-M_{b2})\\ \phi_{R,21} &= A_2F_{21}(M_{b2}-M_{b1}) \quad q_{R,21} = F_{21}(M_{b2}-M_{b1})\end{aligned} \tag{1.69}$$

For a number of black bodies radiating to each other, the flow and flux per body are:

$$\phi_{R1n} = A_1\sum_{j=2}^{n}\left[F_{1j}\left(M_{b1}-M_{bj}\right)\right] \qquad q_{R1n} = \sum_{j=2}^{n}\left[F_{1j}\left(M_{b1}-M_{bj}\right)\right] \tag{1.70}$$

with 1 being the body considered, and 2 to n the $n-1$ others.

1.5.4.2 Grey

Related laws are similar to those for black bodies. Only the radiant exchange differs. For each wavelength and direction, a grey body emits a constant fraction compared with a black body. The ratio between both is called the emissivity (e). Conservation of energy tells us that the absorptivity (α) must equal that emissivity, giving as reflectivity (ρ):

$$\rho = 1 - \alpha = 1 - e$$

Grey bodies with reflectivity of 1 are blank. Lambert's law holds ($L = C^t$). The radiant heat flux obeys the cosine law, while the emittance is:

$$M = \pi L$$

The spectral emittance obeys Planck's law, but multiplied by the emissivity e. The total emittance thus is:

$$M = e\, C_\mathrm{b} \left(\frac{T}{100} \right)^4 \tag{1.71}$$

Each grey body reflects radiation. If eM_b is the emittance of one and E the irradiation by all others, the radiosity of that one equals:

$$M' = e\, M_\mathrm{b} + \rho E \tag{1.72}$$

The difference between radiosity and irradiation defines the emitted flux:

$$q_\mathrm{R} = M' - E \tag{1.73}$$

Eliminating the unknown irradiation between the two results in:

$$q_\mathrm{R} = M' - \frac{M' - e\, M_\mathrm{b}}{\rho} = -\frac{e}{\rho}(M' - M_\mathrm{b}) \tag{1.74}$$

Or, the radiant heat flux received equals:

$$q_\mathrm{R} = \frac{e}{\rho}(M' - M_\mathrm{b}) \tag{1.75}$$

Otherwise said, a grey surface looks black but with a grey filter in front, coupled to the black by a radiant resistance, equal to the ratio between the grey reflectivity and its emissivity (ρ/e). The black body has an emittance M_b, and the grey filter a radiosity M'. For two grey bodies 1 and 2, separated by a transparent medium, the resulting radiant flow interchanged is:

From black body 1 to grey filter 1: $\Phi_{R,11} = \frac{e_1}{\rho_1}(M_{\mathrm{b}1} - M'_1)\,\mathrm{A}_1$
From grey filter 1 to grey filter 2: $\Phi_{R,12} = F_{12}(M'_1 - M'_2)\,\mathrm{A}_1$
From grey filter 2 to black body 2: $\Phi_{R,22} = \frac{e_2}{\rho_2}(M'_2 - M_{\mathrm{b}2})\,\mathrm{A}_2$

The second equation is identical to the radiant heat flow between two black bodies. Indeed, as well the emittance of a black body as the radiosity of a grey body obeys Lambert's law for diffuse radiation. In the three equations, the flows $\Phi_{R,11}$, $\Phi_{R,12}$ and

$\Phi_{R,22}$ must be identical, or, elimination of the unknown radiosities M'_1 and M'_2 gives as flows from 1 to 2 and 2 to 1:

$$\Phi_{R,12} = \left[\frac{1}{\dfrac{\rho_1}{e_1} + \dfrac{1}{F_{12}} + \dfrac{\rho_2 A_1}{e_2 A_2}}\right](M_{b1} - M_{b2})\,A_1 \tag{1.76}$$

$$\Phi_{R,21} = \left[\frac{1}{\dfrac{\rho_2}{e_2} + \dfrac{1}{F_{21}} + \dfrac{\rho_1 A_2}{e_1 A_1}}\right](M_{b2} - M_{b1})\,A_2 \tag{1.77}$$

The term between the brackets stands for the radiation factor, symbol F_R. If A_1 is seen as emitting, we write $F_{R,12}$, while if A_2 is the emitter, $F_{R,21}$. Dividing both equations by the surface seen as emitting gives the radiant heat fluxes.

A common configuration consists of two infinitely large, parallel isothermal surfaces. Then $F_{12} = F_{21} = 1$ and $A_1 = A_2$, giving as flux ($\rho = 1 - e$):

$$q_{R,12} = \frac{\Phi_{R,12}}{A_1} = \left[\frac{1}{\dfrac{1}{e_1} + \dfrac{1}{e_2} - 1}\right](M_{b1} - M_{b2}) \tag{1.78}$$

In building construction, the term between the brackets represents the radiation factor in an infinite cavity. If one of the faces is blank, for example face 1, then $F_{R,12} = 1/(1/0 + 1/e_2 - 1) = 0$. If one is black, for example face 2, then $F_{R,12} = 1/(1/e_1 + 1/1 - 1) = e_1$

Another common configuration is an isothermal surface (1) surrounded by another isothermal surface (2). F_{12} now is 1 and the radiant flow becomes:

$$\Phi_{R,12} = \frac{e_1 e_2}{e_2 \rho_1 + e_1 e_2 + \dfrac{e_1 \rho_2 A_1}{A_2}}(M_{b1} - M_{b2})\,A_1$$

If both surfaces are almost black ($e > 0.9$), the denominator nears 1, giving:

$$\Phi_{R,12} = e_1 e_2 (M_{b1} - M_{b2})\,A_1$$

If, moreover, the surrounded surface (A_1) is very small compared with the surrounding one (A_2), which brings their ratio close to zero, the equation further simplifies to:

$$\Phi_{R,12} = e_1 (M_{b1} - M_{b2})\,A_1 \tag{1.79}$$

In such a case, the resulting radiant heat flow only depends on the emissivity of the surface (A_1) that is surrounded.

When multiple isothermal grey surfaces face each other, all at different temperatures, the radiant heat flow between one of them (A_1) and all others ($n-1$) is written as:

From the equivalent black body 1 to grey filter 1: $\Phi_{R,11} = \frac{e_1}{\rho_1}(M_{b1} - M'_1)\,A_1$

From grey filter 1 to the n − 1 other grey filters: $\Phi_{R,1\ \text{to}\ j} = \sum_{j=2}^{n} F_{1j}\left(M'_1 - M'_j\right)\,A_1$

As both flows are equal, the black body emissivity is given by:

$$M_{b1} = \left(1 + \frac{\rho_1}{e_1}\sum_{j=2}^{n} F_{1j}\right) M'_1 - \frac{\rho_1}{e_1}\sum_{j=2}^{n}\left(F_{1j}\,M'_j\right) \tag{1.80}$$

For the 2 to n surrounding bodies, this equation converts into:

$$M_{b1} = \frac{M'_1}{e_1} - \frac{\rho_1}{e_1}\sum_{j=2}^{n}\left(F_{1j}M'_j\right) \tag{1.81}$$

Per grey body, the radiosities M'_j are unknown but the black body emittances M_{b1} are known. The result is a system of n equations with n unknowns:

$$\left[M_{bj}\right]_n = [F]_{n.n}\left[M'_j\right]_n \tag{1.82}$$

where $[F]_{n.n}$ is the radiation matrix for the n isothermal grey bodies. Solving it gives the radiosities M'_j as functions of the black body emittances M_{bj}. The radiant heat flows then follow from inserting M'_j in the equations given above for the heat exchange between the black body and its grey filter.

1.5.4.3 Coloured

For coloured bodies, emissivity, absorptivity and reflectivity depend on the wavelength, which changes with temperature and sometimes direction. Kirchhoff's law ($e = \alpha$) still applies, but Lambert's law does not, as it requires the same emission per direction. The spectral emittance per wavelength thus differs from black, although the ratio between the coloured and black emittance at the same temperature still gives the average emissivity for that temperature. To simplify things, coloured bodies are often considered grey, albeit with temperature-dependent emissivity. For the emittance and irradiance at strongly different temperatures, ambient and solar for example, Kirchoff's law no longer applies because the short-wave absorptivity (α_S) differs from the ambient long wave emissivity (e_L). To give an example, for polished aluminium, α_S equals 0.2 to 0.4, while e_L is 0.05.

1.5.5 Simple formulae

Thermal radiation seems quite straightforward to model. However, calculating all angle factors is cumbersome, while the system of equations for multiple grey bodies can be very large. A simpler approach, therefore, is welcomed. In a first step, reality is reduced to two radiant surfaces: the one considered (surface 1) and the remaining $n-1$ others shaping the environment. In a second step, the environment is supposed black at a radiant temperature θ_r, which is the temperature it should have as a black body to interchange the same radiant flow as in reality. Solving the system of equations for all surfaces present gives the radiosity for surface 1 (M'_1) as a linear combination of the (black body) emittances of all surfaces present:

$$M'_1 = \sum_{i=1}^{n} a_{\mathrm{ri}} M_{\mathrm{bi}}$$

Insertion into the equation for the grey body radiant heat flow received and equating with the radiant heat flow found for the surrounded surface 1, which is usually very small compared to the $n-1$ surfaces in the environment, fixes that radiant temperature:

$$\theta_r = \sqrt[4]{\frac{1}{\rho_1}\left(\sum_{i=1}^{n} a_{\mathrm{r}i} T_i^4 - e_1 T_1^4\right)} - 273.15$$

Analagously with convection, the radiant heat flow and heat flux can then be written as:

$$q_r = h_r(\theta_{s1} - \theta_r) \qquad \Phi_r = h_r(\theta_{s1} - \theta_r)\mathrm{A} \tag{1.83}$$

In both, h_r represents the surface film coefficient for radiation (W/(m^2.K)), while θ_{s1} is the temperature of surface 1. The surface film coefficient varies with the configuration considered. If the environment surrounds surface 1, the value follows from equalling the equation above to the one derived for where the surrounded surface is small compared with the one surrounding:

$$h_r = e\,C_b\left[\frac{\left(\frac{T_{s1}}{100}\right)^4 - \left(\frac{T_r}{100}\right)^4}{\theta_{s1} - \theta_r}\right] \tag{1.84}$$

The term between brackets is called the temperature ratio for radiation (F_T):

$$F_T = \frac{T_m}{5000}\left[\left(\frac{T_{s1}}{100}\right)^2 + \left(\frac{T_r}{100}\right)^2\right] \approx \frac{4}{100}\left(\frac{T_m}{100}\right)^3 \tag{1.85}$$

As that ratio hardly varies for temperatures between −10 and 50 °C, the simple expression on the right-hand side usually suffices. It considers the flux equation as being more or less linear. Insertion in the surface film coefficient for radiation gives:

$$h_\mathrm{r} = e_1\, C_\mathrm{b}\, F_\mathrm{T} \tag{1.86}$$

Another case is parallel bodies. If the other surface is isothermal, temperature θ_{s2}, the detour via the radiant temperature becomes superfluous and the surface film coefficient can be directly written as:

$$q_\mathrm{r} = h_\mathrm{r}(\theta_{s1} - \theta_{s2})$$

with:

$$h_\mathrm{r} = \frac{5.67 F_\mathrm{T}}{\dfrac{1}{e_1} + \dfrac{1}{e_2} - 1}$$

When surfaces in the environment have the same temperature as the one surrounded, only part of that environment will participate in the radiant exchange, while the radiant temperatures should only include those at different temperature ($= \theta'_r$). In such a case, the surface film coefficient for radiation becomes:

$$q_\mathrm{r} = h_\mathrm{r}(\theta_{s1} - \theta'_r)$$

with:

$$h_\mathrm{r} = \frac{e_1\, C_\mathrm{b}\, F_{12}\, F_T}{e_1 + \rho_1 F_{12}}$$

where F_{12} is the view factor between surface A_1 and those in the environment at different temperatures. If surface A_1 is almost black, the denominator tends to 1 and:

$$h_\mathrm{r} = e_1\, C_\mathrm{b}\, F_{12}\, F_T = 5.67 e_1\, F_{12}\, F_T \tag{1.87}$$

Consider, for example a corner formed by two identical outer walls. Both are equally warm. The result is a radiant exchange with half the space, containing surfaces at a different temperature. The view factor is 0.5. The surface film coefficient for radiation then equals:

$$h_\mathrm{r} = e_1\, C_\mathrm{b}\, F_T/2 = 2.84\, e_1\, F_T \tag{1.88}$$

Of course, surfaces at the same temperature can also be included in the radiant temperature. The view factor then remains 1, but the radiant temperature will change.

1.6 Building-related applications

1.6.1 Surface film coefficients and reference temperatures

In real buildings, conduction, convection and radiation combine. Consider an outside wall without a cavity. Heat is conducted across the wall. Between the inside surface and indoors, convection and radiation take over. The same happens between the outside surface and outdoors. Both can be considered the product of a temperature difference with a surface film coefficient (h_c, h_r). However, convection and radiation are so intertwined that, when possible, the related surface film coefficients are combined into one value inside (h_i) and another outside (h_e), both linked to a specific reference temperature. Of course, the two modes may also remain separated. Then, an air and a radiant temperature will characterize both ambients.

1.6.1.1 Indoors

Assume an isothermal surface at temperature θ_{si}. The convective heat flux exchanged with the air is:

$$\left[q_{ci}\right] = h_{ci,s}\left(\theta_{i,ob} - \theta_{si}\right)$$

with $h_{ci,s}$ the average convective surface film coefficient and $\theta_{i,ob}$ the average air temperature directly outside the boundary layer. If not that temperature, but the air temperature in the centre of the room 1.7 m above the floor (θ_i) is taken as the reference, the flux changes to:

$$q_{ci} = h_{ci}(\theta_i - \theta_{si})$$

with h_{ci} the average convective surface film coefficient, now linked to that new reference temperature:

$$h_{ci} = h_{ci,s}\left(\frac{\theta_{i,ob} - \theta_{si}}{\theta_i - \theta_{si}}\right) \tag{1.89}$$

The radiant heat flux at the inside surface equals:

$$q_{ri} = h_{ri}(\theta_{ri} - \theta_{si})$$

with θ_{ri} the radiant temperature characterizing the indoor environment. The total heat flux at the surface thus becomes:

$$q_i = q_{ci} + q_{ri} = h_{ci}(\theta_i - \theta_{si}) + h_{ri}(\theta_{ri} - \theta_{si}) = (h_{ci} + h_{ri})\left[\left(\frac{h_{ci}\theta_i + h_{ri}\theta_{ri}}{h_{ci} + h_{ri}}\right) - \theta_{si}\right] \tag{1.90}$$

The sum $h_{ci} + h_{ri}$ stands for the inside surface film coefficient for heat, symbol h_i, units W/(m^2.K). The weighted average between the central air temperature and the radiant temperature, named $\theta_{ref,i}$, serves as reference temperature for the surface considered. Standard values for the convective part h_{ci} were given when discussing convection. The radiant part, which convenes with a surface facing a surrounding environment, equals:

$$h_{ri} = 5.67 e_L F_T$$

with e_L the long wave emissivity of the surface and F_T the temperature ratio for radiation in the interval $\theta_{si} - \theta_{ri}$, mostly a value around 0.95. Since most finishes have a long wave emissivity of 0.8 to 0.9, the result is $4.3 \leq h_{ri} \leq 4.95\ \mathrm{W/(m^2.K)}$

The standardized inside surface film coefficients (W/(m^2.K)) thus become:

Vertical surfaces		Horizontal surfaces	
Any flow direction	$\theta \approx 7.7$	Heat flow upwards	10
		Heat flow downwards	6

The 2017 *ASHRAE Handbook of Fundamentals* gives a more complete set of values:

Position	Heat flow direction	h_i (W/(m^2.K)) for a surface emissivity		
		0.9	0.2	0.05
Horizontal	Upward	9.26	5.17	4.32
Sloping 45°	Upward	9.09	5.00	4.15
Vertical	Horizontal	8.29	4.20	3.35
Sloping 45°	Downward	7.50	3.41	2.56
Horizontal	Downward	6.13	2.10	1.25

None of these is accurate for cases that deviate substantially from the assumptions made. Back to the theory, then, to define case-relevant inside surface film coefficients. If necessary, radiation and convection must remain decoupled.

How do we determine the reference temperature? Calculating the radiant temperature is quite complex, which is why, provided the room is beam-shaped and all surfaces are grey with long-wave emissivity ≈0.9, the area-weighted average surface temperature is used as an acceptable estimate:

$$\theta_{ri} = \sum_{k=1}^{n} (A_k \theta_{sk}) \Big/ \sum_{k=1}^{n} A_k \tag{1.91}$$

If the surface considered is part of a vertical, sloped or horizontal envelope assembly, the last with upward heat flow, and if it behaves as a grey body, the reference temperature becomes:

$$\theta_{ref,i} = 0.44\,\theta_i + 0.56\,\theta_{ri}$$

a result close to the average between the central air and the radiant temperature, a value governing thermal comfort in buildings and called the operative temperature, symbol θ_o:

$$\theta_{ref,i} = \theta_o = (\theta_i + \theta_{ri})/2 \tag{1.92}$$

For reflective surfaces, the reference temperature nears the central air temperature: $\theta_o \approx \theta_i$, as convection then dominates. For grey horizontal inside partition and envelope assemblies where the heat flows downward, the reference turns into:

$$\theta_{ref,i} = 0.2\,\theta_i + 0.8\,\theta_{ri} \tag{1.93}$$

The larger the impact envelope assemblies with really cold inside surfaces have on the radiant temperature vertical, sloped and horizontal inside partitions face, the less evident is the use of the inside reference temperatures just defined.

1.6.1.2 Outdoors

Outdoors, three heat fluxes strike the surface. The first is convection to the outside air:

$$q_{ce} = h_{ce,j}\left(\overline{\theta}_{e,j} - \theta_{se}\right)$$

with $h_{ce,j}$ the average convective surface film coefficient and $\overline{\theta}_{e,j}$ the average temperature of the air outside the boundary layer, which is usually replaced by the outside temperature measured under a thermometer hut, 1.7 m above grade, in the nearest weather station (θ_e). The average flux thus changes to:

$$q_{ce} = h_{ce}(\theta_e - \theta_{se})$$

where h_{ce} is the average convective surface film coefficient for that weather station value:

$$h_{ce} = h_{ce,j}\frac{\overline{\theta}_{e,j} - \theta_{se}}{\theta_e - \theta_{se}}$$

A second heat flux comes from long-wave radiation between the surface, the terrestrial environment and the sky, which is assumed black. The black body emittance from the surface (s) to the other two (e and sk) equals:

$$M_{bs} = \left[1 + \frac{\rho_{Ls}}{e_{Ls}}(F_{se} + F_{ssk})\right]\mathrm{M}'_s - \frac{\rho_{Ls}}{e_{Ls}}(F_{se}\mathrm{M}'_e + F_{ssk}\mathrm{M}_{bsk})$$

From the environment (e) to the surface (s) and the sky (sk), it is:

$$M_{be} = \left[1 + \frac{\rho_{Le}}{e_{Le}}(F_{es} + F_{esk})\right]M'_e - \frac{\rho_{Le}}{e_{Le}}(F_{es}M'_s + F_{esk}M_{bsk})$$

In both, e_{Ls} and ρ_{Ls} are the long-wave emissivity and reflectivity of the surface, e_{Le} and ρ_{Le} the average long wave emissivity and reflectivity of the terrestrial environment, F_{se} the view factor between the surface and the environment, F_{ssk} the view factor between the surface and the sky, F_{es} the view factor between the terrestrial environment and the surface, and F_{esk} the view factor between the terrestrial environment and the sky. M_{bsk} is the black body emittance of the sky, while M'_s and M'_e are the radiosities of the surface and the terrestrial environment. As the terrestrial environment and the sky surround the surface completely, the sum of the view factors F_{se} and F_{ssk} is 1, while the view factor F_{es} is close to 0 as the surface is infinitely small compared with the terrestrial environment. The view factor F_{esk} is close to 1 since nearly all radiation from the terrestrial environment reaches the sky. The two balances thus simplify to:

$$M_{\mathrm{bs}} = \frac{1}{e_{\mathrm{Ls}}} M'_{\mathrm{s}} - \frac{\rho_{\mathrm{Ls}}}{e_{\mathrm{Ls}}} (F_{\mathrm{se}} M'_{\mathrm{e}} + F_{\mathrm{ssk}} M_{\mathrm{bsk}}) \qquad M_{\mathrm{be}} = \frac{1}{e_{\mathrm{Le}}} M'_{\mathrm{e}} - \frac{\rho_{\mathrm{Le}}}{e_{\mathrm{Le}}} M_{\mathrm{bsk}}$$

Solving both for M'_s and inserting the result in the equation $q_{\mathrm{rse}} + q_{\mathrm{rssk}} = e_{\mathrm{Ls}}(M_{\mathrm{bs}} - \mathrm{M'}_{\mathrm{s}})/\rho_{\mathrm{Ls}}$, knowing that $e_{\mathrm{Ls}}(F_{\mathrm{se}} + F_{\mathrm{ssk}}) M_{\mathrm{bs}} = e_{\mathrm{Ls}} M_{\mathrm{bs}}$, gives:

$$q_{\mathrm{rse}} + q_{\mathrm{rssk}} = q_{\mathrm{re}} = e_{\mathrm{Ls}} F_{\mathrm{se}} (M_{\mathrm{bs}} - e_{\mathrm{Le}} M_{\mathrm{be}}) + e_{\mathrm{Ls}} F_{\mathrm{ssk}} \left[M_{\mathrm{bs}} - \left(\rho_{\mathrm{Le}} \frac{F_{\mathrm{se}}}{F_{\mathrm{ssk}}} + 1 \right) M_{\mathrm{bsk}} \right]$$

Presume now that the terrestrial environment is a black body at outdoor temperature. Using this assumption, and the experimental fact that during clear nights the sky temperature drops some 21 °C below the air temperature in the atmospheric boundary layer, simplifies the radiant heat flux between the surface and the overall environment outdoors to:

$$q_{\mathrm{rs}} = e_{\mathrm{Ls}} C_{\mathrm{b}} [(F_{\mathrm{se}} F_{\mathrm{Tse}} + F_{\mathrm{ssk}} F_{\mathrm{Tssk}})(\theta_{\mathrm{e}} - \theta_{\mathrm{se}}) - 21 F_{\mathrm{ssk}} F_{\mathrm{Tssk}} (1 - 0.87c)]$$

where F_{Tse} is the temperature ratio for radiation between the surface and the terrestrial environment, F_{Tws} is the temperature ratio for radiation between the surface and the sky, and c the cloudiness factor (0 for a clear sky and 1 for an overcast sky).

A third heat flux comes from the sun. Each square metre of exterior surface absorbs the solar beam, diffuse and reflected short-wave radiation (E_{ST}) proportionally to its short-wave absorptivity (α_{S}):

$$q_{\mathrm{se}} = \alpha_{\mathrm{S}} E_{\mathrm{ST}}$$

Summing up the three heat fluxes gives:

$$\begin{aligned} q_{\mathrm{e}} &= h_{\mathrm{ce}}(\theta_{\mathrm{e}} - \theta_{\mathrm{se}}) + 5.67 e_{\mathrm{Ls}} (F_{\mathrm{se}} F_{\mathrm{Tse}} + F_{\mathrm{ssk}} F_{\mathrm{Tssk}})(\theta_{\mathrm{e}} - \theta_{\mathrm{se}}) \\ &\quad -120\, e_{\mathrm{Ls}} F_{\mathrm{ssk}} F_{\mathrm{Tssk}} \left(1 - f_{\mathrm{c}}\right) + \alpha_{\mathrm{K}} E_{\mathrm{ST}} \end{aligned}$$

With the outside surface film coefficient for radiation (h_{re}) equal to:

$$h_{\mathrm{re}} = 5.76 e_{\mathrm{Ls}} (F_{\mathrm{se}} F_{\mathrm{Tse}} + F_{\mathrm{ssk}} F_{\mathrm{Tssk}})$$

this equation rewrites as:

$$q_{\mathrm{e}} = (h_{\mathrm{ce}} + h_{\mathrm{re}}) \left\{ \left[\theta_{\mathrm{e}} + \frac{\alpha_{\mathrm{K}} E_{\mathrm{ST}} - e_{\mathrm{Ls}} 120\, F_{\mathrm{ssk}} F_{\mathrm{Tssk}} \left(1 - f_{\mathrm{c}}\right)}{h_{\mathrm{ce}} + h_{\mathrm{re}}} \right] - \theta_{\mathrm{se}} \right\} \tag{1.94}$$

The term between brackets ([]) with units °C may act as the reference temperature and is called the (average) sol-air temperature, symbol θ_e^*, over a given time interval (1 hour, 1 day, 1 week, 1 month). Consider it as the fictive air temperature, which ensures that the heat exchanged with the outside surface equals the value obtained by solar irradiation, long-wave radiation and convection, provided that the outside convective surface film coefficient is 19 W/(m^2.K). The sol-air temperature depends on the radiant properties of the surface, its inclination, its orientation, the weather, the time interval considered, and so on. Its value differs between applications. The sum $h_{ce} + h_{re}$ yet shapes the outside surface film coefficient h_e, units W/(m^2.K).

The flux equation above then becomes:

$$q_e = h_e\left(\theta_e^* - \theta_{se}\right) \tag{1.95}$$

Replacement of the temperature ratios for radiation F_{Tse} and F_{Tssk} by one value F_T, and from the fact that $F_{se} + F_{ssk} = 1$, the surface film coefficient for radiation (h_{re}) simplifies to:

$$h_{re} = 5.67\, e_L F_T \tag{1.96}$$

As the temperature factor F_T ranges between 0.8 and 0.9, a probable interval for its value is $4.4 e_L \le h_{re} \le 5.1 e_L$ W/(m^2.K).

Provided that outside surfaces are grey with a long-wave emissivity of 0.9, h_{re} of 4–4.6 W/(m^2.K) looks most likely. Thus, the standard outside surface film coefficient for heat transfer could be some 23 W/(m^2.K). The EN standard takes 25 W/(m^2.K), while the 2017 *ASHRAE Handbook of Fundamentals* makes a distinction between winter and summer:

	Direction of heat flow	h_e (W/(m^2.K)
Winter (wind speed 6.7 m/s)	Any	34.0
Summer (wind speed 3.4 m/s)	Any	22.7

For more accurate numerics one should return to the complete heat balance, including a more precise evaluation of the mean wind velocity. Long-term measurements in the early 1980s by the Laboratory of Building Physics, KULeuven, at the leeward side of an existing building for example gave a much lower average than 25 W/(m^2.K).

1.6.2 Steady state: flat assemblies

1.6.2.1 Thermal transmittance of envelope parts and partitions

The use of surface film coefficients simplifies the calculation of the steady-state heat flux ambient-to-ambient across flat assemblies. Consider the outer wall in Figure 1.31.

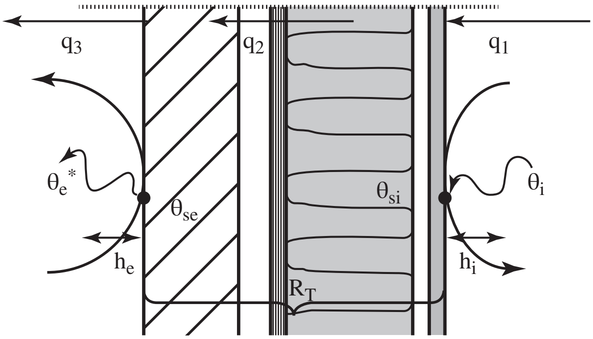


Fig. 1.31 Outer wall, thermal transmittance

Indoors, the reference temperature is θ_o, while outdoors it is θ_e^*. Assume a heated indoors and cold outdoors. From indoors to the inside surface the heat flux is:

$$q_1 = h_i(\theta_o - \theta_{si})$$

with θ_{si} the inside surface temperature. Across the assembly the heat flux equals:

$$q_2 = (\theta_{si} - \theta_{se})/R_T$$

with θ_{se} the outside surface temperature and R_T the total thermal resistance of the assembly. From the outside surface to outdoors, the heat flux is:

$$q_3 = h_e\left(\theta_{se} - \theta_e^*\right)$$

In steady state, the three must be equal, with a common value q. Rearrangement and addition gives:

$$\begin{aligned} q/h_i &= \theta_o - \theta_{si} \\ qR_T &= \theta_{si} - \theta_{se} \\ q/h_e &= \theta_{se} - \theta_e^* \\ \hline \text{Sum:}\quad q(1/h_i + R_T + 1/h_e) &= \theta_o - \theta_e^* \end{aligned}$$

a result that is rewritten as $q = U\left(\theta_o - \theta_e^*\right)$ with:

$$U = \frac{1}{1/h_e + R + 1/h_i} \tag{1.97}$$

The quantity U is called the thermal transmittance of the assembly, units W/(m^2.K). The lower that value, the less heat passes across, or the property reflects the insulation quality of a wall, roof or floor, separating the indoors from outdoors.

Fig. 1.32 Masonry, clearly not a layer where the heat flux develops one-dimensionally

Thus a calculation for ambient-to-ambient, accounting for radiation and convection at the inside and outside surfaces, looks simple. It suffices to add two surface resistances to the total resistance:

- indoors, a value $1/h_i$, denoted R_i, equal to 0.13 m^2.K/W for vertical surfaces, 0.1 m^2.K/W for sloped and horizontal surfaces if the heat flows upwards, and 0.17 m^2.K/W if downwards.
- outdoors, one value suffices for $1/h_e$, denoted R_e: 0.04 m^2.K/W independent of the slope and flow direction.

The inverse of the thermal transmittance is called the thermal resistance ambient-to-ambient, symbol R_a, units m^2.K/W. The thermal transmittance as defined has the prefix 'clear wall', because possible two- and three-dimensional effects are not considered. However, due to voids in the mortar joints and vertical perforations in the bricks, heat transfer across a masonry wall for example already develops three-dimensionally (see Figure 1.32).

For inside partitions, the surface film coefficient at both sides is the value inside (h_i), giving, as thermal transmittance:

$$U = 1/(R + 2/h_i) \tag{1.98}$$

Reference temperatures are operative on both sides of the partition.

1.6.2.2 Average thermal transmittance of parts in parallel

Consider an assembly with surface A_T, composed of n parts in parallel with surfaces A_i that all face the outdoors (Figure 1.33).

Each part is different. If lateral conduction between parts is negligible and if all face the same reference temperature indoors, the heat flow across each is:

$$\Phi_i = U_i \, A_i \, \Delta\theta$$

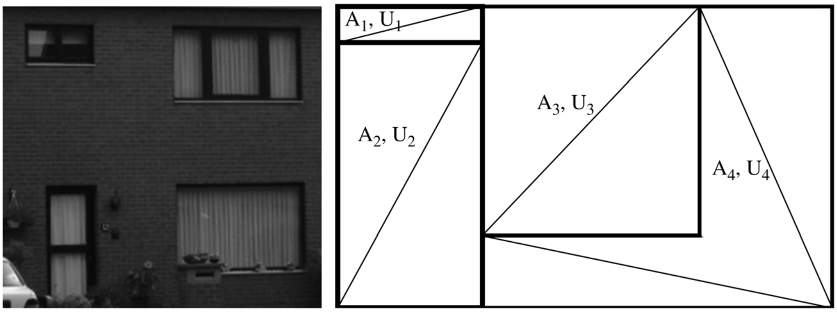


Fig. 1.33 Assembly composed of *n* parallel parts with hardly any lateral heat exchange

In case they face different operative temperatures indoors ($\theta_{o,j}$), that equation converts to:

$$\Phi_i = a_i U_i \, A_i \, \Delta\theta$$

with a_i a reduction factor equal to:

$$a_i = \frac{\theta_{o,j} - \theta_e}{\theta_{o,ref} - \theta_e}$$

with $\theta_{o,ref}$ the indoor operative temperature taken as the reference. The heat flow across the whole now equals the sum of the flows across each, or:

$$\Phi_T = \sum_{i=1}^{n} \Phi_i = \Delta\theta \sum_{i=1}^{n} (a_i U_i A_i) \tag{1.99}$$

Rewriting gives:

$$\Phi_T = \Delta\theta \, U_m \sum_{i=1}^{n} A_i \tag{1.100}$$

where U_m is the average clear-wall thermal transmittance of *n* parts in parallel, a value equal to the surface weighted average of their (clear wall) thermal transmittances:

$$U_m = \sum_{i=1}^{n} (a_i U_i A_i) \Big/ \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} (a_i U_i A_i) / A_T \tag{1.101}$$

Conversion to resistances gives:

$$R_{am} = A_T \Big/ \sum_{i=1}^{n} \left(\frac{a_i A_i}{R_{ai}} \right) \tag{1.102}$$

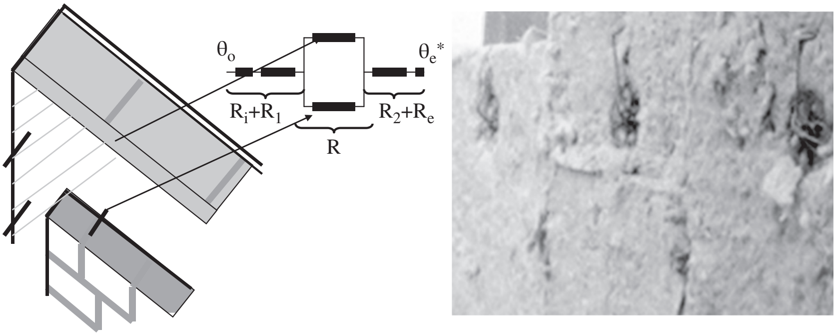


Fig. 1.34 Cavity wall, with an electrical analogy accounting for the ties that perforate the fill

1.6.2.3 Electrical analogy

As long as lateral conduction between parts is negligible, an electrical analogy allows us to solve quite complex cases. Consider a cavity wall. The ties perforate the fill. If $\mathrm{A_t}$ is the tie section, R_t their thermal resistance, $\mathrm{A_{is}}$ is the wall area and R_is is the thermal resistance of the insulation, then the overall thermal resistance (R) of the insulation equals (Figure 1.34):

$$R = \frac{\mathrm{A_{is}}}{\dfrac{\mathrm{A_{is}} - \mathrm{A_t}}{R_\mathrm{is}} + \dfrac{\mathrm{A_t}}{R_\mathrm{t}}}$$

With $R_1 + R_\mathrm{i}$ the thermal resistance from the insulation to indoors and $R_2 + R_\mathrm{e}$ the thermal resistance from the insulation to outdoors, the value using the two ambients becomes:

$$R_\mathrm{T} = (R_\mathrm{i} + R_1) + R + (R_2 + R_\mathrm{e}) = (R_\mathrm{i} + R_1) + \frac{\mathrm{A_i}}{\dfrac{\mathrm{A_i} - \mathrm{A_t}}{R_\mathrm{is}} + \dfrac{\mathrm{A_t}}{R_\mathrm{t}}} + (R_2 + R_\mathrm{e})$$

1.6.2.4 Thermal resistance of an unvented cavity

In an infinitely extending unvented cavity, the nature, distance and temperature difference between the bounding surfaces, the slope, the heat flow direction, and the mean temperature of the gas fill will all affect conduction, convection and radiation, giving as the total heat flux:

$$q_\mathrm{T} = \left(\frac{\lambda_\mathrm{g}\,\mathrm{Nu}}{d} + \frac{C_\mathrm{b}\,F_\mathrm{T}}{1/e_\mathrm{L1} + 1/e_\mathrm{L2} - 1}\right)(\theta_\mathrm{c} - \theta_\mathrm{c2})$$

with λ_g the thermal conductivity of the gas, Nu the case-specific Nusselt number, e_L1 and e_L2 the long-wave emissivities and θ_c1 and θ_c2 the temperatures of the bounding

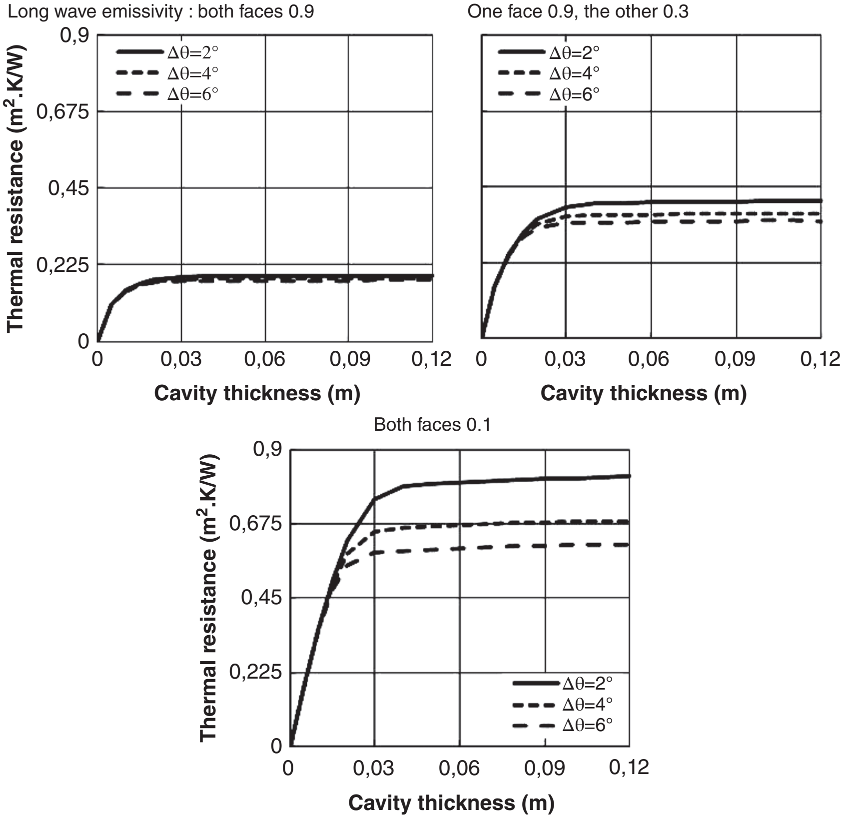


Fig. 1.35 Thermal resistance of an infinite vertical cavity for three different sets of parameters

surfaces. The thermal resistance then is:

$$R_c = \left(\frac{\lambda_g\,\mathrm{Nu}}{d} + \frac{C_b F_T}{1/e_{L1} + 1/e_{L2} - 1}\right)^{-1} \tag{1.103}$$

Figure 1.35 gives values for a vertical air-filled cavity wherein the air temperature is 10 °C, depending on the width, the temperature difference across and the long-wave emissivity of the bounding surfaces.

The thermal resistance increases considerably with lower long-wave emissivity of the bounding surfaces, while at low emissivity the temperature difference across gains influence, underlining the dominance of radiation. The absence, in turn, of any additional gain once the cavity width passes 20–30 mm illustrates that radiation does not depend on this variable, while more convection gradually compensates for the drop in conduction. For non-vented cavities with limited dimensions, the values in Table 1.4 allow first-order calculations.

Table 1.4 Thermal resistance of cavities

Thickness (mm)	Vertical cavity		Horizontal cavity	
	R_c (m^2 K/W), both surfaces grey	R_c (m^2 K/W), one surface reflecting	R_c (m^2 K/W), heat flow up	R_c (m^2 K/W), heat flow down
$0 < d < 5$	0.00		0.00	0.00
$5 \leq d < 7$	0.11		0.11	0.11
$7 \leq d < 10$	0.13		0.13	0.13
$10 \leq d < 15$	0.15		0.15	0.15
$15 \leq d < 25$	0.16	0.35	0.17	0.17
$25 \leq d < 50$	0.16	0.35	0.17	0.19
$50 \leq d < 100$	0.16	0.35	0.18	0.21
$100 \leq d < 300$	0.16	0.35	0.18	0.22
$d > 300$	0.16	0.35	0.18	0.23

1.6.2.5 Interface temperatures

Both surface resistances R_i and R_e resemble the thermal resistance of a 1 m thick air layer with thermal conductivity h_i or h_e, whereby the reference temperatures act as fictitious 'surface temperatures' that activate the steady-state heat transfer. So for any composite envelope assembly, temperatures in the $[R, \theta]$ plane lie on a straight line linking $[0,\theta_e^*]$ to $[R_a, \theta_o]$ or, for inside partitions, linking $[0,\theta_{o,1}]$ to $[R_a, \theta_{o,2}]$ with both slopes, giving the heat flux When tracing the layer sequence, both surface resistances must be respected (see Figure 1.36).

The temperature on the inside surface is:

$$\text{Envelope assembly:} \quad \theta_{si} = \theta_o - R_i \frac{\theta_o - \theta_e^*}{R_a} = \theta_o - \frac{U_{h_i}}{h_i}\left(\theta_o - \theta_e^*\right) \tag{1.104}$$

$$\text{Inside partition:} \quad \theta_{si} = \theta_{i.1} - R_i \frac{\theta_{o,1} - \theta_{o,2}}{R_a} = \theta_{o,1} - \frac{U_{h_i}}{h_i}\left(\theta_{o,1} - \theta_{o,2}\right) \tag{1.105}$$

The suffix h_i underlines that the clear wall thermal transmittance must be calculated using the surface film coefficient in the denominator. The temperatures at the interfaces equal:

$$\theta_x = \theta_i - q\left(R_i + R_{si}^x\right)$$

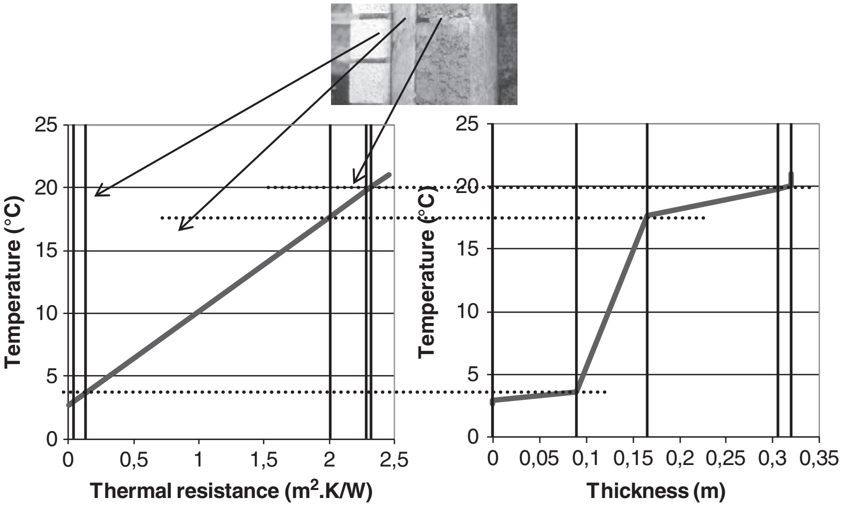


Fig. 1.36 Composite envelope assembly (filled cavity wall): temperature curve

1.6.2.6 Solar transmittance

The solar heat flux across any envelope part can be written as:

$$q_S = gE_{ST} \tag{1.106}$$

with E_{ST} the total incident solar radiation on the outside face and q_S the heat flux transmitted, both in W/m^2. The factor g, called the solar transmittance, encompasses the direct and indirect solar gains. The direct ones are:

$$q_{Sd} = \tau_S E_{ST}$$

with τ_S the total shortwave transmissivity of the part. For opaque parts, transmissivity and the direct gains are zero, but not so for transparent parts.

Indirect gains occur because opaque and transparent parts absorb a fraction of the solar flux impinging. They so warm up and conduct absorbed heat to the inside, where convection and long wave radiation dissipates it. For single glass with shortwave absorptivity α_S, estimating the indirect gains is easy as they are part of the heat flux dissipated by convection and radiation from the inside surface to the indoors:

$$q_{Si} = h_i(\theta_{si} - \theta_o) \tag{1.107}$$

where θ_{si} is the unknown inside surface temperature. Assuming that the glass is equally warm gives $\theta_{si} = \theta_x = \theta_{se}$ with θ_{se} the outside surface temperature and θ_x the glass temperature. The thermal balance for 1 m^2 of glass thus becomes (sum of the absorbed solar radiation, the heat flux outdoors and the heat flux indoors to the glass being zero):

$$\alpha_S E_{ST} + h_e(\theta_e - \theta_x) + h_i(\theta_o - \theta_x) = 0$$

with θ_e the outdoor air and θ_o the indoor operative temperature. The glass temperature thus is:

$$\theta_x = \theta_{si} = \frac{\alpha_S E_{ST}}{h_i + h_e} + \frac{h_e\theta_e + h_i\theta_i}{h_i + h_e}$$

The second term on the right-hand side stands for that temperature if single glass could not absorb solar radiation, and the first for the increase due to the fact that it does. Combining this with the equation for the heat flux to the indoors gives:

$$q_{Si} = \frac{h_i \alpha_S E_{ST}}{h_i + h_e} + \frac{h_i h_e(\theta_e - \theta_i)}{h_i + h_e} \tag{1.108}$$

Only the first term on the right is linked to the sun, thus representing the indirect gains, or:

$$q_{Si} = \frac{h_i \alpha_S E_{ST}}{h_i + h_e}$$

The solar transmittance for single glass thus equals:

$$g = \frac{q_{Sd} + q_{Si}}{E_{ST}} = \tau_S + \frac{\alpha_S}{1 + h_e/h_i} \tag{1.109}$$

The gains not only depend on the short-wave transmissivity but also on the short-wave absorptivity of the glass and the ease by which the absorbed heat dissipates to the indoors. In fact, a lower ratio between the outside and inside surface film coefficient increases the transmittance.

For double glass, calculating the solar transmittance is more demanding, Let τ_{S1}, ρ_{S1} and α_{S1} be the transmissivity, reflectivity and absorptivity, all shortwave, of one pane, and τ_{S2}, ρ_{S2} and α_{S2} for the other pane. Reflection in the cavity breaks the transmission of solar radiation (Figure 1.37) into a geometric series with ratio $\rho_{S1}\rho_{S2}$, whose sum gives:

$$q_{Sd} = \frac{\tau_{S1}\tau_{S2}}{1 - \rho_{S1}\rho_{S2}} E_{ST} \tag{1.110}$$

In general, the denominator $1 - \rho_{S1}\rho_{S2}$ nears 1. So the guide that the product of the transmissivities of both panes fixes the transmissivity of double glass is quite correct as a rule of thumb.

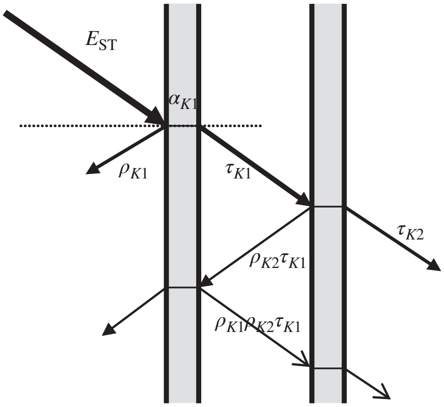


Fig. 1.37 Double glass, solar transmittance

Assuming $\theta_e = \theta_o = 0\,°\text{C}$ and both panes are isothermal, the indirect gains can be written as:

$$q_{Si} = h_i \theta_{x2}$$

with θ_{x2} the sun-induced temperature of the inside pane, a value ensuing from the heat balance per pane (1 is outside, 2 inside):

$$\text{Pane 1:}\quad \alpha_{S1} \underbrace{\frac{1 - \rho_{S1}\rho_{S2} + \tau_{S1}\rho_{S2}}{1 - \rho_{S1}\rho_{S2}}}_{f_1} E_{ST} - h_e \theta_{x1} + \frac{\theta_{x2} - \theta_{x1}}{R_c} = 0$$

$$\text{Pane 2:}\quad \alpha_{S2} \underbrace{\frac{\tau_{S1}}{1 - \rho_{S1}\rho_{S2}}}_{f_2} E_{ST} + \frac{\theta_{x1} - \theta_{x2}}{R_c} - h_i \theta_{x2} = 0$$

This system has as a solution:

$$\theta_{x2} = \frac{\dfrac{\alpha_{S1} f_1}{R_c} + \alpha_{S2} f_2 \left(h_e + \dfrac{1}{R_c}\right)}{\left(h_i + \dfrac{1}{R_c}\right)\left(h_e + \dfrac{1}{R_c}\right) - \dfrac{1}{R_c^2}} E_{ST} \tag{1.111}$$

Inserting the outcome into the equation for the indirect gains gives, as solar transmissivity:

$$g = \tau_{S1}\tau_{S2} + h_i \frac{\dfrac{\alpha_{S1} f_1}{R_c} + \alpha_{S2} f_2 \left(h_e + \dfrac{1}{R_c}\right)}{\left(h_i + \dfrac{1}{R_c}\right)\left(h_e + \dfrac{1}{R_c}\right) - \dfrac{1}{R_c^2}} \tag{1.112}$$

The result shows how to decrease the gains across double glazing: limit the direct transmission and lower either the inside surface film coefficient or the short-wave absorptivity of both panes. For multiple glazing including a shade, a same approach applies, with the short-wave transmissivity equal to the product of the short-wave transmissivities of all panes and the shade, and the indirect gains derived from a system of heat balances per pane and the shade.

1.6.3 Local inside surface film coefficients

Average surface film coefficients are usable when area-average phenomena such as the heat lost or gained are at stake. To quantify temperatures or heat fluxes at specific spots, local values are preferred. In general, the following holds:

$$h_{\mathrm{ix}}\left(\theta_{\mathrm{ref,i}}-\theta_{\mathrm{six}}\right)=h_{\mathrm{cix}}(\theta_{\mathrm{ix}}-\theta_{\mathrm{six}})+h_{\mathrm{rix}}(\theta_{\mathrm{rix}}-\theta_{\mathrm{six}}) \tag{1.113}$$

with h_{ix} the local surface film coefficient linked to a reference temperature $\theta_{\mathrm{ref,i}}$, h_{cix} the local convective surface film coefficient, θ_{ix} the local air temperature outside the boundary layer, θ_{six} the local surface temperature, h_{rix} the local radiant surface film coefficient, and θ_{rix} the radiant temperature the spot faces. If R' is an equivalent thermal resistance that links the inside surface of that spot to the environment on the other side, then:

$$h_{\mathrm{ix}}\left(\theta_{\mathrm{ref,i}}-\theta_{\mathrm{six}}\right)=\left(\theta_{\mathrm{six}}-\theta_{\mathrm{j}}\right)/R' \tag{1.114}$$

This equation is an approximation. In fact, the equivalent thermal resistance depends on the distribution of the local inside surface film coefficients (h_{ix}) over the whole surface. For the envelope, θ_{j} is the temperature outdoors ($\mathrm{j}=\mathrm{e}$), while for inside partitions, it is the reference temperature in the neighbouring space. Eliminating the local surface temperature θ_{six} from both equations and solving for the local surface film coefficient gives:

$$h_{\mathrm{ix}}=\frac{h_{\mathrm{cix}}+h_{\mathrm{rix}}-p_{\mathrm{T}}}{1+R'p_{\mathrm{T}}} \tag{1.115}$$

where:

$$\mathrm{p_T}=\frac{\mathrm{h_{cix}}\left(\theta_{\mathrm{ref,.i}}-\theta_{\mathrm{ix}}\right)+\mathrm{h_{rix}}\left(\theta_{\mathrm{ref.,i}}-\theta_{\mathrm{rix}}\right)}{\theta_{\mathrm{ref,i}}-\theta_{\mathrm{j}}} \tag{1.116}$$

If the reference temperature indoors ($\theta_{\mathrm{ref,i}}$), the relationship with the local air temperature just outside the boundary layer (θ_{ix}), the relationship with the radiant temperature facing the spot (θ_{rix}) and the local inside surface film coefficients h_{cix} and h_{rix} are known, then combining the equations above gives the heat flux at the spot considered, on condition that the equivalent thermal resistance is known. The questions left are how to link the local reference temperature indoors to the overall reference indoors ($\theta_{\text{ref,i}}$), and what values to use for the local surface film coefficients.

As overall reference indoors, the air temperature in the room's centre, 1.7 m above floor level, is chosen (θ_i). Assuming the local air temperature increases linearly along the room's height with little slope if well insulated and less convective heated, then the relation with the overall reference could be:

$$\frac{\theta_{\text{ix}} - \theta_\text{j}}{\theta_\text{i} - \theta_\text{j}} = 1 + 0.2 p_\text{c} U_\text{m}(y - 1.7) \tag{1.117}$$

with y the height ordinate, θ_j the reference temperature in the neighbouring room or outdoors, p_c a convection factor (1 for air heating, 0.9 for convectors, 0.4 to 0.8 for radiators, 0.4 for floor heating), and U_m the weighted average thermal transmittance of all walls in the room. The relationship reflects the outcome of a series of measurements in a test room, where the heating system and the insulation of the outside walls could be varied.

For the local radiant temperature (θ_{rix}) a uniform value is assumed, proportional to the overall reference value, with a gradient depending on the local convective surface film coefficient, the convection factor and the weighted average thermal transmittance of all walls in the room:

$$\frac{\theta_{\text{rix}} - \theta_\text{i}}{\theta_\text{i} - \theta_\text{j}} = \frac{h_{\text{cix}}}{h_{\text{cix}} + \dfrac{(p_\text{c} - 0.4) U_\text{m}}{0.6}} \tag{1.118}$$

This equation followed from computer simulations of the radiant heat exchange in rooms of different shapes.

The local convective surface film coefficient (h_{cix}) is set at 2.5 W/(m^2.K), while the following values are used for the local radiant surface film coefficient (h_{rix}):

Corner between three envelope assemblies or two and an inside partition:	
5.5 e_L	Surfaces more than 0.5 m from the edge lines
3.4 e_L	Surfaces less than 0.5 m from the edge lines but more than 0.5 m from the corner
2.2 e_L	Surfaces less than 0.5 m from the corner
Edge between two envelope assemblies or one and an inside partition:	
5.5 e_L	Surfaces more than 0.5 m from the edge line
3.4 e_L	Surfaces less than 0.5 m from the edge line
Envelope assembly or inside partition:	
5.5 e_L	–

Where furniture hides a wall, a combined inside surface film coefficient of 2 W/(m^2.K) applies.

1.6.4 Steady state: two and three dimensions

1.6.4.1 Pipes

At the outside surface of a pipe, the heat flow equals:

$$\Phi_{n+1} = 2\pi R_{n+1} h_2 \left(\theta_{s,2} - \theta_{ref,2}\right) \tag{1.119}$$

with h_2 the surface film coefficient outside the pipe, $\theta_{ref,2}$ the reference temperature there, $\theta_{s,2}$ the outside surface temperature and R_{n+1} the outside radius. At the inside surface the flow is:

$$\Phi_1 = 2\pi R_1 h_1 \left(\theta_{ref,1} - \theta_{s,1}\right) \tag{1.120}$$

with h_1 the surface film coefficient between fluid and pipe, $\theta_{ref,1}$ the temperature of the fluid, $\theta_{s,1}$ the inside surface temperature and R_1 the inside radius. Across the pipe, the flow numbers:

$$\Phi_{1,n+1} = \frac{\theta_{s,1} - \theta_{s,2}}{\sum\limits_{i=1}^{n} \left[\dfrac{\ln(R_{i+1}/R_i)}{2\pi\lambda_i}\right]}$$

with Σ indicating that the pipe wall may consist of several layers. In steady state, the three heat flows are equal. Rearrangement and addition gives:

$$\begin{aligned}
\frac{\Phi_{n+1}}{2\pi R_{n+1} h_2} &= \theta_{s,2} - \theta_{ref,2} \\
\Phi_{1,n+1} \sum_{i=1}^{n} \left[\frac{\ln(R_{i+1}/R_i)}{2\pi\lambda_i}\right] &= \theta_{s,1} - \theta_{s,2} \\
\frac{\Phi_1}{\theta_{ref,1} - \theta_{s,1}} &= 2\pi R_1 h_1
\end{aligned}$$

$$\text{Sum:} \quad \Phi_{1,n+1} \left\{ \frac{1}{2\pi R_{n+1} h_2} + \sum_{i=1}^{n} \left[\frac{\ln(R_{i+1}/R_i)}{2\pi\lambda_i}\right] + \frac{1}{2\pi R_1 h_1} \right\} = \theta_{ref,1} - \theta_{ref,2}$$

For flat assemblies, this sum is rewritten as:

$$\Phi_{1,n+1} = U_{pipe}\left(\theta_{ref,1} - \theta_{ref,2}\right)$$

where U_{pipe} stands for the thermal transmittance, now per metre run, of the pipe:

$$U_{\text{pipe}} = \frac{1}{\dfrac{1}{2\pi R_{n+1} h_2} + \sum_{i=1}^{n} \left[\dfrac{\ln(R_{i+1}/R_i)}{2\pi\lambda_i}\right] + \dfrac{1}{2\pi R_1 h_1}} \quad (\text{W/(m.K)}) \tag{1.121}$$

Insulation will lower the heat loss of pipes transporting warm fluids, or the gains of pipes transporting cold fluids. A difference is that the additional benefit of thicker insulation drops off more rapidly than for flat assemblies.

1.6.4.2 Floors on grade

Calculation of the thermal transmittance of a floor on grade is a typical example of a three-dimensional heat flow problem solved using a simplified method. The thermal transmittance is written as:

$$U = aU_{\text{o,floor}} \tag{1.122}$$

with a being a reduction factor and $U_{\text{o,floor}}$ the thermal transmittance of the floor as if it were a flat assembly facing the outside. Valuing the reduction factor starts with fixing what is called the characteristic floor dimension:

$$B' = 2A_{\text{fl}}/P \quad (\text{m}) \tag{1.123}$$

with A_{fl} the floor's surface area and P that part of the floor's perimeter touching the outdoors and called the free perimeter (see Figure 1.38).

Then the equivalent soil thickness (d_{t}) of the floor that replaces its thermal resistance is fixed:

$$d_{\text{t}} = d_{\text{fw}} + \lambda_{\text{gr}}\left(\frac{1}{h_{\text{e}}} + R_{\text{T,fl}} + \frac{1}{h_{\text{i}}}\right) \quad (\text{m}) \tag{1.124}$$

Fig. 1.38 The arrow shows a part of the free perimeter

with d_{fw} the average thickness of the foundation walls along the free perimeter in m, λ_{gr} the thermal conductivity of the soil, $R_{\mathrm{T,fl}}$ the thermal resistance of the floor, h_{i} the surface film coefficient indoors, 6 W/(m^2.K), and h_{e} the surface film coefficient outdoors, 25 W/(m^2.K).

Finally, the reduction factor, which depends on the ratio between the equivalent soil thickness and the characteristic floor dimension, follows from:

$$\begin{aligned}&\text{For } d_{\mathrm{t}} < B': \quad a = \frac{1}{U_{\mathrm{o,floor}}}\left(\frac{2\lambda_{\mathrm{gr}}}{\pi B' + d_{\mathrm{t}}}\right)\ln\left(\frac{\pi B'}{d_{\mathrm{t}}} + 1\right)\\&\text{For } d_{\mathrm{t}} \geq B': \quad a = \frac{1}{U_{\mathrm{o,floor}}}\left(\frac{\lambda_{\mathrm{gr}}}{0.457\,B' + d_{\mathrm{t}}}\right)\end{aligned} \tag{1.125}$$

1.6.4.3 Thermal bridges

The term thermal bridge applies to all spots in the envelope where heat flows two- or three-dimensionally. The term may be taken literally: not only is more heat lost or gained than across neighbouring flat parts, except for single glazing, but during the heating season the inside surface will also stay colder there. CVM is used to calculate the heat exchange with the ambient using surface film resistances, through which the heat moves normally to the end faces. Where local surface temperatures are of interest, local values apply. When it is the overall heat loss or gain that matters, the standard values are used. The energy balance for a control volume with a centre point on an end face combines six heat flows: four from the neighbouring centre points on the face, one from the neighbouring control volume in the material, and one across the surface film resistance with the ambient reference temperature as source. Consider Figure 1.39.

The surface of a control volume touching the end face extends parallel to the [y, z] plane. The heat flow from the ambient temperature θ_1 in (i, m, n) to its centre point (s, m, n) equals:

$$\Phi_{s,m,n}^{i,m,n} = h_{\mathrm{i}}\left(\theta_{i,m,n} - \theta_{s,m,n}\right)a^2$$

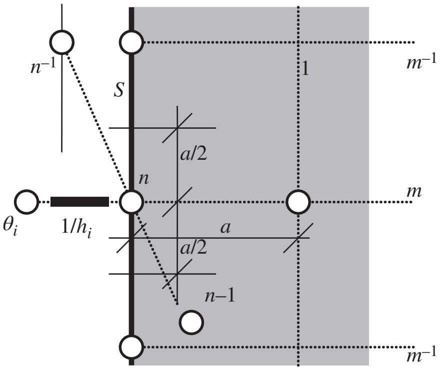


Fig. 1.39 CVM method, with control volumes at the inside or outside surface

The heat flows from the four neighbouring centre points on the end face to that point are:

$$\Phi^{s,m,n}_{s,m+1,n} = \lambda_1 \left(\theta_{s,m+1,n} - \theta_{s,m,n}\right) \frac{a}{2} \quad \Phi^{s,m,n}_{s,m-1,n} = \lambda_1 \left(\theta_{s,m-1,n} - \theta_{s,m,n}\right) \frac{a}{2}$$

$$\Phi^{s,m,n}_{s,m,n+1} = \lambda_1 \left(\theta_{s,m,n+1} - \theta_{s,m,n}\right) \frac{a}{2} \quad \Phi^{s,m,n}_{s,m,n-1} = \lambda_1 \left(\theta_{s,m,n-1} - \theta_{s,m,n}\right) \frac{a}{2}$$

The heat flow from the neighbouring control volume in the material to that point is:

$$\Phi^{s,m,n}_{l,m,n} = \lambda_1 \left(\theta_{1,m,n} - \theta_{s,m,n}\right) a$$

Setting to zero gives:

$$a\,h_{\mathrm{i}}\theta_{i,m,n} + \lambda_1 \frac{\theta_{s,m+1,n} + \theta_{s,m-1,n} + \theta_{s,m,n+1} + \theta_{s,m,n-1}}{2} + \lambda_1\theta_{l,m,n} - (a\,h_{\mathrm{i}} + 3\lambda_1)\theta_{s,l,m,n} = 0$$

with $a\,h_{\mathrm{i}}\theta_{i,m,n}$ the known term. Mesh points in the corners and others give analogous equations. Figure 1.40 shows the results of a CVM calculation for a wall with a load-bearing outer leaf, the cavity closed at the window, and a non-load-bearing inner leaf.

In practice, a distinction is made between geometric and structural thermal bridges. The former follow from the three-dimensional nature of building enclosures. The latter are a consequence of structural decisions, such as concrete girders and columns penetrating the envelope, or discontinuities in the thermal insulation. Structural integrity often explains their existence. For example, take a balcony – continuity with the floor slab is needed to balance the cantilever moment (Figure 1.41).

Neutralizing both thermal bridge types demands continuity of the thermal insulation. Ideally, the insulation should be traceable on the drawings without crossing parts that create easy heat flow paths. Complete avoidance is often not possible, although the impact must remain manageable.

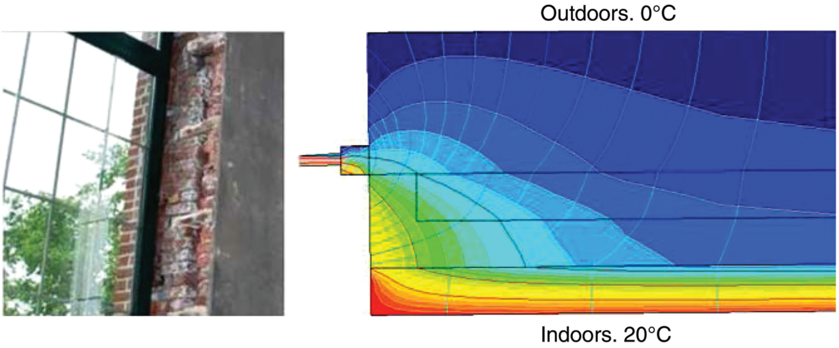


Fig. 1.40 CVM calculation result

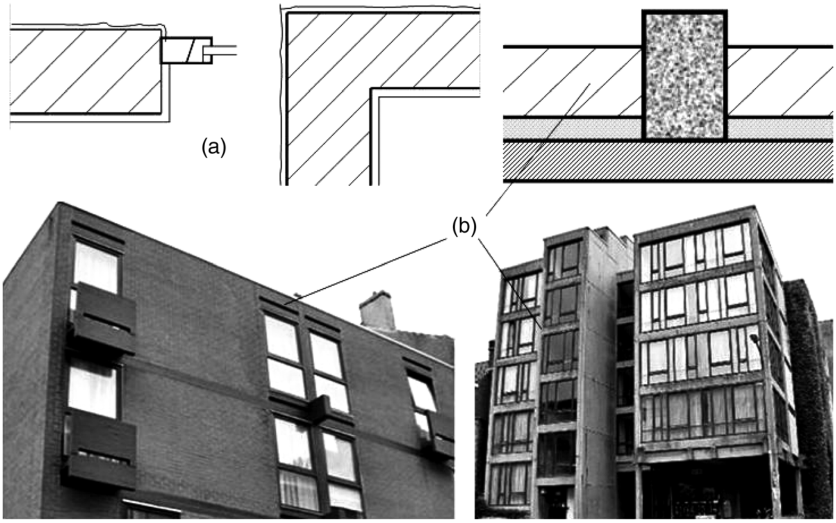


Fig. 1.41 (a) Geometric thermal bridges; (b) structural thermal bridges

Using CVM at the design stage is demanding. Therefore the concepts of a linear and a local thermal transmittance have been introduced. The first, symbol ψ, units W/(m.K), stands for the extra heat flow a two-dimensional thermal bridge gives per metre run and per kelvin temperature difference between the ambient on both sides. The second, symbol χ, units W/K, quantifies the extra heat flow a three-dimensional thermal bridge induces per kelvin temperature difference between the ambient on both sides. Calculating the first demands a well-defined one-dimensional reference and agreement on what surface to consider, inside or outside, with a preference for outside as it allows use of the facade drawings (Figure 1.42). The detail that acts as a thermal bridge is ignored at first. Otherwise,

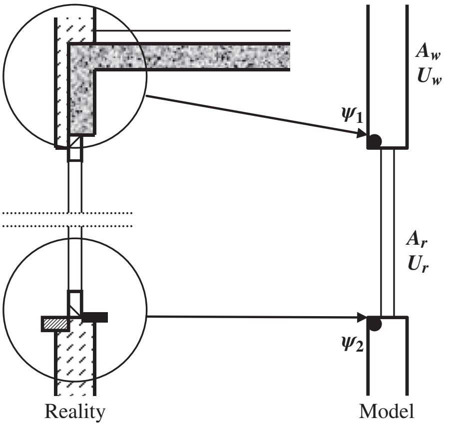


Fig. 1.42 Linear thermal transmittances. The dummy consists of flat parts with lines perpendicular to the section representing the linear thermal bridges

a flat dummy replaces reality, with a dot where the thermal bridge sits, and the one-dimensional heat loss is calculated.

Then, using the correct structural drawings the real two-dimensional heat transfer and inside surface temperatures are calculated. When Φ_{2D} is the two-dimensional heat flow and Φ_o the heat flow across the flat dummy, the linear thermal transmittance equals:

$$\psi = \frac{\Phi_{2D} - \Phi_o}{L\,\Delta\theta} \tag{1.126}$$

with L the length of the thermal bridge. If an assembly only contains a local thermal bridge, the local thermal transmittance (χ) becomes:

$$\psi = \frac{\Phi_{3D} - \Phi_o}{L\Delta\theta} \tag{1.127}$$

Often, a local thermal bridge emerges where linear ones cross. If so, two references must be calculated: first the one-dimensional, and then including the linear thermal bridges. Then the local transmittance is extracted from a three-dimensional calculation:

$$\chi = \frac{\Phi_{3D} - \Phi_{2D}}{\Delta\theta} \tag{1.128}$$

Once all linear and local thermal transmittances are known, the whole-wall thermal transmittance of a flat assembly with thermal bridges follows from:

$$U = U_o + \frac{\sum_{i=1}^{n} (\psi_i L_i) + \sum_{j=1}^{m} \chi_j}{A} \tag{1.129}$$

where U_o is its clear wall thermal transmittance, A the surface area considered, n the number of linear thermal bridges, L_i their length and m the number of local thermal bridges.

The lowest inside surface temperature ($\theta_{s,min}$) that an envelope-related thermal bridge calculation gives is mostly transposed into a non-dimensional temperature factor:

$$f_{h_i} = \frac{\theta_{s,min} - \theta_e}{\theta_o - \theta_e} \tag{1.130}$$

with θ_o the reference temperature indoors and θ_e the one outdoors. The suffix h_i is a reminder that the local surface film coefficient must be used when calculating surface temperatures. A CVM calculation with a 1 K temperature difference between the environments on either side directly gives the temperature factor.

The higher the linear or local thermal transmittance and the lower the temperature factor, the more problematic is a thermal bridge. The inside surface will collect more dirt, have a greater mould risk, may become a preferred spot for surface condensation and see crack sensitivity increase, while taking a disproportionate share in the heat loss or gain.

Thermal bridge catalogues have been published that contain the linear thermal transmittances, local thermal transmittances and temperature factors for reveals, lintels, dormer windows, balconies, and so forth, considering various designs, material combinations and layer thicknesses. Interactive CD-ROMs and software tools to calculate two- and three-dimensional heat flows and temperature fields are also available.

1.6.4.4 Windows

Windows transfer heat three-dimensionally, as the IR picture in Figure 1.43 shows.

Calculating the thermal transmittance of a window (U_{window}) thus requires appropriate software tools. However, as this is not really practical, frames are characterized by an equivalent thermal transmittance ($U_{\text{eq,frame}}$), multi-pane glass by a central thermal transmittance ($U_{\text{o,glass}}$), and the glazing/spacer/frame combination by a linear thermal transmittance (ψ_{spacer}). This allows us to write:

$$U_{\text{window}} = \frac{A_{\text{glass}} U_{\text{o,glass}} + A_{\text{frame}} U_{\text{eq,frame}} + \psi_{\text{spacer}} L_{\text{spacer}}}{A_{\text{window}}} \tag{1.131}$$

The surface taken by the frame (A_{frame}) coincides with its normal projection onto an outside plane parallel to the window. The visible glass surface (A_{glass}) is defined the same way, while the length of the spacer (L_{spacer}) equals the total perimeter of all

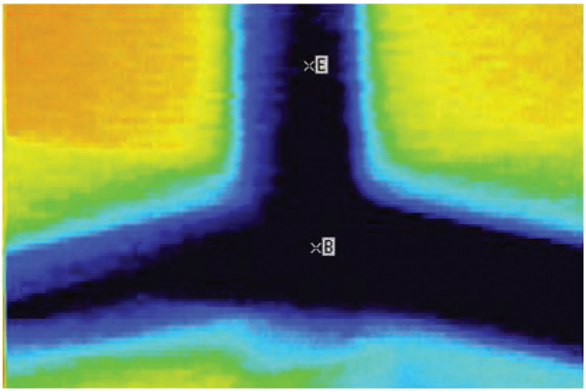


Fig. 1.43 Window: IR picture of the frame and the double glass

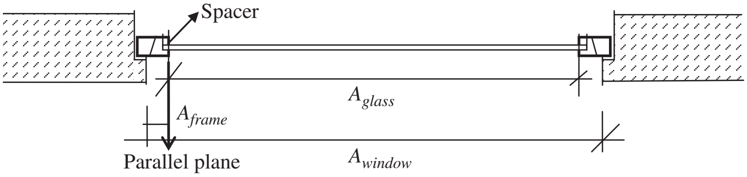


Fig. 1.44 Window, thermal transmittance

glazing parts, measured out to out (see Figure 1.44). Table 1.5 lists approximate thermal transmittances and linear thermal transmittances for different types of frames, glazing and spacers.

1.6.4.5 Building envelopes

Building envelopes, also called building enclosures, shield the indoors from the outdoors, from unheated neighbouring spaces, sometimes from water volumes and from the soil. The assemblies forming the enclosure encompass low-slope and sloped roofs, outside walls, walls separating heated from unheated spaces, glazed surfaces, floors on grade, floors above crawlspaces, floors above unheated basements, floors separating heated from unheated spaces and floors separating the indoors from the outdoors (see Figure 1.45). For party walls, the assumption typically but not necessarily is that both buildings are at a same temperature.

Envelopes are by definition three-dimensional. Quantifying the time-averaged heat flow for a 1 °C difference with the outdoors is done by decomposing the envelope

Table 1.5 Frames, glass and spacers, thermal transmittances and linear thermal transmittances

Window frames	U_{frame} W/(m^2.K)	Glazing	U_{glass} W/(m^2.K)
Hardwood, $d = 70$ mm	2.08	Double	2.8
Aluminium, 20 mm thermal cut	2.75	Double, low-e, argon-filled	1.1
PVC, three room frame	2.00	Triple, low-e, argon-filled	0.6
Spacers			
Metal	ψ W/(m.K)	Insulating	ψ W/(m.K)
$U_{\text{frame}} < 5.9$, $U_{\text{glass}} > 2.0$ W/(m^2.K)	0.06	$U_{\text{frame}} < 5.9$, $U_{\text{glass}} > 2.0$ W/(m^2.K)	0.05
$U_{\text{frame}} < 5.9$, $U_{\text{glass}} < 2.0$ W/(m^2.K)	0.11	$U_{\text{frame}} < 5.9$, $U_{\text{glass}} < 2.0$ W/(m^2.K)	0.07

Fig. 1.45 Building envelope

into flat and curved parts with area A_j and clear thermal transmittance $U_{o,j}$ coupled in parallel. The contact lines and linear details, each with length L_k, are represented by linear thermal transmittances, while all spots where heat flows three-dimensionally get local thermal transmittances. This gives, as average thermal transmittance of an envelope:

$$U_\mathrm{m} = \frac{\sum\limits_{j=1}^{n}\left(a_j A_j U_{\mathrm{o},j}\right) + \sum\limits_{k=1}^{m}\left(a_k L_k \psi_k\right) + \sum\limits_{l=1}^{p}\left(a_l \chi_l\right)}{A_\mathrm{T}} \tag{1.132}$$

In this formula, a_i, a_k and a_l are reduction factors. A value of 1 stands for parts separating the indoors from the outdoors, a value below 1 for parts separating the building from unheated neighbouring spaces, for floors on grade (see above), floors above unheated basements, floors above crawlspaces and vertical walls contacting the soil. For party walls the value is typically 0. Contact with water gives:

$$a = \frac{1}{1 - 0.04 U_\mathrm{o}}$$

Still, how to measure surfaces and lengths has to be decided upon. The outside dimensions are handy, as these are available from the facade drawings. When for reasons of simplicity, linear and local thermal transmittances are overlooked, then out-to-out gives the smallest error, although it is best not to overlook thermal bridging.

Bad workmanship can seriously degrade the real average thermal transmittance, compared with what was calculated. A formula that reflects this has the form:

$$U_\mathrm{m} = \frac{\sum\limits_{j=1}^{n}\left(a_j A_j U_{\mathrm{o},j}/\eta_{\mathrm{ins},j}\right) + \sum\limits_{k=1}^{m}\left(a_k L_k \psi_k/\eta_{\mathrm{ins},k}\right) + \sum\limits_{l=1}^{p}\left(a_l \chi_l/\eta_{\mathrm{ins},l}\right)}{A_\mathrm{T}} \tag{1.133}$$

with $\eta_{\text{ins},j}$, $\eta_{\text{ins},k}$ and $\eta_{\text{ins},l}$ the insulation efficiencies, with values of 1 for perfect workmanship, but far below 1 for poor workmanship allowing air looping around, wind washing behind and indoor air washing in front of the insulation.

1.6.5 Heat balances

The use of surface film coefficients does not reflect reality in an exact way. In case the concepts do not work, a return to and solution of the separate heat balances is preferred. First, the surfaces or interfaces where the temperature and heat flows or fluxes are the unknowns are selected. These form the calculation points, with their number defining the number of balances needed. Then, in each calculation point, conservation of energy applies: the sum of all heat flows coming from the environment or neighbouring volumes equals zero. In this way, each calculation point supplies an equation in which its temperature and some or all unknown temperatures of the neighbouring points feature as variables, and the known temperatures are given. Solving the system then gives the requested temperatures and heat flows or fluxes. The challenge lies in not overlooking any intervening heat flows or fluxes.

1.6.6 Transient

1.6.6.1 Periodic: flat assemblies

To calculate the periodic response from environment to environment, the surface film resistances are again assumed to represent 1 m thick air layers with thermal conductivity h_i and h_e, but a volumetric specific heat capacity of 0. The reference temperature of either thus becomes a fictitious surface temperature. For both, the following apply:

$$\omega_\text{n} = \sqrt{\frac{2in\pi\rho c\lambda}{T}} = 0 \qquad \cosh(\omega_\text{n}R) = 1$$

$$\omega_\text{n}\sinh(\omega_\text{n}R) = 0 \qquad \frac{\sinh(\omega_\text{n}R)}{\omega_\text{n}} = \frac{0}{0} = \lim_{\text{n}\to\infty}\left(\frac{\sinh(\omega_\text{n}R)}{\omega_\text{n}}\right) = R$$

turning the complex surface matrixes into:

$$W_\text{i} = \begin{bmatrix} 1 & 1/h_\text{i} \\ 0 & 1 \end{bmatrix} \qquad W_\text{e} = \begin{bmatrix} 1 & 1/h_\text{e} \\ 0 & 1 \end{bmatrix}$$

Transposition into real matrices gives:

$$W_\text{i} = \begin{bmatrix} 1 & 0 & 1/h_\text{i} & 0 \\ 0 & 1 & 0 & 1/h_\text{i} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad W_\text{e} = \begin{bmatrix} 1 & 0 & 1/h_\text{e} & 0 \\ 0 & 1 & 0 & 1/h_\text{e} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.134}$$

For an envelope assembly, the system matrix environment to environment thus becomes:

$$W_{\mathrm{na}} = W_{\mathrm{i}}\, W_{\mathrm{n1}}\, W_{\mathrm{n2}}\, W_{\mathrm{n3}} \ldots W_{\mathrm{nn}}\, W_{\mathrm{e}}$$

For an inside partition, it changes to:

$$W_{\mathrm{na}} = W_{\mathrm{i}}\, W_{\mathrm{n1}}\, W_{\mathrm{n2}}\, W_{\mathrm{n3}} \ldots W_{\mathrm{nn}}\, W_{\mathrm{i}}$$

For single-layer assemblies, these products reduce to $W_{\mathrm{na}} = W_{\mathrm{i}}\, W_{\mathrm{n}}\, W_{\mathrm{e}}$ when part of the envelope, and $W_{\mathrm{na}} = W_{\mathrm{i}}\, W_{\mathrm{n}}\, W_{\mathrm{i}}$ when an inside partition.

All this, of course, is a simplification. On the one hand, the radiant part of the surface film resistance involves all other faces seen by the surface; on the other hand, due to the volumetric specific heat capacity of air, limited air velocity and the interactions with other surfaces and furniture, some inertia is involved.

1.6.6.2 Periodic: spaces

Assume that the envelope and partitions enclosing a space can be decomposed into parallel flat assemblies whereby the windows lack thermal inertia. To simplify the calculations, ventilation accounts for a constant outside airflow, while air exchanges with neighbouring spaces are lacking and the solar and internal gains get injected in the space's centre. That centre's operative temperature θ_o is thermally linked to all assemblies by surface film coefficients h_i, which combine convection and radiation (Figure 1.46).

The response to a periodic heat input consists of a zero harmonic, which equals the average response as steady-state reality, a first harmonic with as period (T) the time span of, for example, 1 day, and higher harmonics with periods $T/2$, $T/3$, and so on.

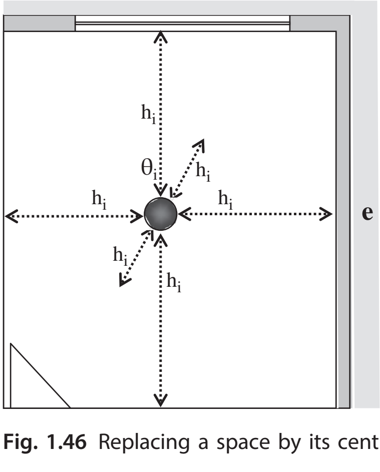


Fig. 1.46 Replacing a space by its centre point

The zero harmonic with the operative temperature in the centre (θ_o) as an unknown equals:

$$\sum_{j=1}^{n}\left[a_{\mathrm{e},j}U_{\mathrm{e},j}A_{\mathrm{e},j}\left(\theta^{*}_{\mathrm{e},j}-\theta_\mathrm{o}\right)\right]+\sum_{k=1}^{m}\left[U_{\mathrm{w},k}A_{\mathrm{w},k}\left(\theta'_{\mathrm{e},k}-\theta_\mathrm{o}\right)\right]+\sum_{l=1}^{p}\left[U_{\mathrm{i},l}A_{\mathrm{i},l}(\theta_\mathrm{i}-\theta_\mathrm{o})\right]$$
$$+\underbrace{0.34\,nV(\theta_\mathrm{e}-\theta_\mathrm{i})}_{(4)}+\underbrace{\sum_{k=1}^{m}\left(g_{\mathrm{w},k}f_{\mathrm{w},k}r_{\mathrm{w},k}E_{\mathrm{sun,w},k}\right)}_{(5)}+\overline{\Phi}_\mathrm{intern}=0 \tag{1.135}$$

The suffix e stands for all opaque envelope assemblies, the suffix w for the windows, and the suffix i for the inside partitions. $\theta^{*}_{\mathrm{e},j}$ is the sol-air temperature for envelope assembly j and $\theta'_{\mathrm{e},k}$ the specific sol-air temperature for windows:

$$\theta'_\mathrm{e}=\theta_\mathrm{e}-\frac{120\,e_\mathrm{L}F_\mathrm{w,sk}\left(1-f_\mathrm{c}\right)}{h_\mathrm{e}}$$

wherein θ_e is the outside air temperature, e_L the long-wave emissivity of the glass, $F_\mathrm{w,sk}$ the view factor between window and sky, f_c the cloudiness factor and h_e the outside surface film coefficient. Term (5) in the balance equation gives the solar gains across the windows. θ_i in the ventilation term (4) is the air temperature in the space, assumed equal to the operative temperature θ_o. The θ_l's are the operative temperatures in the neighbouring spaces, the A's the surface areas and the U's the clear wall thermal admittances. V is the air volume in the space, n the ventilation rate (ach), g the solar transmittance of the windows including their shading devices, E_sun the solar irradiation on the windows, f the ratio between glass and total area per window, and r a shadow factor. The product $g_{\mathrm{w},k}f_{\mathrm{w},k}r_{\mathrm{w},k}E_{\mathrm{sun,w},k}$ gives the average solar and Φ_intern the average internal gains over the base period – all in SI units!

The harmonic reponses look like:

$$\sum_{j=1}^{n}\Phi^{n}_{\mathrm{e},j}+\sum_{k=1}^{m}\Phi^{n}_{\mathrm{w},k}+\sum_{l=1}^{p}\Phi^{n}_{\mathrm{i},l}+\Phi^{n}_\mathrm{vent}+\sum_{k=1}^{m}\Phi^{n}_{\mathrm{sun,w},k}+\Phi^{n}_\mathrm{intern}=\left(\rho_\mathrm{a}c_\mathrm{a}+\frac{c_\mathrm{f}M_\mathrm{f}}{V}\right)V_j\frac{\mathrm{d}\theta^{n}_\mathrm{o}}{\mathrm{d}t} \tag{1.136}$$

with $\Phi^{n}_{\mathrm{e},j}$ the nth harmonic of the heat flow across the opaque envelope assemblies, $\Phi^{n}_{\mathrm{i},k}$ the nth harmonic of the heat flow across the inside partitions, $\Phi^{n}_{\mathrm{w},l}$ the nth harmonic of the heat flow across the windows, Φ^{n}_vent the nth harmonic of the enthalpy flow by ventilation, $\Phi^{n}_{\mathrm{sun,w},k}$ the nth harmonic of the solar gains, Φ^{n}_intern the nth harmonic of the internal gains, θ^{n}_o the nth harmonic of the operative temperature, c_f specific heat capacity and M_f the weight of all furniture and furnishings. The operative temperature and the heat flows can now be written as:

Operative (and air) temperature	$\theta_{\mathrm{o}}^{n} = \alpha_{\mathrm{o}}^{n} \exp(2in\pi t/T)$
Transmission	$\Phi^{n} = \hat{\Phi}^{n} \exp(2in\pi t/T)$
Ventilation	$\Phi_{\mathrm{vent}}^{n} = \hat{\Phi}_{\mathrm{vent}}^{n} \exp(2in\pi t/T)$
Solar gains	$\Phi_{\mathrm{sun}}^{n} = \hat{\Phi}_{\mathrm{sun}}^{n} \exp(2in\pi t/T)$
Internal gains	$\Phi_{\mathrm{internal}}^{n} = \hat{\Phi}_{\mathrm{internal}}^{n} \exp(2in\pi t/T)$

In these formulae, α_{o}^{n} is the complex operative temperature, $\hat{\Phi}_{\mathrm{x}}^{n}$ the complex heat flow, T the base period, n the order of the harmonic and i the imaginary unit. Entering these expressions in the harmonic heat balance gives:

$$\sum_{j=1}^{n} \hat{\Phi}_{\mathrm{e},j}^{n} + \sum_{k=1}^{m} \hat{\Phi}_{\mathrm{w},k}^{n} + \sum_{l=1}^{p} \hat{\Phi}_{\mathrm{i},l}^{n} + \hat{\Phi}_{\mathrm{V},j}^{n} + \sum_{k=1}^{m} \hat{\Phi}_{\mathrm{sun,w},k}^{n} + \hat{\Phi}_{\mathrm{intern}}^{n} = i(\omega_{n} \rho_{\mathrm{a}} c V)\alpha_{\mathrm{o}}^{n}$$

with ω_n the pulsation of the nth harmonic and c the equivalent specific heat capacity in the space, often set as five times the specific heat capacity of air:

$$c = c_{\mathrm{a}} + c_{\mathrm{f}}\, M_{\mathrm{f}}/(\rho_{\mathrm{a}} V) \approx 5c_{\mathrm{a}} \approx 5000 \tag{1.137}$$

If necessary, a more accurate value can be calculated, using the weight and specific heat capacity of the materials comprising the furniture and furnishings.

Applying the definitions of temperature damping, dynamic thermal resistance and admittance allows the rewriting of the separate complex heat flows. To keep it simple, the results are given for the first harmonic only. Higher harmonics give identical expressions, but with the transient properties, complex temperatures and complex heat flow rates for the harmonic considered. Assuming that heat goes from outside to inside, the heat flow across the opaque envelope assemblies can be written as:

$$\hat{\Phi}_{\mathrm{e},j}^{n} = \alpha'_{\mathrm{e},j} A_{\mathrm{e},j} = \left(\frac{1}{D_{\mathrm{q,e},j}} \alpha_{\mathrm{e},j}^{*} - \frac{D_{\theta,\mathrm{e},j}}{D_{\mathrm{q,e},j}} \alpha_{\mathrm{o}} \right) A_{\mathrm{e},j} = \left(\frac{1}{D_{\mathrm{q,e},j}} \alpha_{\mathrm{e},j}^{*} - Ad_{\mathrm{e},j} \alpha_{\mathrm{o}} \right) A$$

For windows the thermal transmittance remains the intervening property, giving as heat flow:

$$\hat{\Phi}_{\mathrm{w},k}^{n} = \alpha'_{\mathrm{w},k} A_{\mathrm{i},l} = \left[U_{\mathrm{w},k} \left(\alpha'_{\mathrm{e},k} - \alpha_{\mathrm{o}} \right) \right] A_{\mathrm{w},k}$$

The heat flows across the opaque partitions with neighbouring spaces are written as:

$$\hat{\Phi}_{\mathrm{i},l}^{n} = \alpha'_{\mathrm{i},l} A_{\mathrm{i},l} = \left(\alpha_{l}/D_{\mathrm{q,i},l} - Ad_{\mathrm{i},l} \alpha_{\mathrm{o}} \right) A_{\mathrm{i},l}$$

The constant ventilation rate gives:

$$\Phi_{\mathrm{vent}} = 0.34\, nV(\alpha_{\mathrm{e}} - \alpha_{\mathrm{o}})$$

If, besides the solar irradiation, the solar transmittance of the window with shading is also variable, the complex component of the solar gains equals:

$$\hat{\Phi}_{\text{sun,w},k} = f_{\text{w},k} A_{\text{w},k} \left(\alpha'_{\text{sun,w},k}\right)$$

with:

$$\alpha'_{\text{sun,w},k} = \text{Harm}\left(g_{\text{w},k} f_{\text{w},k} r_{\text{w},k} q_{\text{sun,w},k}\right)$$

wherein $q_{\text{sun,w},k}$ is the flux touching the outside face of the shading. Harm(. . .) indicates that the product between brackets forms a Fourier series. The complex components of the internal gains finally follow from a Fourier analysis:

$$\hat{\Phi}^{n}_{\text{intern}} = \text{Harm}(\Phi_{\text{intern}})$$

Transposing all these equations into the balance and solving for the complex operative temperature gives:

$$\alpha_{\text{o}} = \frac{\begin{array}{c}\sum\limits_{j=1}^{n} \left(\frac{A_{\text{e},j}}{D_{\text{q,e},j}} \alpha^{*}_{\text{e},j}\right) + \sum\limits_{k=1}^{m} \left(U_{\text{w},k} A_{\text{w},k} \alpha'_{\text{e},k}\right) + \sum\limits_{l=1}^{p} \left(\frac{A_{\text{i},l}}{D_{\text{q,i},l}} \alpha_{l}\right) + 0.34 n V \alpha_{\text{e}} \\ + \sum\limits_{k=1}^{m} f_{\text{w},k} A_{\text{w},k} \, \text{Harm}\left(g_{\text{w},k} r_{\text{w},k} q_{\text{sun,w},k}\right) + \text{Harm}\left(\hat{\Phi}_{\text{intern}}\right)\end{array}}{\sum\limits_{j=1}^{n} \left(A_{\text{e},j} A d_{\text{e},j}\right) + \sum\limits_{k=1}^{m} \left(U_{\text{w},k} A_{\text{w},k}\right) + \sum\limits_{l=1}^{p} \left(A_{\text{i},l} A d_{\text{i},l}\right) + 0.34 n V + i(6000 \omega V)}$$

The solution presumes a transposition of complex to real numbers.

If the sol-air and specific sol-air temperatures for glazing are assumed equal to the outdoor temperature ($\theta'_{\text{e}} = \theta^{*}_{\text{e}} = \theta_{\text{e}}$ and $\alpha'_{\text{e}} = \alpha^{*}_{\text{e}} = \alpha_{\text{e}}$), which means neither solar radiation nor under-cooling, then, for a ventilation rate of zero and all neighbouring spaces at the same operative temperature as the one considered, that formula simplifies to:

$$\alpha_{\text{o}} = \left\{ \frac{\sum\limits_{j=1}^{n} \left(\frac{A_{\text{e},j}}{D_{\text{q,e},j}}\right) + \sum\limits_{k=1}^{m} \left(U_{\text{w},k} A_{\text{w},k}\right)}{\sum\limits_{j=1}^{n} \left(A_{\text{e},j} A d_{\text{e},j}\right) + \sum\limits_{k=1}^{m} \left(U_{\text{w},k} A_{\text{w},k}\right) + \sum\limits_{l=1}^{q} \left[A_{\text{i},l}\left(A d_{\text{i},l} - \frac{1}{D_{\text{i},l}}\right)\right] + i(6000\, \omega V)} \right\} \alpha_{\text{e}} \tag{1.138}$$

The term between the large brackets contains only construction-related characteristics: surface areas and the inverse of the dynamic thermal resistances and the admittances of all opaque envelope parts fixing the thermal inertia and storage capacity, surface area and thermal transmittance of the windows, for all inside partitions the surface area and thermal storage capacity, and this last also for the air, the furniture and furnishings. The inverse stands for the ratio between the complex outdoor air and complex indoor operative temperatures:

$$D_{\theta,\text{space}} = \left\{ \frac{\sum_{j=1}^{n} \left(A_{\text{e},j} Ad_{\text{e},j}\right) + \sum_{k=1}^{m} \left(U_{\text{w},k} A_{\text{w},k}\right) + \sum_{l=1}^{q} \left[A_{\text{i},l}\left(Ad_{\text{i},l} - \frac{1}{D_{\text{i},l}}\right)\right] + i(6000\,\omega V)}{\sum_{j=1}^{n} \left(\frac{A_{\text{e},j}}{D_{\text{q,e},j}}\right) + \sum_{k=1}^{m} \left(U_{\text{w},k} A_{\text{w},k}\right)} \right\} \tag{1.139}$$

That inverse is called the room damping for the harmonic considered, and reflects how well a space dampens the temperature swings outdoors. The first harmonic usually suffices to classify a building space as dampening well, or dampening poorly.

1.6.6.3 Thermal bridges

For the consideration of thermal bridges, see Section 1.2.4 on transient conduction. Combine what is advanced there with the surface resistance approach explained above, under steady-state thermal bridges. As a reminder, surface resistances lack capacitance.

1.7 Problems and solutions

Problem 1.1

Calculate the thermal transmittance of an outside wall, inside to outside, assembled as follows ($h_\text{i} = 7.7\ \text{W/(m}^2\text{.K)}$, $h_\text{e} = 25\ \text{W/(m}^2\text{.K)}$):

Layer	Thickness, cm	Thermal conductivity, W/(m.K)	Thermal resistance, m^2.K/W
Plaster	1	0.3	
Inside leaf	14	0.5	
Cavity fill	8	0.04	
Unvented air cavity	4		0.17
Brick veneer	9	0.9	

Solution

All quantities must be expressed in SI units. So, metres (m), not centimetres (cm):

$$\begin{aligned} U_\text{o} &= \frac{1}{1/h_\text{i} + \sum R_j + 1/h_\text{e}} \\ &= \frac{1}{1/8 + 0.01/0.3 + 0.14/0.5 + 0.08/0.04 + 0.17 + 0.09/0.9 + 1/25} \\ &= 0.36\ \text{W/(m}^2\text{.K)} \end{aligned}$$

Never give more than two digits.

Problem 1.2

Calculate the thermal transmittance of a low-slope roof, inside to outside, assembled as follows ($h_i = 10$ W/(m^2.K), $h_e = 25$ W/(m^2.K)):

Layer	Thickness, cm	Thermal conductivity, W/(m.K)
Plaster	1	0.3
Concrete floor	14	2.5
Screed	10	0.6
Vapour barrier	1	0.2
Thermal insulation	12	0.028
Membrane	1	0.2

Solution

$U_o = 0.21$ W/(m^2.K).

Problem 1.3

Calculate the clear wall thermal transmittance of a timber frame outer wall, inside to outside, assembled as follows ($h_i = 7.7$ W/(m^2.K), $h_e = 25$ W/(m^2.K), studs not considered):

Layer	Thickness (d) cm	Thermal conductivity (λ), W/(m.K)	Thermal resistance (R), m^2.K/W
Gypsum board	1.2	0.2	
Air space	2		0.17
Airflow retarder	0.02	0.2	
Thermal insulation	20	0.04	
Outside sheathing	2	0.14	
Unvented air cavity	2		0.17
Brick veneer	9	0.9	

Solution

$U_o = 0.17$ W/(m^2.K).

Problem 1.4

Calculate the sol-air temperature for a horizontal surface subjected to a solar irradiation of 750 W/m^2. The outdoor air temperature is 30 °C, the outside surface film coefficient 12 W/(m^2.K). The surface has a short-wave absorptivity of 0.9 and a long-wave emissivity of 0.8. The long-wave losses to the clear sky reach 100 W/m^2 (low-slope roof).

Repeat the calculation for a daily mean outdoor air temperature of 24 °C, a daily mean solar irradiation of 169 W/m^2 and a daily mean long-wave loss to the clear sky of 50 W/m^2. This is representative of a south-oriented vertical wall during a hot summer's day in a temperate climate. Redo the exercise for a cold winter's day with a daily mean outdoor air temperature of −15 °C, a daily mean solar irradiation of 109 W/m^2 and a daily mean long-wave loss to the clear sky of 50 W/m^2. The short-wave absorptivity and long-wave emissivity of the outside wall face are 0.5 and 0.8, respectively, while the outside surface film coefficient reaches 16 W/(m^2.K) and the cloudiness reaches 0.8.

Solution

The equivalent temperature in the first situation is:

$$\theta_\mathrm{e}^* = \theta_\mathrm{e}^* + \frac{a_\mathrm{K} E_\mathrm{S} - e_\mathrm{L} q_\mathrm{L}}{h_\mathrm{e}} = 30 + \frac{0.9 \times 750 - 0.8 \times 100}{12} = 79.6°\mathrm{C}$$

which is high. The mean equivalent temperature for the south-oriented wall during the hot summer's day touches:

$$\theta_\mathrm{e}^* = \theta_\mathrm{e}^* + \frac{a_\mathrm{K} E_\mathrm{S} - e_\mathrm{L} q_\mathrm{L}}{h_\mathrm{e}} = 24 + \frac{0.5 \times 169 - 0.8 \times 50}{16} = 26.8°\mathrm{C}$$

During the cold winter's day, one has:

$$\theta_\mathrm{e}^* = \theta_\mathrm{e}^* + \frac{a_\mathrm{K} E_\mathrm{S} - e_\mathrm{L} q_\mathrm{L}}{h_\mathrm{e}} = -15 + \frac{0.5 \times 109 - 0.8 \times 50}{16} = -13.6°\mathrm{C}$$

Problem 1.5

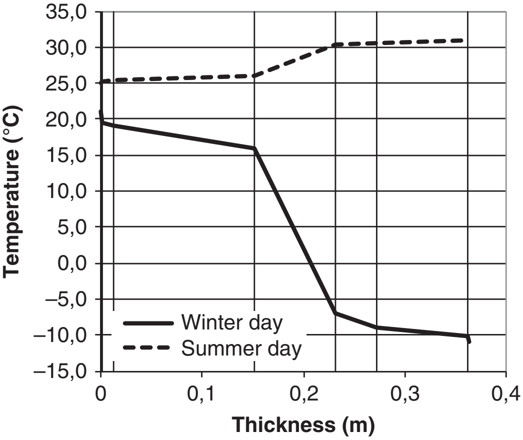
Return to Problem 1.1. Calculate the highest and lowest daily mean temperatures for all interfaces, knowing that the equivalent outdoor temperature has the same value as in the repeat part of Problem 1.4. The operative temperature indoors is 21 °C in winter and 25 °C in summer. The surface film coefficient outside is 16 W/(m^2.K), and the surface film resistance inside 0.13 m^2.K/W. Draw the result.

Solution

The temperatures are given by $\theta_j = \theta_\mathrm{i} - \left(\theta_\mathrm{i} - \theta_\mathrm{e}^*\right) \sum_{i=1}^{j} R/R_\mathrm{a}$. As a table and a figure:

Layer	ΣR, m^2.K/W	Temp, cold winter's day, °C	Temp, warm summer's day, °C
	0	21.0	25.0
	0.13	19.5	25.3
1	0.16	19.1	25.4
2	0.44	15.9	26.0
3	2.44	−7.1	30.3

4	2.61	−9.0	30.6
5	2.71	−10.2	30.9
	2.78	−10.9	31.0



The insulation backs the temperature difference. It is as if the wall splits into a part leaning to the indoors and a part leaning to the outdoors which experiences the greatest temperature change.

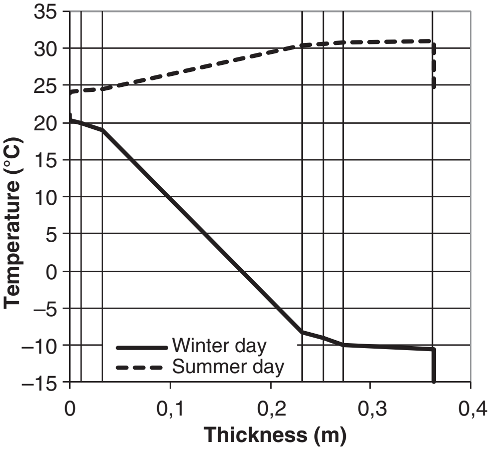
Problem 1.6

Now repeat Problem 1.5 for the timber frame wall of Problem 1.3. The outdoor sol-air and air temperatures in winter and summer, the operative temperatures indoors in winter and summer and the inside and outside surface film coefficients are as given above. Draw the result.

Solution

As a table and a figure:

Interface	ΣR_j, m^2.K/W	Temp, cold winter's day, °C	Temp, warm summer's day, °C
$1/h_i$	0	21.0	24.0
Gypsum board	0.13	20.3	24.2
Air space	0.19	20.0	24.2
Airflow retarder	0.36	19.0	24.4
Thermal insulation	0.36	19.0	24.4
Outside sheathing	5.36	−8.3	30.4
Air cavity	5.50	−9.1	30.6
Brick veneer	5.67	−10.0	30.8
$1/h_e$	5.77	−10.6	30.9



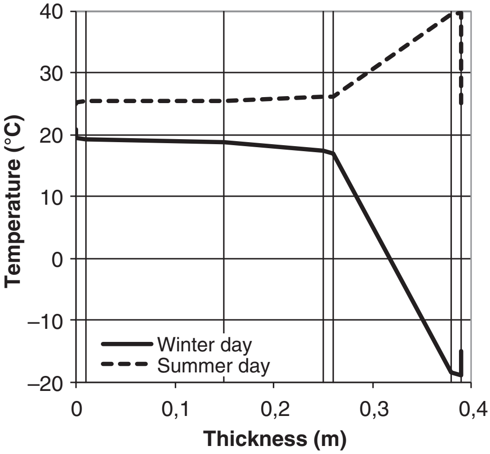
Problem 1.7

Take the low-sloped roof of Problem 1.2. Calculate the highest and lowest daily mean temperatures for all interfaces, knowing that the daily mean outdoor equivalent temperature in summer reaches 40 °C for a daily mean air temperature of 24 °C, while in winter these values are −19.5 °C and −15 °C respectively. The average surface film coefficient outside during windless days is 12 W/(m^2.K). Inside, the operative temperature is 21 °C in winter and 25 °C in summer. The inside surface film coefficient equals 6 W/(m^2.K) in summer and 10 W/(m^2.K) in winter. Draw the result.

Solution

As a table and a figure:

Interface	Winter		Summer	
	ΣR_j, m^2.K/W	Temp, °C	ΣR_j, m^2.K/W	Temp, °C
$1/h_i$	0	21.0	0	25.0
Render	0.17	19.6	0.10	25.3
Concrete floor	0.20	19.3	0.13	25.4
Screed	0.26	18.9	0.19	25.6
Vapour barrier	0.42	17.5	0.36	26.1
Thermal insulation	0.47	17.1	0.41	26.3
Membrane	4.76	−18.4	4.69	39.6
$1/h_e$	4.81	−18.8	4.74	39.7



Problem 1.8

A manufacturer introduces a new sandwich panel with, as a section, inside to outside:

Layer	Thickness (d), cm	Thermal conductivity (λ), W/(m.K)	Thermal resistance (R), m^2.K/W
Aluminium	0.2	230	
VIP (vacuum insulation)	2	0.006	
Air cavity	2		0.15
Glass pane	1	Assume ∞	

The panel fills a curtain wall. Assume temperatures of 35 °C outdoors and 24 °C indoors. Solar irradiation on the glass pane reaches 500 W/m^2. No long-wave radiation needs to be considered. The surface film coefficient outdoors is 15 W/(m^2.K), and indoors is 7.7 W/(m^2.K). Short-wave radiant properties of the glass are: $a_S = 0.05$, $r_S = 0.20$, $\tau_S = 0.75$. The cavity face of the VIP has a short-wave absorptivity of 1. What will be the temperature in the glass pane? How large will the heat flux be entering the building across the panel?

Solution

The problem is solved by writing two heat balances: one for the glass and one for the VIP cavity face:

$$\text{Glass (temperature } \theta_1\text{): } h_e(\theta_e - \theta_1) + a_S E_S + \frac{\theta_{s2} - \theta_1}{R_{cav}} = 0$$

$$\text{VIP (temperature } \theta_{s2}\text{): } \frac{\theta_1 - \theta_{s2}}{R_{cav}} + \tau_S E_S + \frac{\theta_i - \theta_{s2}}{R_{VIP} + R_{alu} + 1/h_i} = 0$$

or:

$$\begin{cases} -(15+1/0.15)\theta_1 + \dfrac{\theta_{s2}}{0.15} = -0.05\times 500 - 15\times 35 \\ \dfrac{\theta_1}{0.15} - \theta_{s2}\left(\dfrac{1}{0.15} + \dfrac{1}{0.02/0.005 + 0.002/230 + 1/7.7}\right) \\ \qquad = -0.75\times 500 - \dfrac{1}{0.02/0.005+0.002/230+1/7.7}\times 24 \end{cases}$$

Solving this system of two equations gives $\theta_1 = 60.2$ °C, $\theta_{s2} = 113.2$ °C. The heat flow rate to the inside is 21.6 W/m^2. The high temperatures underline that the panel acts as a solar collector. The heat flux to the inside equals that transferred in the absence of solar irradiation by an assembly with thermal transmittance of 1.96 W/(m^2.K), whereas the clear wall thermal transmittance of the manufactured panel is only 0.23 W/(m^2.K). What measures could lower the temperatures within and heat flux across the panel?

Problem 1.9

Solve Problem 1.8 for the case when heat-absorbing glass, $a_S = 0.3$, $r_S = 0.19$, $\tau_S = 0.51$, is used, and the short-wave absorptivity and reflectivity of the VIP's cavity face is 0.5.

Solution

The temperature of the glass is 52.8 °C, and the temperature at the cavity side of the VIP, 70.2 °C. The heat flux to the inside equals 11.2 W/m^2, which corresponds to a *U* value of 1.02 W/(m^2.K). The real *U* value remains 0.23 W/(m^2.K)

Problem 1.10

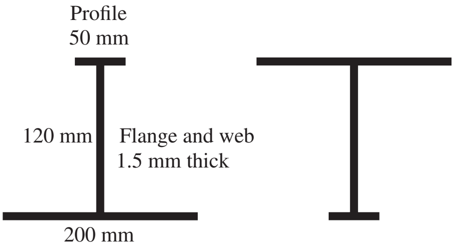
The roof of a mountain chalet is covered with 40 cm of snow ($\lambda = 0.07$ W/(m.K), $a_S = 0.15$). The outdoor temperature is −15 °C, while indoors it is 22 °C. Solar irradiation reaches 600 W/m^2. The surface film coefficients are 15 W/(m^2.K) outside and 10 W/(m^2.K) inside. What insulation thickness is needed to stop the snow from melting in contact with the membrane, for an insulation material with apparent thermal conductivity of 0.023 W/(m.K)? What heat flux will be noted across the roof? The thermal resistance face-to-face of the roof without insulation is 0.5 m^2.K/W.

Solution

The thickness needed is 21 cm. The heat flux across the roof equals 2.24 W/m^2.

Problem 1.11

An intensely ventilated attic receives an insulated ceiling, composed of metal girders, mounted 60 cm centre-to-centre with a 120 mm thick thermal insulation in between. The girder section is given in the figure below. Suppose the insulation has a thermal conductivity of 0 W/(m.K), while for the metal the value is ∞ W/(m.K). The surface film coefficients are 25 W/(m^2.K) at the attic side and 6 W/(m^2.K) inside. The attic temperature is −10 °C, and the temperature indoors 20 °C. Does the heat loss differ between the profile mounted with the broader flange indoors (a) or vice versa (b)? What is the metal temperature in both cases? Calculate the *U* value of the ceiling.



Solution

The heat balance for the profile in case (a) is: $0.2 \times 6 \times (20 - \theta_x) + 0.05 \times 25 \times (-10 - \theta_x) = 0$. In case (b) it is: $0.05 \times 6 \times (20 - \theta_x) + 0.2 \times 25 \times (-10 - \theta_x) = 0$. So yes, the heat losses do differ. In case (a) the metal temperature is 4.7 °C, while in case (b) it is −8.3 °C. For 1/0.6 girders per metre run, in case (a), $U = 1.02$ W/(m^2.K), while in case (b) it is 0.47 W/(m^2.K)

Problem 1.12

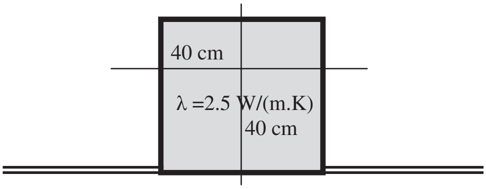
Solve Problem 1.11 for a metal profile with flanges of 100 mm each.

Solution

The temperature of the steel profiles is 4.2 °C, and the U value of the ceiling is 0.53 W/(m^2.K).

Problem 1.13

A reinforced concrete column with sides of 0.4 m is positioned between two glass panels in such a way that the glass lines up with the column's inside face. The glass is considered a surface with thickness zero. The temperature indoors is 21 °C, and the temperature outdoors 0 °C. The inside surface film coefficient equals 8 W/(m^2.K), and the outside surface film coefficient 25 W/(m^2.K). Calculate the temperature field in and the heat loss across the column.



Solution

Assume the column reacts as a flat wall. The U value then is:

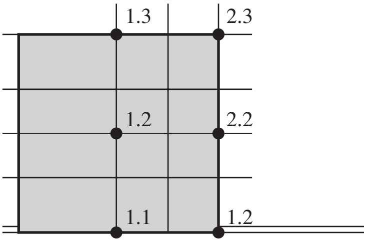
$$\frac{1}{1/25 + 0.4/2.5 + 1/8} = 3.1\ \mathrm{W/(m^2K)}$$

The temperature at the inside surface thus equals $21 - 3.1 \times (21 - 0)/8 = 12.9$ °C, giving a temperature factor of 0.65. The heat loss becomes $3.1 \times 0.4 \times 21 = 25.8$ W/m. Or, the thermal transmittance and temperature factor are close to those for double glass. In the column's centre, the temperature is 7.8 °C.

A first upgrade consists of applying a very simple CVM grid with the centre of the column as the calculation point (point 1, 2). The heat balance there is:

$$\frac{0.2(21-\theta_x)}{1/8+0.2/2.5}+3\frac{0.2(0-\theta_x)}{1/25+0.2/2.5}=0$$

giving as the central temperature 3.4 °C, and as the inside surface temperature, 10.3 °C, which gives a temperature factor of 0.49, a value 24.6% lower than just calculated. The heat loss now is 34.3 W/m – 33.7% higher than with the flat wall assumption. In a second upgrade, the grid over half the column is refined to 6 calculation points, of which 5 lie on the perimeter.



The heat balances are:

Point 1,1

$$8\times 0.1\times\left(21-\theta_{1,1}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{1,2}-\theta_{1,1}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{2,1}-\theta_{1,1}\right)=0$$

Point 2,1

$$8\times 0.1\times\left(21-\theta_{2,1}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{1,1}-\theta_{2,1}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{2,2}-\theta_{2,1}\right)$$
$$+25\times 0.1\times\left(0-\theta_{2,1}\right)=0$$

Point 1,2

$$\frac{2.5\times 0.1}{0.2}\left(\theta_{1,1}-\theta_{1,2}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{1,3}-\theta_{1,2}\right)+\frac{2.5\times 0.2}{0.2}\left(\theta_{2,2}-\theta_{1,2}\right)=0$$

Point 2,2

$$\frac{2.5\times 0.1}{0.2}\left(\theta_{2,1}-\theta_{2,2}\right)+\frac{2.5\times 0.2}{0.2}\left(\theta_{1,2}-\theta_{2,2}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{2,3}-\theta_{2,2}\right)$$
$$+25\times 0.2\times\left(0-\theta_{2,2}\right)=0$$

Point 1,3

$$\frac{2.5\times 0.1}{0.2}\left(\theta_{1,2}-\theta_{1,3}\right)+25\times 0.1\times\left(0-\theta_{1,3}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{2,3}-\theta_{1,3}\right)=0$$

Point 2,3

$$2\times 25\times 0.1\times\left(0-\theta_{2,3}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{2,2}-\theta_{2,3}\right)+\frac{2.5\times 0.1}{0.2}\left(\theta_{1,3}-\theta_{2,3}\right)=0$$

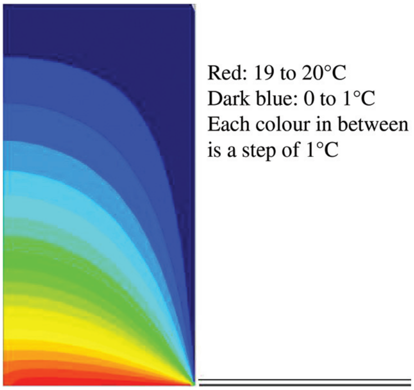
Solving this system gives as temperatures in the column:

0.4 0.9 0.4
1.4 2.9 1.4
4.9 8.1 4.9

The lowest temperature factor inside now sits at the corners, 0.25 – as poor as single glass. The heat loss equals:

$$\Phi = 2 \times 0.1 \times 8 \times (21 - 4.9) + 0.2 \times 8 \times (21 - 8.1) = 46.4\,\mathrm{W/m}$$

that is 80% higher than given by the flat wall assumption. The last upgrade comes by including more control volumes, and using software for two-dimensional heat transport. The figure below shows the nearly correct answer in terms of temperatures:



Red: 19 to 20°C
Dark blue: 0 to 1°C
Each colour in between
is a step of 1°C

Problem 1.14

Aerated concrete is chosen as the envelope material. The manufacturer promises a very good transient thermal response, which is formulated in terms of a much higher effective thermal resistance than calculated steady state. Is this true? The material properties are:

Situation	Density, kg/m^3	Thermal conductivity (λ), W/(m.K)	Specific heat capacity, J/(kg.K)
Just applied (humid)	450	0.30	2700
After some years (air-dry)	450	0.13	1120

The wall thicknesses to consider are 10, 20 and 30 cm. The outside surface film coefficient is 25 W/(m^2.K), and the inside surface film coefficient 8 W/(m^2.K).

Solution

A way to evaluate the claim is through calculating the harmonic properties. A high dynamic thermal resistance seems to confirm it, but a low admittance indicates that this will not suffice to stabilize the indoor climate in the case of significant solar and internal gains.

To show how the harmonic calculation works, temperature damping, the dynamic thermal resistance and the admittance are calculated for the initially humid aerated concrete wall with $d = 10\,\text{cm}$ and $\lambda = 0.3\,\text{W/(m.K)}$.

Thermal diffusivity: $a = \frac{\lambda}{\rho c} = 2.469 \times 10^{-7}\ \text{m}^2/\text{s}$

X_n value: $X_n = d\sqrt{\frac{n\pi}{aT}} = 0.1\sqrt{\frac{3.1418}{2.469\times10^{-7}\times3600\times24}} = 1.2135$

Functions $G_{n1}(X_n)$ to $G_{n6}(X_n)$:

G_{n1}	0.640
G_{n2}	1.437
G_{n3}	0.928
G_{n4}	0.486
G_{n5}	−1.431
G_{n6}	2.733

Layer matrices:

$$\textit{Inside surface}\begin{bmatrix} 1 & 0 & 0.125 & 0 \\ 0 & 1 & 0 & 0.125 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad |1|$$

$$\textit{Layer}\begin{bmatrix} 0.640429 & 1.437199 & 0.309307 & 0.161937 \\ -1.437199 & 0.640429 & -0.161937 & 0.309307 \\ -4.29249 & 8.198851 & 0.640429 & 1.437199 \\ -8.198851 & -4.29249 & -1.437199 & 0.640429 \end{bmatrix} \qquad |2|$$

$$\textit{Outside surface}\begin{bmatrix} 1 & 0 & 0.04 & 0 \\ 0 & 1 & 0 & 0.04 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad |3|$$

Matrix multiplication:

$$|2| \times |1| = \begin{bmatrix} 0.640429 & 1.437199 & 0.38936 & 0.341587 \\ -1.437199 & 0.640429 & -0.341587 & 0.38936 \\ -4.29249 & 8.198851 & 0.103868 & 2.462055 \\ -8.198851 & -4.29249 & -2.462055 & 0.103868 \end{bmatrix} \qquad |4|$$

$$|3| \times |4| = \begin{bmatrix} 0.46873 & 1.765153 & 0.3993515 & 0.440069 \\ -1.765153 & 0.46873 & -0.440069 & 0.3993515 \\ -4.29249 & 8.198851 & 0.103868 & 2.462055 \\ -8.198851 & -4.29249 & -2.462055 & 0.103868 \end{bmatrix}$$

Harmonic properties:

Temperature damping $$|D_\theta| = \sqrt{0.46873^2 + 1.765133^2} = 1.83$$

$$\phi_\theta = a\tan\left(\frac{1.765133}{0.46873}\right)\frac{12}{\pi} = 5\,\text{h}$$

Dynamic thermal resistance $$\left|D_q\right| = \sqrt{0.3993515^2 + 0.38936^2} = 0.59\ \text{m}^2\text{K/W}$$

$$\phi_q = a\tan\left(\frac{0.38936}{0.3993515}\right)\frac{12}{\pi} = 3.2\,\text{h}$$

Admittance $$|Ad| = \frac{|D_\theta|}{\left|D_q\right|} = 3.09\ \text{W/(m}^2\text{K)}$$

$$\phi_{Ad} = \phi_\theta - \phi_q = 1.8\,\text{h}$$

It is for the reader to calculate the other cases, using a spreadsheet or program. The results are:

	Case 1 $d = 10$ cm $\lambda = 0.3$ W/(m.K)	Case 2 $d = 20$ cm $\lambda = 0.3$ W/(m.K)	Case 3 $d = 30$ cm $\lambda = 0.3$ W/(m.K)	Case 4 $d = 10$ cm $\lambda = 0.13$ W/(m.K)	Case 5 $d = 20$ cm $\lambda = 0.13$ W/(m.K)	Case 6 $d = 30$ cm $\lambda = 0.13$ W/(m.K)
$D\theta$	1.83	6.56	23.1	1.63	5.72	19.1
ϕ_θ (h)	5 h 00′	9 h 48′	14 h 36′	4 h 33′	9 h 19′	13 h 57′
D_q (m^2.K/W)	0.59	1.93	6.56	1.02	3.16	10.4
ϕ_q (h)	3 h 18′	7 h 54′	12 h 42′	2 h 24′	6 h 53′	11 h 28′
Ad (W/(m^2.K))	3.09	3.39	3.53	1.60	1.81	1.84
ϕ_{Ad} (h)	1 h 48′	1 h 52′	1 h 57′	2 h 09′	2 h 26′	2 h 29′

So aerated concrete is clearly not the wonder promised. To get sufficient temperature damping air-dry (D_θ>15), a thickness beyond 20 cm is needed. The same holds for the dynamic thermal resistance if a value beyond 4 m^2.K/W is the target. The admittance is low, surely when the aerated concrete is airdry. Therefore the material does not function in the way the manufacturers claim.

Problem 1.15

Take a living room with surface 4 × 6.5 m and ceiling height 2.5 m. The room has two exterior walls, one of 4 × 2.5 m^2 and the other 6.5 × 2.5 m^2, completely glazed with gas-filled, low-e double glazing, U value 1.3 W/(m^2.K) for $h_\text{i} = 7.7$ W/(m^2.K) and $h_\text{e} = 25$ W/(m^2.K). Both partition walls and the ceiling have a thermal resistance of 0.505 m^2.K/W from the surface in the living room to the ambient in the neighbouring space. The living room has floor heating with the network of pipes covered by a screed having a thermal resistance of 0.1 m^2.K/W. Walls, floor and ceiling are grey bodies with emissivity 0.9. The glass has a grey body emissivity of 0.92. The ventilation rate in the room is 1 ach and the surface film coefficient for convection reaches 3.5 W/(m^2.K). Calculate the glass, wall and ceiling temperatures, knowing that the indoor and outdoor air temperatures are 21 °C and −8 °C respectively.

Solution

The room is considered as a system with six grey surfaces: the windows with surface A_1 and A_2 and temperatures T_{s1} and T_{s2}, the two inside walls with surface A_3 and A_4 and surface temperatures T_{s3} and T_{s4}, the ceiling with surface A_5 and surface temperature T_{s5} and, the floor with surface A_6 and surface temperature T_{s6}. The floor heating has a temperature T_{fl}. Seven heat balances so are needed, convective for the room and one per wall surface.

Room balance: $Q_v + \sum_{j=1}^{6} h_c A_j \left(21 - \theta_{sj}\right) = 0$

or, with $Q_v = \rho_a c_a V(\theta_e - \theta_i)$, $\theta_i = 21$ °C, $\theta_e = -8$ °C, $V = 65\,\text{m}^3$, $A_1 = A_3 = 10\,\text{m}^2$, $A_2 = A_4 = 16.25\,\text{m}^2$, $A_5 = A_6 = 26\,\text{m}^2$, $c_a = 1008$ J/(kg.K), ρ_a and $h_c = 3.5$ W/(m^2.K):

$$-633.36 + 35T_{s1} + 56.875T_{s2} + 35T_{s3} + 56.875T_{s4} + 91T_{s5} + 91T_{s6} - 7680.75 = 0$$

Surface balances: the radiant heat flow rate at each surface is written as $q_R = e_L(M' - M_b)/\rho_L$. Linearization of the black body emittance M_b in a temperature interval 10–25 °C gives:

$$M_b = 307.75 + 5.57\theta_s, \quad r^2 = 0.999$$

The radiosity M' equals:

$$M'_j = \frac{M_{bj}}{e_j} - \frac{\rho_j}{e_j}\sum_{i=2}^{6} F_{ji}\, M'_j$$

with F_{ji} the view factor between each surface and the other five:

	Surface 1	Surface 2	Surface 3	Surface 4	Surface 5	Surface 6
Surface 1	–	0.187	0.070	0.187	0.278	0.278
Surface 2	0.115	–	0.115	0.210	0.280	0.280
Surface 3	0.070	0.187	–	0.187	0.278	0.278
Surface 4	0.115	0.210	0.115	–	0.280	0.280
Surface 5	0.107	0.175	0.107	0.175	–	0.436
Surface 6	0.107	0.175	0.107	0.175	0.436	–

The black body emittance of each surface thus becomes:

s1 $M_{\mathrm{b1}} = \frac{1}{0.92} M'_{\mathrm{s1}} - \frac{0.08}{0.92}(0.187 M'_{\mathrm{s2}} + 0.07 M'_{\mathrm{s3}} + 0.187 M'_{\mathrm{s4}} + 0.278 M'_{\mathrm{s5}} + 0.278 M'_{\mathrm{s6}})$

s2 $M_{\mathrm{b2}} = \frac{1}{0.92} M'_{\mathrm{s2}} - \frac{0.08}{0.92}(0.115 M'_{\mathrm{s1}} + 0.115 M'_{\mathrm{s3}} + 0.21 M'_{\mathrm{s4}} + 0.28 M'_{\mathrm{s5}} + 0.28 M'_{\mathrm{s6}})$

s3 $M_{\mathrm{b3}} = \frac{1}{0.9} M'_{\mathrm{s3}} - \frac{0.1}{0.9}(0.07 M'_{\mathrm{s1}} + 0.187 M'_{\mathrm{s2}} + 0.187 M'_{\mathrm{s4}} + 0.278 M'_{\mathrm{s5}} + 0.278 M'_{\mathrm{s6}})$

s4 $M_{\mathrm{b4}} = \frac{1}{0.9} M'_{\mathrm{s4}} - \frac{0.1}{0.9}(0.115 M'_{\mathrm{s1}} + 0.210 M'_{\mathrm{s2}} + 0.115 M'_{\mathrm{s3}} + 0.28 M'_{\mathrm{s5}} + 0.28 M'_{\mathrm{s6}})$

s5 $M_{\mathrm{b5}} = \frac{1}{0.9} M'_{\mathrm{s5}} - \frac{0.1}{0.9}(0.107 M'_{\mathrm{s1}} + 0.175 M'_{\mathrm{s2}} + 0.107 M'_{\mathrm{s3}} + 0.175 M'_{\mathrm{s4}} + 0.436 M'_{\mathrm{s6}})$

s6 $M_{\mathrm{b6}} = \frac{1}{0.9} M'_{\mathrm{s6}} - \frac{0.1}{0.9}(0.107 M'_{\mathrm{s1}} + 0.175 M'_{\mathrm{s2}} + 0.107 M'_{\mathrm{s3}} + 0.175 M'_{\mathrm{s4}} + 0.436 M'_{\mathrm{s5}})$

Inverting the matrix of this system of six equations gives the radiosities of the six as functions of the black body emittances:

$$\text{Matrix:} \begin{bmatrix} 1.0870 & -0.0163 & -0.0061 & -0.0163 & -0.0242 & -0.0242 \\ -0.0100 & 1.0870 & -0.0100 & -0.0183 & -0.0243 & -0.0243 \\ -0.0078 & -0.0208 & 1.1111 & -0.0208 & -0.0309 & -0.0309 \\ -0.0128 & -0.0233 & -0.0128 & 1.1111 & -0.0311 & -0.0311 \\ -0.0119 & -0.0194 & -0.0119 & -0.0194 & 1.1111 & -0.0485 \\ -0.0119 & -0.0194 & -0.0119 & -0.0194 & -0.0485 & 1.1111 \end{bmatrix}$$

$$\text{Inverted:} \begin{bmatrix} H'_1 \\ H'_2 \\ H'_3 \\ H'_4 \\ H'_5 \\ H'_6 \end{bmatrix} = \begin{bmatrix} 0.9208 & 0.0150 & 0.0058 & 0.0146 & 0.0219 & 0.0219 \\ 0.0092 & 0.9214 & 0.0090 & 0.0162 & 0.0221 & 0.0221 \\ 0.0074 & 0.0187 & 0.9010 & 0.0182 & 0.0273 & 0.0273 \\ 0.0115 & 0.0207 & 0.0112 & 0.9017 & 0.0275 & 0.0275 \\ 0.0108 & 0.0176 & 0.0105 & 0.0172 & 0.9032 & 0.0408 \\ 0.0108 & 0.0176 & 0.0105 & 0.0172 & 0.0408 & 0.9032 \end{bmatrix}$$

$$x \begin{bmatrix} 307.75 + 5.57\theta_{\mathrm{s1}} \\ 307.75 + 5.57\theta_{\mathrm{s2}} \\ 307.75 + 5.57\theta_{\mathrm{s3}} \\ 307.75 + 5.57\theta_{\mathrm{s4}} \\ 307.75 + 5.57\theta_{\mathrm{s5}} \\ 307.75 + 5.57\theta_{\mathrm{s6}} \end{bmatrix}$$

Introducing this result into the radiant heat flow rate equation allows the elimination of the constant 307.75. The combined heat balance per surface, including radiation, convection and conduction, is now

$$q_{\mathrm{R}} + q_{\mathrm{C}} + q_{\mathrm{cond}} = 0$$

or:

$$-10.1367\theta_{s1} + 0.9599\theta_{s2} + 0.3726\theta_{s3} + 0.9352\theta_{s4} + 1.4023\theta_{s5} + 1.4023\theta_{s6} + 0\theta_{fl} = 60.985$$

$$0.5907\theta_{s1} - 10.0972\theta_{s2} + 0.5771\theta_{s3} + 1.0391\theta_{s4} + 1.4129\theta_{s5} + 1.4129\theta_{s6} + 0\theta_{fl} = 60.985$$

$$0.3726\theta_{s1} + 0.9378\theta_{s2} - 10.444\theta_{s3} + 0.9136\theta_{s4} + 1.3699\theta_{s5} + 1.3699\theta_{s6} + 0\theta_{fl} = 115.084$$

$$0.5755\theta_{s1} + 1.0391\theta_{s2} + 0.5622\theta_{s3} - 10.410\theta_{s4} + 1.3765\theta_{s5} + 1.3765\theta_{s6} + 0\theta_{fl} = 115.084$$

$$0.5393\theta_{s1} + 0.8831\theta_{s2} + 0.5269\theta_{s3} + 0.8603\theta_{s4} - 10.335\theta_{s5} + 2.0454\theta_{s6} + 0\theta_{fl} = 115.084$$

$$0.5393\theta_{s1} + 0.8831\theta_{s2} + 0.5269\theta_{s3} + 0.8603\theta_{s4} + 2.0454\theta_{s5} - 18.355\theta_{s6} + 10\theta_{fl} = -73.5$$

In these equations, the diagonal terms consist of:

$$\theta_{s1}, \theta_{s2} - \left[\frac{1}{(1/1.3 - 0.13)} + 3.5 + 0.9208\left(\frac{0.92}{0.08}\right)\right]$$

$$\theta_{s3}, \theta_{s4}, \theta_{s5} - \left[\frac{1}{0.505} + 3.5 + (0.901, 0.9017, 0.9032)\left(\frac{0.9}{0.1}\right)\right]$$

$$\theta_{s6} - \left[\frac{1}{0.1} + 3.5 + 0.9032\left(\frac{0.9}{0.1}\right)\right]$$

Solving this system of six surface and one room balance equation gives temperatures:

Window $4 \times 2.5\ \mathrm{m}^2$	$\theta_{s1} = 17.6\ °\mathrm{C}$
Window $6.5 \times 2.5\ \mathrm{m}^2$	$\theta_{s2} = 17.8\ °\mathrm{C}$
Wall $4 \times 2.5\ \mathrm{m}^2$	$\theta_{s3} = 21.9\ °\mathrm{C}$
Wall $6.5 \times 2.5\ \mathrm{m}^2$	$\theta_{s4} = 21.8\ °\mathrm{C}$
Ceiling	$\theta_{s5} = 22.3\ °\mathrm{C}$
Floor	$\theta_{s6} = 29.2\ °\mathrm{C}$
Floor heating	$\theta_{fl} = 36.1\ °\mathrm{C}$

Problem 1.16

Repeat Problem 1.15 assuming normal double glass with $U = 2.9\ \mathrm{W/(m^2.K)}$ for $h_i = 7.7\ \mathrm{W/(m^2.K)}$ and $h_e = 25\ \mathrm{W/(m^2.K)}$, while all other data remain the same.

Solution

With normal double glazing, the temperatures become:

Window $4 \times 2.5\ \mathrm{m}^2$	$\theta_{s1} = 11.8\ °\mathrm{C}$
Window $6.5 \times 2.5\ \mathrm{m}^2$	$\theta_{s2} = 12.1\ °\mathrm{C}$
Wall $4 \times 2.5\ \mathrm{m}^2$	$\theta_{s3} = 21.9\ °\mathrm{C}$
Wall $6.5 \times 2.5\ \mathrm{m}^2$	$\theta_{s4} = 21.7\ °\mathrm{C}$
Ceiling	$\theta_{s5} = 22.6\ °\mathrm{C}$
Floor	$\theta_{s6} = 34.7\ °\mathrm{C}$
Floor heating	$\theta_{fl} = 47.0\ °\mathrm{C}$

The floor is much warmer now than acceptable for feet comfort (28 °C), or, heat loss is too high to only install floor heating. The room also needs a radiator or a convector.

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