1 Introduction

Ante mare et terras et, quod tegit omnia,
caelum Unus erat toto naturae vultus in
orbe, Quem dixere Chaos, rudis
indigestaque moles Nec quicquam nisi
pondus iners congestaque eodem Non
bene iunctarum discordia semina rerum.
Ovid

It seems appropriate to begin a book which is entitled "Deterministic Chaos" with an explanation of both terms. According to the Encyclopaedia Britannica the word "chaos" is derived from the Greek "χαοσ" and originally meant the infinite empty space which existed before all things. The later Roman conception interpreted chaos as the original crude shapeless mass into which the Architect of the world introduces order and harmony. In modern usage which we will adopt here, chaos denotes a state of disorder and irregularity.

In the following, we shall consider physical systems whose time dependence is deterministic, i. e., there exists a prescription, either in terms of differential or difference equations, for calculating their future behavior from given initial conditions. One could assume naively that deterministic motion (which is, for example, generated by continuous differential equations) is rather regular and far from being chaotic because successive states evolve continuously from each other. But it was already discovered at the turn of the century by the mathematician H. Poincaré (1892) that certain mechanical systems, whose time evolution is governed by Hamilton's equations, could display chaotic motion. Unfortunately, this was considered by many physicists as a mere curiosity, and it took another 70 years until, in 1963, the meteorologist E. N. Lorenz found that even a simple set of three coupled, first-order, nonlinear differential equations can lead to completely chaotic trajectories. Lorenz's paper, the general importance of which is recognized today, was also not widely appreciated until many years after its publication. He discovered one of the first examples of deterministic chaos in dissipative systems.

In the following, deterministic chaos denotes the irregular or chaotic motion that is generated by nonlinear systems whose dynamical laws uniquely determine the time evolution of a state of the system from a knowledge of its previous history. In recent years – due to new theoretical results, the availability of high speed computers, and refined experimental techniques – it has become clear that this phenomenon is abundant in nature and has far-reaching consequences in many branches of science (see the long list in Table 1, which is far from complete).

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Table 1: Some nonlinear systems which display deterministic chaos. (For numerals, see "References" on page 259.)

Forced pendulum [1]
Fluids near the onset of turbulence [2]
Lasers [3]
Nonlinear optical devices [4]
Josephson junctions [5]
Chemical reactions [6]
Classical many-body systems (three-body problem) [7]
Particle accelerators [8]
Plasmas with interacting nonlinear waves [9]
Biological models for population dynamics [10]
Stimulated heart cells (see Plate IV at the beginning of the book) [11]

We note that nonlinearity is a necessary, but not a sufficient condition for the generation of chaotic motion. (Linear differential or difference equations can be solved by Fourier transformation and do not lead to chaos.) The observed chaotic behavior in time is neither due to external sources of noise (there are none in the Lorenz equations) nor to an infinite number of degrees of freedom (in Lorenz's system there are only three degrees of freedom) nor to the uncertainty associated with quantum mechanics (the systems considered are purely classical). The actual source of irregularity is the property of the nonlinear system of separating initially close trajectories exponentially fast in a bounded region of phase space (which is, e. g., three-dimensional for Lorenz's system).

It becomes therefore practically impossible to predict the long-time behavior of these systems, because in practice one can only fix their initial conditions with finite accuracy, and errors increase exponentially fast. If one tries to solve such a nonlinear system on a computer, the result depends for longer and longer times on more and more digits in the (irrational) numbers which represent the initial conditions. Since the digits in irrational numbers (the rational numbers are of measure zero along the real axis) are irregularly distributed, the trajectory becomes chaotic.

Lorenz called this sensitive dependence on the initial conditions the butterfly effect, because the outcome of his equations (which describe also, in a crude sense, the flow of air in the earth's atmosphere, i. e., the problem of weather forecasting) could be changed by a butterfly flapping wings. This also seems to be confirmed sometimes by daily experience.

The results described above immediately raise a number of fundamental questions:

- Can one predict (e. g., from the form of the corresponding differential equations) whether or not a given system will display deterministic chaos?
- Can one specify the notion of chaotic motion more mathematically and develop quantitative measures for it?
- What is the impact of these findings on different branches of physics?
- Does the existence of deterministic chaos imply the end of long-time predictability in physics for some nonlinear systems, or can one still learn something from a chaotic signal?

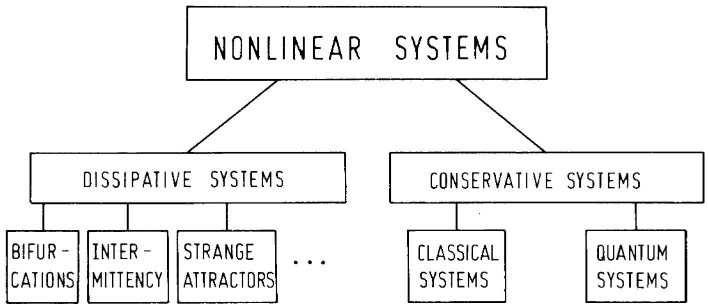

Figure 1: Classification of systems which display deterministic chaos. (We consider in the following only classical dissipative systems, i. e., no quantum systems with dissipation.)

The last question really goes to the fundaments of physics, namely the problem of predictability. The shock which was associated with the discovery of deterministic chaos has therefore been compared with that which spread when it was found that quantum mechanics only allows statistical predictions.

Those questions mentioned above, to which some answers already exist, will be discussed in the remainder of this book. It should be clear, however, that there are still many more unsolved than solved problems in this relatively new field.

The rest of the introduction takes the form of a short survey which summarizes the contents of this book. Figure 1 shows that one has to distinguish between deterministic chaos in dissipative systems (e. g., a forced pendulum with friction) and conservative systems (e. g., planetary motion which is governed by Hamilton's equations).

The first six chapters are devoted to dissipative systems. We begin with a review of some representative experiments in which deterministic chaos has been observed by different methods. As a next step, we explain the mechanism which leads to deterministic chaos for a simple model system and develop quantitative measures to characterize a chaotic signal. This allows us to distinguish different types of chaos, and we then show that, up to now, there are at least three routes or transitions in which nonlinear systems can become chaotic if an external control parameter is varied. Interestingly enough, all these routes can be realized experimentally, and they show a fascinating universal behavior which is reminiscent of the universality found in second-order equilibrium phase transitions. (Note that the transitions to chaos in dissipative systems only occur when the system is driven externally, i. e., is open.) In this context, universality means that there are basic properties of the system (such as critical exponents near the transition to chaos) that depend only on some global features of the system (for example, the dimensionality).

The most recent route to chaos has been found by Grossmann and Thomae (1977), Feigenbaum (1978), and Coullett and Tresser (1978). They considered a simple difference equation which, for example, has been used to describe the time dependence of populations in biology, and found that the population oscillated in time between stable values (fixed points) whose number doubles at distinct values of an external parameter. This continues until the number of fixed points becomes infinite at a finite parameter value, where the variation in time of the population becomes irregular. Feigenbaum has shown, and this was a major achievement, that these results are not restricted to this special model but are in fact universal and hold for a large variety of physical, chemical, and biological systems. This discovery has triggered an explosion of theoretical and experimental activity in the field. We will study this route in Chapter 4 and show that its universal properties can be calculated using the functional renormalization group method.

A second approach to chaos, the so-called intermittency route, has been discovered by Manneville and Pomeau (1979). Intermittency means that a signal which behaves regularly (or laminarly) in time becomes interrupted by statistically distributed periods of irregular motion (intermittent bursts). The average number of these bursts increases with the variation of an external control parameter until the motion becomes completely chaotic. It will be shown in Chapter 5 that this route also has universal features and provides a universal mechanism for $1/f$-noise in nonlinear systems.

Yet a third possibility was found by Ruelle and Takens (1971) and Newhouse (1978). In the seventies they suggested a transition to turbulent motion which was different from that proposed much earlier by Landau (1944, 1959). Landau considered turbulence in time as the limit of an infinite sequence of instabilities (Hopf bifurcations) each of which creates a new basic frequency. However, Ruelle, Takens, and Newhouse showed that after only two instabilities in the third step, the trajectory becomes attracted to a bounded region of phase space in which initially close trajectories separate exponentially, such that the motion becomes chaotic. These particular regions of phase space are called strange attractors. We will explain this concept in Chapter 6, where we will also discuss several methods of extracting information about the structure of the attractor from the measured chaotic time signal. The Ruelle–Takens–Newhouse route is (as are the previous two routes) well verified experimentally, and we will present some experimental data which show explicitly the appearance of strange attractors in Chapter 7.

To avoid the confusion which might arise by the use of the word turbulence, we note that what is meant here, is only turbulence in time. The results of Ruelle, Takens, and Newhouse also concern the onset of turbulence or chaotic motion in time. It is in fact one of the aims (but not yet the result) of the study of deterministic chaos in hydrodynamic systems, to understand the mechanisms for fully developed turbulence, which implies irregular behavior in time and space.

We now come to the second branch in Fig. 1, which denotes chaotic motion in conservative systems. Many textbooks give the incorrect impression that most systems in classical mechanics can be integrated. But as mentioned above, Poincaré (1892) was already aware that, e. g., the nonintegrable three-body problem of classical mechanics can lead to completely chaotic trajectories. About sixty years later, Kolmogorov (1954), Arnold (1963), and Moser (1967) proved, in what is now known as the KAM theorem, that the motion in the phase space of classical mechanics is neither completely regular nor completely irregular, but that the type of trajectory depends sensitively on the chosen initial conditions. Thus, stable regular classical motion is the exception, contrary to what is implied in many texts.

A study of the long-time behavior of conservative systems, which will be discussed in Chapter 8, is of some interest because it touches on such questions as: Is the solar system stable? How can one avoid irregular motion in particle accelerators? Is the self-generated deterministic chaos of some Hamiltonian systems strong enough to prove the ergodic hypothesis? (The ergodic hypothesis lies at the foundation of classical statistical mechanics and implies that the trajectory uniformly covers the energetically allowed region of classical phase space such that time averages can be replaced by the average over the corresponding phase space.)

In Chapter 8 we consider the behavior of quantum systems whose classical limit displays chaos. Such investigations are important, for example, for the problem of photodissociation, where a molecule is kicked by laser photons, and one wants to know how the incoming energy spreads over the quantum levels. (The corresponding classical system could show chaos because the molecular forces are highly nonlinear.) For several examples we show that the finite value of Planck's constant leads, together with the boundary conditions, to an almost-periodic behavior of the quantum system even if the corresponding classical system displays chaos. Although the difference between integrable and nonintegrable (chaotic) classical systems is still mirrored in some properties of their quantum counterparts (for example in the energy spectra), many problems in this field remain unsolved.

As already mentioned in the preface, the field of deterministic chaos continued to grow after the last edition of this book in 1989. Especially the concept of unstable periodic orbits has been rediscovered and developed further by Cvitanovich *et al.* (1990). Unstable periodic orbits are the building blocks of chaotic dynamics, and their importance was already known by Poincaré (1892) and Ruelle (1978).

Exploiting this concept, Ott, Grebogi and Yorke showed in 1990 that deterministic chaos can be controlled. As we will show in Chapter 10, unstable periodic orbits, which are contained in all chaotic systems, can be stabilized by small time-dependent changes of the control parameter of the system in such a way that the dynamical behavior becomes non-chaotic. Control of chaos is, because of its possible technical applications, an extremely active field where one would hope to make interesting new progress, especially for spatially coupled chaotic systems.

Synchronization phenomena are related to control problems as, in very simple cases, a desired state can be reached by external periodic modulation. But synchronization includes much richer phenomena, especially when nonlinear and chaotic dynamical systems are considered. Chapter 11 presents a brief introduction into this field.

The study of spatiotemporal chaotic motion is still in its infancy. In Chapter 12 we present a selection of phenomena and approaches which may turn out to be among the cornerstones on which a consistent and systematic exposition of this topic may be based. However, an ultimate answer is currently not available.