5 Transport in Magnetoplasmas

To complete the system of macroscopic equations in plasma physics we need to determine values for the transport tensors, $\boldsymbol{\kappa}_e$, $\boldsymbol{\kappa}_i$, $\boldsymbol{\delta}$, $\boldsymbol{\eta}$, and $\varrho\nu\mathsf{W}$ appearing in the following equations:

$$\mathbf{q}_e = -\boldsymbol{\kappa}_e \cdot \boldsymbol{\nabla} T_e - T_e \boldsymbol{\delta} \cdot \mathbf{j} \,, \qquad \mathbf{q}_i = -\boldsymbol{\kappa}_i \cdot \boldsymbol{\nabla} T_i \,,$$

$$\boldsymbol{\eta} \cdot \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{\mathbf{j} \times \mathbf{B}}{e n_e} + \frac{\boldsymbol{\nabla} p_e}{e n_e} + \boldsymbol{\delta} \cdot \boldsymbol{\nabla} T_e \,,$$

and

$$\boldsymbol{\pi} = -2\varrho\nu \, \mathsf{W} : \boldsymbol{\nabla}\mathbf{v} \,,$$

which are equations (2.68), (2.59), and (2.72).

The tensorial character of the transport coefficients is due to the anisotropy caused by the presence of the magnetic field, the measure of which is the magnitude of the dimensionless number $\varpi \equiv \omega_c \tau$, where ω_c is the cyclotron frequency and τ is the appropriate collision interval. In the simplest treatment of transport it is possible to separate the effect of anisotropy from that of isotropic transport ($\varpi = 0$). Hence we shall begin by determining the scalar coefficients κ_e, κ_i, δ, η, and $\varrho\nu$, for particles following linear paths, the first step being to calculate the value of τ for the transport in question. Then the scalar coefficient, which is a function of τ, is obtained and finally it is modified to accommodate the influence of the magnetic field on the particle trajectories.

5.1 Coulomb collisions

5.1.1 Particle diffusion in electric microfields

As remarked in §2.1.3, the electric field $\mathbf{E}(\mathbf{r}, t)$ appearing in Maxwell's equations is a macroscopic variable, defined by averaging over a macroscopic point, by which we mean an infinitesimal volume containing a vast number of particles. We shall now reduce the time- and length- scales of the description, and consider the effect of random microfields on the motion of a typical particle, moving in a plasma without a (macroscopic) magnetic field. With neutral particles, the forces at collisions are due to highly localized electric microfields, but in a plasma the long range of the Coulomb force between charged particle extends the range of the microfields to a Debye length λ_D, with an associated time-scale equal to the reciprocal of the plasma frequency, ω_{pe}^{-1}. These parameters, namely (see §§2.2.3 and 2.2.4)

$$\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{n_e e^2}\right)^{\frac{1}{2}} \,, \qquad \omega_{pe} = \left(\frac{n_e e^2}{\epsilon_0 m_e}\right)^{\frac{1}{2}} \,, \tag{5.1}$$

play central roles in microfield processes.

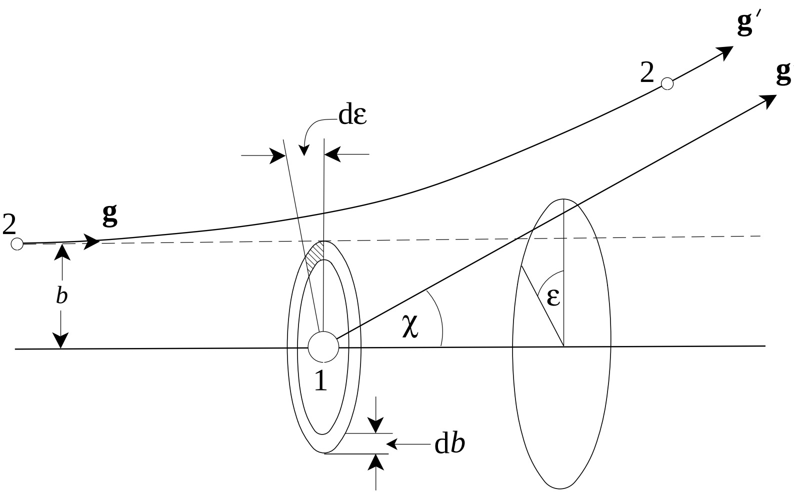

Figure 5.1: Binary encounter, repulsive force

A basic parameter in collision theory is the so-called 90° deflection impact parameter $\bar{b}_0$. By an 'impact parameter' is meant the perpendicular distance of a pre-scattered particle trajectory from the stationary reference particle in a binary encounter. In Fig. 5.1 '1' is the reference particle, and '2', the scattered particle, has an initial velocity $\mathbf{g}$ relative to '1' and, when scattered through an angle χ, a final velocity $\mathbf{g}'$. It is easily verified that $|\mathbf{g}| = |\mathbf{g}'| = g$, say. The impact parameter b is defined in the figure; the azimuthal angle ϵ specifies the plane in which '2' moves. Clearly χ depends on both g and b. When χ = 90° for an average value of g, b is equal to $\bar{b}_0$.

To find $\bar{b}_0$ one equates the electrostatic potential energy between colliding particles at their closest, to their average kinetic energy. If the particles carry charges Ze and $Z'e$, the potential energy is $Z'Ze^2/(4\pi\epsilon_0\bar{b}_0)$ (see §5.1.2). By (1.19) the average kinetic energy for the two particles is $3k_BT$. Hence

$$\bar{b}_0 = \frac{|ZZ'|e^2}{12\pi\epsilon_0 k_B T}\,. \tag{5.2}$$

The number of particles in a 'Debye sphere' is

$$n_D = \tfrac{4}{3}\pi\lambda_D^3 n\,, \tag{5.3}$$

and so an alternative expression for $\bar{b}_0$ is

$$\bar{b}_0 = \frac{|ZZ'|\lambda_D}{9n_D}\,. \tag{5.4}$$

For particles carrying the same charge, say $Z = Z' = 1$, we can interpret $\bar{b}_0$ as being the diameter of an equivalent solid, neutral particle, since colliding particles can penetrate no closer. By (1.1) a particle P will therefore move a distance

$$\lambda_{90^\circ} = \frac{1}{\pi \bar{b}_0^2 n} = \frac{\pi}{n}\left(\frac{12\epsilon_0 k_B T}{e^2}\right)^2 = 108 n_D \lambda_D \tag{5.5}$$

between 90° collisions.

The average inter-particle distance d is

$$d = (2n)^{-1/3} = \left(\frac{2\pi}{3}\right)^{1/3} \frac{\lambda_D}{n_D^{1/3}} \, .$$

The plasmas of interest to us have temperatures high enough and densities low enough for the following inequalities to be well satisfied:

$$\frac{\lambda_D}{9n_D} \ll \left(\frac{2\pi}{3}\right)^{1/3} \frac{\lambda_D}{n_D^{1/3}} \ll \lambda_D \ll 108 n_D \lambda_D \, ,$$

i.e.
$$\bar{b}_0 \ll d \ll \lambda_D \ll \lambda_{90^\circ} \, . \tag{5.6}$$

It follows that a particle must travel a relatively long way to experience a 90° deflection, and since it is influenced via its electric field by all the particles within a Debye sphere, the vast majority of its interactions with other particles will involve quite small deflections. In fact we shall find that the average electron collision frequency for momentum transfer is typically more than a hundred times the 90° collision frequency, $(2k_B T_e/m_e)^{\frac{1}{2}} \lambda_{90^\circ}^{-1}$, which shows how much more important are the accumulated effects of grazing collisions than is the occasional abrupt collision. The Debye cylinder swept by a particle P in the distance $\sim 10^{-2}\lambda_{90^\circ}$ required for this momentum transfer, contains $\pi\lambda_D^2 10^{-2}\lambda_{90^\circ} n \sim n_D^2$ particles ($\geq 10^{13}$ in a typical laboratory plasma). Thus P interacts with a vast number of particles per momentum transfer time, and the binary event called a 'collision' in a neutral gas is replaced by an essentially continuous process in a plasma.

5.1.2 Particle orbits

We start by considering the relative motion of two particles p_1, p_2 of masses m_1, m_2 moving in each other's field of force. Let the particles be at position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and exert forces $\mathbf{F}$, $-\mathbf{F}$ on each other. Then $m_1\ddot{\mathbf{r}}_1 = -\mathbf{F}$, $m_2\ddot{\mathbf{r}}_2 = \mathbf{F}$, so that $m_1 m_2\big(\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1\big) = (m_1 + m_2)\mathbf{F}$, or

$$M\ddot{\mathbf{r}} = \mathbf{F} \qquad \big(\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1, \; M \equiv m_1 m_2/(m_1 + m_2)\big) \, .$$

Thus the motion of p_2 relative to p_1 is the same as the motion of a particle of mass M (the 'reduced' mass) about a fixed centre of force $\mathbf{F}$. Specifying $\mathbf{r}$ by polar coordinates r, θ in the plane of the orbit, we can write the conservation laws for momentum and energy as

$$r^2\dot{\theta} = \text{const.} = gb \, , \quad \tfrac{1}{2}M\big(\dot{r}^2 + r^2\dot{\theta}^2\big) + V = \text{const.} = \tfrac{1}{2}Mg^2 \, , \tag{5.7}$$

where b is the impact parameter defined in Fig. 5.1, g is the constant relative velocity and V is the potential energy (zero at $r = \infty$) of the force $\mathbf{F}$.

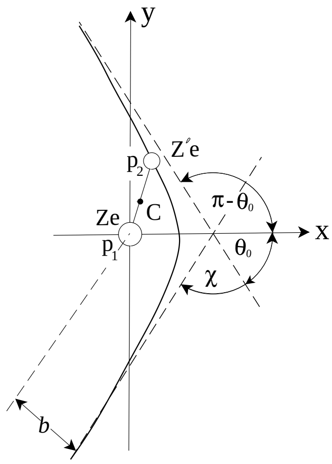

Figure 5.2: Hyperbolic orbit of an electron $(Z' = -1)$ in the Coulomb field of an ion.

Let the electric charges on $p_1,\ p_2$ be $Ze,\ Z'e$, then since (2.26), viz. $\epsilon_0 \nabla \cdot \mathbf{E} = Q$, has the solution

$$\mathbf{E} = \frac{\mathsf{q}}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3}\,, \tag{5.8}$$

for the electric field at a vector distance $\mathbf{r}$ from a charge q, the force on p_2 due to p_1 is

$$\mathbf{F} = Z'e\mathbf{E} = ZZ'e^2\mathbf{r}/\left(4\pi\epsilon_0 r^3\right).$$

Hence

$$V = ZZ'e^2/\left(4\pi\epsilon_0 r\right).$$

The *impact* parameter b_0 is the positive number

$$b_0 \equiv |ZZ'|e^2/\left(4\pi\epsilon_0 M g^2\right). \tag{5.9}$$

To be definite in the following, we shall assume that Z and Z' have opposite signs; for the other case it is necessary only to change the sign of b_0 in the final expressions. Eliminating $\dot{\theta}$ from (5.7) we get

$$\frac{dr}{dt} = \pm g\left(1 - \frac{b^2}{r^2} + \frac{2b_0}{r}\right)^{\frac{1}{2}},$$

the positive (negative) sign applying to the outgoing (incoming) trajectory, and using $(5.7)_1$ to remove the time dependence, we find that

$$\frac{d\theta}{dr} = \frac{\pm b\, dr}{r^2[1 - b^2/r^2 + 2b_0/r]^{\frac{1}{2}}}.$$

Choosing OX to be an axis of symmetry for the orbit of p, we obtain the integral

$$\frac{b^2}{rb_0} = 1 + \epsilon\cos\theta \qquad \left(\epsilon \equiv \left(1 + (b/b_0)^2\right)^{\frac{1}{2}}\right). \tag{5.10}$$

Equation (5.10) describes a conic with eccentricity ϵ and focus at the origin. Since $\epsilon > 1$, it is a hyperbola, as illustrated for the case of an electron being scattered by an ion in Fig. 5.2. Let θ_0 be the angle between OX and the asymptotes of (5.10), then for the upper branch of the conic, $\theta \to \pi - \theta_0$ as $r \to \infty$, whence $\cos\theta_0 = \epsilon^{-1}$ or

$$\tan\theta_0 = b/b_0\,. \tag{5.11}$$

The scattering angle is given by (see Fig. 5.2)

$$\chi = \pi - 2\theta_0\,, \tag{5.12}$$

so that if $b < b_0,\ \chi > \pi/2$ and we have 'close' collisions, while if $b > b_0$, then $\chi < \pi/2$ and we have grazing or 'distant' collisions. The centre of mass C of the particles is at a distance $r_c = m_1 r/(m_1 + m_2)$ from p_2, so if r in (5.10) is replaced by $(m_1 + m_2)r_c/m_1$, it becomes the polar equation of the orbit of p_2 in the centre-of-mass frame. The limit $\theta \to \pi - \theta_0$ as $r_c \to \infty$ is unchanged, so χ is also the scattering angle in the CM frame.

5.1.3 The Rutherford scattering cross-section

Now suppose that instead of precise knowledge of the impact parameter b of p_2, we know only that it is incident on an element of area $b\,db\,d\epsilon$ as shown in Fig. 5.3. The probability $\sigma\,d\Omega$ that it is deflected into the solid angle $d\Omega = \sin\chi\,d\chi\,d\epsilon$ is termed the *differential cross-section* for the scattering collision. Of N incident particles per unit area per second, $|b\,db\,d\epsilon|N$ will be scattered into $d\Omega$. By definition this number also equals $\sigma\,d\Omega\,N$. Hence $\sigma\,d\Omega = -b\,db\,d\epsilon$, the negative sign being necessary since $db/d\chi$ is negative.

From (5.11) and (5.12)

$$b = b_0\cot\tfrac{1}{2}\chi, \qquad b\frac{db}{d\chi} = -\tfrac{1}{2}b_0^2\,\frac{\cos\frac{1}{2}\chi}{\sin^3\frac{1}{2}\chi}\,, \tag{5.13}$$

therefore

$$\sigma = -\frac{b}{\sin\chi}\frac{db}{d\chi} = \frac{b_0^2}{2\sin\chi}\,\frac{\cos\frac{1}{2}\chi}{\sin^3\frac{1}{2}\chi}\,,$$

i.e.

$$\sigma(g,\chi) = \frac{b_0^2}{4\sin^4\frac{1}{2}\chi} = \left(\frac{ZZ'e^2}{8\pi\epsilon_0 Mg\sin^2\frac{1}{2}\chi}\right)^2. \tag{5.14}$$

This is the Rutherford scattering cross-section; it is evident that small-angle scattering is far more probable than large deflections.

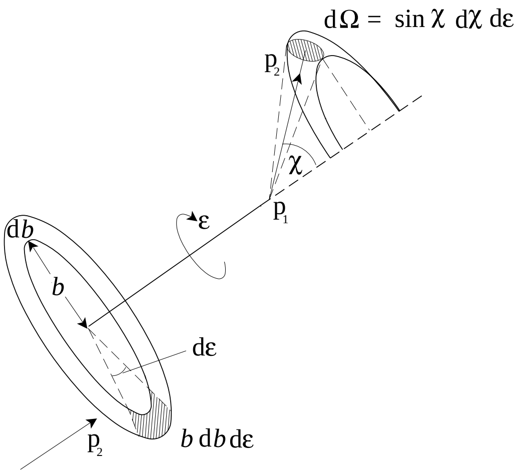

Figure 5.3: Coulomb scattering

5.2 The Fokker-Planck equation

5.2.1 Friction and diffusion coefficients

Consider the particles comprising a fully-ionized plasma. Choose one of them to be a 'test' particle and suppose that the species to which it belongs has a velocity distribution $f(\mathbf{r},\,\mathbf{w},\,t)$. The long-range Coulomb forces between the test particle p and the 'field' particles within a Debye distance will cause p to experience a multiplicity of 'distant' collisions, which as noted above, will be far more numerous than close collisions. Consequently, almost all the changes in direction and speed experienced by p will be small. Let p have an initial velocity $\mathbf{w}$, then after a small time interval Δt, distant collisions will generate a random walk motion in p, producing a cumulative change $\Delta\mathbf{w}$ satisfying $|\Delta\mathbf{w}| \ll |\mathbf{w}|$. We are ignoring here the non-random collisional contributions to $\Delta\mathbf{w}$, allowing them to enter in the usual way via the convective terms in the kinetic equation (see remarks in §1.3.3).

Let $P(\mathbf{w}|\Delta\mathbf{w})$ denote the transition probability density that p experiences the change $\Delta\mathbf{w}$ in Δt, then

$$f(\mathbf{r},\,\mathbf{w}-\Delta\mathbf{w},\,t-\Delta t)\times P(\mathbf{w}-\Delta\mathbf{w}|\Delta\mathbf{w})\,d(\Delta\mathbf{w})$$

is the number of particles like p that are deflected from the range $(\mathbf{w}-\Delta\mathbf{w},\,\mathbf{w})$ into the element $(\mathbf{w},\,\mathbf{w}+d\mathbf{w})$ owing to interactions occurring in $(t-\Delta t,\,t)$. These particles contribute to the number $f(\mathbf{r},\,\mathbf{w},\,t)\,d\mathbf{w}$, and assuming that the process is Markovian, i.e. that no earlier time-intervals contribute to this number, we obtain the Chapman-Kolmogorov equation,

$$f(\mathbf{r},\,\mathbf{w},\,t)=\int f(\mathbf{r},\mathbf{w}-\Delta\mathbf{w},\,t-\Delta t)\,P(\mathbf{w}-\Delta\mathbf{w}|\Delta\mathbf{w})\,d(\Delta\mathbf{w})\,, \tag{5.15}$$

where the integration is over all possible changes in the velocity vector. The time Δt must be small enough for $\Delta\mathbf{w}$ to remain quite small compared with $\mathbf{w}$, but long enough compared with the transit time of p over the correlation length for microfield fluctuations. For electrons this lower limit is about ω_{pe}^{-1} (see §2.2.3 and §4.6.3).

Expanding the integrand in (5.15) in a Taylor series to first order in Δt and to second order in $\Delta\mathbf{w}$, we obtain the approximate form

$$f(\mathbf{r},\,\mathbf{w},\,t) \;=\; \int \Big\{ \big(f - \Delta t\,\mathbb{D}f\big) P(\mathbf{w}|\Delta\mathbf{w}) - \Delta\mathbf{w} \cdot \frac{\partial}{\partial\mathbf{w}}\big[f\,P(\mathbf{w}|\Delta\mathbf{w})\big] + \tfrac{1}{2}\Delta\mathbf{w}\Delta\mathbf{w} : \frac{\partial^2}{\partial\mathbf{w}\partial\mathbf{w}}\big[f\,P(\mathbf{w}|\Delta\mathbf{w})\big] \Big\}\, d(\Delta\mathbf{w})\,,$$

where $\mathbb{D}f$ is the rate of change of f following the bunch of particles through phase space (see (1.90)) and the terms in the integrand are evaluated at $(\mathbf{r},\,\mathbf{w},\,t)$. As the probability of a transition of any kind occurring is unity,

$$\int P(\mathbf{w}|\Delta\mathbf{w})\,d(\Delta\mathbf{w}) = 1\,.$$

Hence the leading term in the expression for f cancels with the left-side and the remaining terms can be arranged as a kinetic equation for f:

$$\mathbb{D}f = \mathbb{C} \equiv -\frac{\partial}{\partial\mathbf{w}} \cdot \big(\mathbf{A}f\big) + \frac{1}{2}\frac{\partial^2}{\partial\mathbf{w}\partial\mathbf{w}} : \big(\mathbf{B}f\big)\,, \tag{5.16}$$

in which

$$\mathbf{A} = \langle\Delta\mathbf{w}\rangle \equiv \frac{1}{\Delta t}\int \Delta\mathbf{w}\,P(\mathbf{w}|\Delta\mathbf{w})\,d(\Delta\mathbf{w})\,, \tag{5.17}$$

and

$$\mathbf{B} = \langle\Delta\mathbf{w}\Delta\mathbf{w}\rangle \equiv \frac{1}{\Delta t}\int \Delta\mathbf{w}\,\Delta\mathbf{w}\,P(\mathbf{w}|\Delta\mathbf{w})\,d(\Delta\mathbf{w})\,. \tag{5.18}$$

The expression in (5.16) for the collision term $\mathbb{C}$ is known as the Fokker-Planck equation. Notice that it may be expressed as the divergence in velocity space of a 'flow' vector $\mathbf{J}$,

$$\mathbb{C} = -\frac{\partial}{\partial\mathbf{w}} \cdot \mathbf{J}, \qquad \mathbf{J} \equiv \mathbf{A}f - \frac{1}{2}\frac{\partial}{\partial\mathbf{w}} \cdot \big(\mathbf{B}f\big)\,. \tag{5.19}$$

This vector describes the continuous flow of phase plane points due to the accumulation of many small-angle collisions. The averages $\langle\Delta\mathbf{w}\rangle$ and $\langle\Delta\mathbf{w}\Delta\mathbf{w}\rangle$ are termed the *friction* and *diffusion* coefficients for reasons that will become clear below. To apply (5.16) to a plasma we must calculate these averages for the case of Coulomb collisions.

In general there will be several types of field particles or scatterers influencing the test particle p, including the species to which p itself belongs. We shall start by considering just one type of scatterer, denoting its distribution function by $f_s\big(\mathbf{r},\,\mathbf{w}_s,\,t\big)$. The probability that a single scatterer deflects p into the solid angle $d\Omega = \sin\chi\,d\chi\,d\epsilon$ (see Fig. 5.3) is $\sigma(g,\,\chi)\,d\Omega$, where $g = |\mathbf{w} - \mathbf{w}_s|$ is the relative speed between the interacting particles and $\sigma(g,\,\chi)$ is the Rutherford scattering cross-section. The corresponding scattering rate is $g\sigma(g,\,\chi)\,d\Omega$, i.e. of a group of N incident particles, $Ng\sigma(g,\,\chi)\,d\Omega$ will appear in $d\Omega$. The assumption of small scattering angles allows us to superimpose linearly the contributions of all the scatterers lying in the appropriate element $d\mathbf{w}$ of velocity space. Thus the total probability that p is scattered into $d\Omega$ per second per unit volume is $f_s\, d\mathbf{w}_s\, g\sigma(g,\, \chi)\, d\Omega$. Hence the averages in (5.17) and (5.18) are equivalent to

$$\left.\begin{aligned} \langle\Delta\mathbf{w}\rangle &= \iint \Delta\mathbf{w}\, g\sigma(g,\,\chi)\, d\Omega\, f_s\, d\mathbf{w}_s = \int \big[\Delta\mathbf{w}\big]_\Omega f_s\, d\mathbf{w}_s\,, \\ \langle\Delta\mathbf{w}\Delta\mathbf{w}\rangle &= \iint \Delta\mathbf{w}\Delta\mathbf{w}\, g\sigma(g,\,\chi)\, d\Omega\, f_s\, d\mathbf{w}_s = \int \big[\Delta\mathbf{w}\Delta\mathbf{w}\big]_\Omega f_s\, d\mathbf{w}_s\,, \end{aligned}\right\} \tag{5.20}$$

where

$$\left.\begin{aligned} \big[\Delta\mathbf{w}\big]_\Omega &\equiv \int_0^{2\pi}\!\!\int_\chi \Delta\mathbf{w}\, g\sigma(g,\,\chi)\sin\chi\, d\chi\, d\epsilon\,, \\ \big[\Delta\mathbf{w}\Delta\mathbf{w}\big]_\Omega &\equiv \int_0^{2\pi}\!\!\int_\chi \Delta\mathbf{w}\Delta\mathbf{w}\, g\sigma(g,\,\chi)\sin\chi\, d\chi\, d\epsilon\,. \end{aligned}\right\} \tag{5.21}$$

When there are several types of scatterer, say $s = 1,\, 2,\, \ldots$, the integrals in (5.20) are required for each value of s and must be summed over s to give the averages. In the next section we shall express these averages as velocity space gradients of two 'super-potential' functions.

5.2.2 Scattering in velocity space

Let the test particle p have mass m and velocities $\mathbf{w},\, \mathbf{w}'$ before and after an elastic collision with a scatterer having mass m_s and velocities $\mathbf{w}_s,\, \mathbf{w}'_s$ before and after the collision. Then with $\mathbf{g} = \mathbf{w} - \mathbf{w}_s,\; g' = \mathbf{w}' - \mathbf{w}'_s$, denoting the relative velocities and

$$\mathbf{G} = (m\mathbf{w} + m_s\mathbf{w}_s)/(m + m_s), \quad \mathbf{G}' = (m\mathbf{w}' + m_s\mathbf{w}'_s)/(m + m_s),$$

denoting the centre of mass velocities, it is readily shown from the conservation of momentum and energy that

$$\mathbf{G} = \mathbf{G}', \quad g = g', \quad \mathbf{w} = \mathbf{G} + \frac{M}{m}\mathbf{g} \qquad \Big(M \equiv \frac{mm_s}{m + m_s}\Big)\,. \tag{5.22}$$

The angle between $\mathbf{g}$ and $\mathbf{g}'$ is the scattering angle χ and since $\mathbf{g}$ is unchanged in magnitude by the collision, $|\Delta\mathbf{g}| = |\mathbf{g}' - \mathbf{g}| = 2g\sin\frac{1}{2}\chi$. Resolving the vector $\Delta\mathbf{g}$ into a component Δg_1 parallel to $\mathbf{g}$ and components Δg_2 and Δg_3 perpendicular to $\mathbf{g}$ as shown in Fig. 5.4, we have

$$\Delta\mathbf{g} = 2g\sin\tfrac{1}{2}\chi\big(-\sin\tfrac{1}{2}\chi,\, \cos\tfrac{1}{2}\chi\, \cos\epsilon,\, \cos\tfrac{1}{2}\chi\, \sin\epsilon\big)\,. \tag{5.23}$$

By (5.22),

$$\mathbf{w}' - \mathbf{w} = (M/m)(\mathbf{g}' - \mathbf{g}), \text{ i.e. } \Delta\mathbf{w} = (M/m)\Delta\mathbf{g}.$$

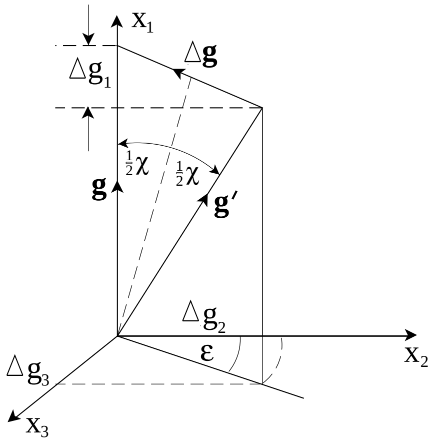

Figure 5.4: Scattering in velocity space

The first average in (5.21) is now calculated using (5.14) and (5.23). The integrals containing $\cos\epsilon$ and $\sin\epsilon$ vanish, leaving only a component parallel to the unit vector $\mathbf{g}/g$. Thus with χ lying in $\chi_{\min} \leq \chi \leq \chi_{\max}$, we get

$$\left[\Delta\mathbf{w}\right]_{\Omega} = -4\pi\frac{M}{m}\left(b_0 g\right)^2 \left[\ln\sin\tfrac{1}{2}\chi\right]_{\chi_{\min}}^{\chi_{\max}} \frac{\mathbf{g}}{g}\,. \tag{5.24}$$

As χ is assumed to be small, $\frac{1}{2}\pi \gg \chi_{\max} \gg \chi_{\min}$. The largest impact parameter b for which the Coulomb force is effective is the Debye length λ_D (§2.2.4), so a reasonable assumption is that $\chi_{\min}$ corresponds to $b = \lambda_D$. Then by (5.13)$_1$, $\cot\frac{1}{2}\chi_{\min} \approx 2/\chi_{\min} = \lambda_D/b_0$. Therefore

$$\left[\ln\sin\tfrac{1}{2}\chi\right]_{\chi_{\min}}^{\chi_{\max}} \approx (1-\alpha)\ln\Lambda \qquad \left(\alpha = -\ln\tfrac{1}{2}\chi_{\max}/\ln\Lambda\right), \tag{5.25}$$

where

$$\Lambda \equiv \lambda_D/\overline{b}_0\,, \tag{5.26}$$

and we have removed the weak dependence of the logarithm on particle velocity by replacing b_0 by an average value $\overline{b}_0$.

5.2.3 Super-potential functions

Provided that $\frac{1}{2}\chi_{\max}$ is large enough, $\alpha \ll 1$, when it follows from (5.9), (5.24), (5.25), and (5.26) that

$$\left[\Delta\mathbf{w}\right]_{\Omega} = -\frac{\left(ZZ'e^2\right)^2\ln\Lambda}{4\pi\epsilon_0^2 Mm}\frac{\mathbf{g}}{g^3} \qquad \left(\mathbf{g} = \mathbf{w}-\mathbf{w}_s\right). \tag{5.27}$$

Finally, by integrating over the field particles we obtain

$$\langle \Delta \mathbf{w} \rangle = -\Gamma \frac{m}{M} \int f_s(\mathbf{w}_s) \frac{\mathbf{w} - \mathbf{w}_s}{|\mathbf{w} - \mathbf{w}_s|^3} \, d\mathbf{w}_s \,, \tag{5.28}$$

where

$$\Gamma \equiv \frac{\left(ZZ'e^2\right)^2 \ln \Lambda}{4\pi\epsilon_0^2 m^2} \,. \tag{5.29}$$

As

$$\mathbf{g} \cdot \frac{\partial \mathbf{g}}{\partial \mathbf{w}} = g \frac{\partial g}{\partial \mathbf{w}} = \mathbf{g} \,,$$

we can write

$$\left[\Delta \mathbf{w}\right]_\Omega = \frac{m}{M} \Gamma \frac{\partial}{\partial \mathbf{w}} \left(\frac{1}{g}\right) , \tag{5.30}$$

and therefore

$$\langle \Delta \mathbf{w} \rangle = \Gamma \frac{\partial \mathcal{H}}{\partial \mathbf{w}} \,, \tag{5.31}$$

where

$$\mathcal{H} \equiv \frac{m}{M} \int \frac{f_s(\mathbf{w}_s)}{|\mathbf{w} - \mathbf{w}_s|} \, d\mathbf{w}_s \,. \tag{5.32}$$

This scalar function $\mathcal{H}$ is the first of the two super-potentials, introduced by Rosenbluth, MacDonald and Judd (1957).

A similar method is applied to the average $\langle \Delta \mathbf{w} \Delta \mathbf{w} \rangle$. First we calculate $\left[\Delta w_i \Delta w_j\right]_\Omega$, $i, j = 1, 2, 3$. The integration over ϵ eliminates the terms in which i and j are not equal, and it is readily found that

$$\left[\Delta w_2 \Delta w_2\right]_\Omega = \left[\Delta w_3 \Delta w_3\right]_\Omega = \Gamma / g = \left[\Delta w_1 \Delta w_1\right]_\Omega \ln \Lambda \,.$$

As $\ln \Lambda$ is typically between 10 and 20 in the plasmas of interest to us, we can neglect $\left[\Delta w_1 \Delta w_1\right]_\Omega$ compared with the other components, and as this quantity is the coefficient of the tensor component $\mathbf{gg}/g^2$, by removing it we get

$$\left[\Delta \mathbf{w} \Delta \mathbf{w}\right]_\Omega = \frac{\Gamma}{g} \left(\mathbb{1} - \mathbf{gg}/g^2\right) . \tag{5.33}$$

By

$$\frac{\partial^2 g}{\partial \mathbf{w} \, \partial \mathbf{w}} = \frac{\partial}{\partial \mathbf{w}} \left(\frac{\mathbf{g}}{g}\right) = \frac{1}{g} \left(\mathbb{1} - \mathbf{gg}/g^2\right) , \tag{5.34}$$

(5.33) can be expressed

$$\left[\Delta \mathbf{w} \Delta \mathbf{w}\right]_\Omega = \Gamma \frac{\partial^2 g}{\partial \mathbf{w} \, \partial \mathbf{w}} \,. \tag{5.35}$$

Now we introduce the second super-potential,

$$\mathcal{G} \equiv \int f_s(\mathbf{w}_s)\,|\mathbf{w}-\mathbf{w}_s|\,d\mathbf{w}_s\,, \tag{5.36}$$

then the average of (5.35) over the field particles gives

$$\langle \Delta\mathbf{w}\Delta\mathbf{w}\rangle = \Gamma \frac{\partial^2 \mathcal{G}}{\partial \mathbf{w}\,\partial \mathbf{w}}\,. \tag{5.37}$$

By (5.17), (5.18), (5.31), and (5.37) the Fokker-Planck equation may be expressed

$$\mathbb{C} = -\Gamma \frac{\partial}{\partial \mathbf{w}} \boldsymbol{\cdot} \Big(f \frac{\partial \mathcal{H}}{\partial \mathbf{w}} \Big) + \frac{\Gamma}{2} \frac{\partial^2}{\partial \mathbf{w}\,\partial \mathbf{w}} \boldsymbol{:} \Big(f \frac{\partial^2 \mathcal{G}}{\partial \mathbf{w}\,\partial \mathbf{w}} \Big)\,. \tag{5.38}$$

In general, when there are several scatterers, we write $\mathcal{H} = \sum_s \mathcal{H}_s$ and $\mathcal{G} = \sum_s \mathcal{G}_s$, and a separate kinetic equation, $\mathbb{D}f_s = \mathbb{C}_s$, is required for each species. As $\mathcal{H}$ and $\mathcal{G}$ are integrals, we now have a set of rather complicated, coupled, integro-differential equations, which can be solved accurately only by numerical computation.

Finally we note two useful relations involving $\mathcal{H}$ and $\mathcal{G}$. The first is

$$\frac{m}{M} \nabla^2_{\mathbf{w}} \mathcal{G} = 2\mathcal{H} \qquad \big(\boldsymbol{\nabla}_{\mathbf{w}} \equiv \frac{\partial}{\partial \mathbf{w}}\big)\,, \tag{5.39}$$

which follows on integrating $\nabla^2_{\mathbf{w}} g = 2/g$ over the field particles. The second is the Poisson equation

$$\nabla^2_{\mathbf{w}} \mathcal{H} = -4\pi \frac{m}{M} f_s(\mathbf{w}_s)\,, \tag{5.40}$$

which is implied by (5.28) and (5.31).

5.3 Lorentzian plasma

5.3.1 Collisional loss rate

A hypothetical, fully-ionized gas with ions of infinite mass and no electron-electron interactions, is known as a ‘Lorentzian’ plasma. This ideal medium approximates the situation when the positive ions have a very high nuclear charge ($Z \gg 1$); it also provides an exact algebraic theory for comparison purposes.

For electron-ion collisions, $M = m_e m_i/(m_e + m_i) \to m_e$ as $m_i \to \infty$. The ions have negligible random velocities, otherwise their temperature would tend to infinity with m_i. We shall take the laboratory frame to be fixed in the ion fluid so that $f_i(\mathbf{w}_i) = n_i \delta(0)$, where n_i is the ion number density and $\delta(\mathbf{w}_i)$ is the delta function. In this case with $m = m_e$ and $f_s = f_i$ in (5.32) and (5.36), we get $\mathcal{H} = n_i/w$ and $\mathcal{G} = n_i w$, where w is the electron speed. Therefore (5.31), (5.33), and (5.37) yield

$$\langle \Delta\mathbf{w}\rangle = -\frac{n_i \Gamma}{w^2}\hat{\mathbf{w}}\,, \qquad \langle \Delta\mathbf{w}\Delta\mathbf{w}\rangle = \frac{n_i\Gamma}{w}\mathbb{1}_\perp\,, \tag{5.41}$$