1 Function Spaces, Linear Operators and Green's Functions

1.1 Function Spaces

Consider the set of all complex valued functions of the real variable x, denoted by $f(x), g(x), \ldots$ and defined on the interval (a, b). We shall restrict ourselves to those functions which are *square-integrable*. Define the *inner product* of any two of the latter functions by

$$(f, g) \equiv \int_a^b f^*(x)\, g(x)\ dx, \tag{1.1.1}$$

in which $f^*(x)$ is the complex conjugate of $f(x)$. The following properties of the inner product follow from the definition (1.1.1).

$$\begin{aligned} (f,g)^* &= (g,f), \\ (f, g+h) &= (f,g)+(f,h), \\ (f,\alpha g) &= \alpha(f,g), \\ (\alpha f, g) &= \alpha^*(f,g), \end{aligned} \tag{1.1.2}$$

with α a complex scalar.

While the inner product of any two functions is in general a complex number, the inner product of a function with itself is a real number and is non-negative. This prompts us to define the *norm of a function* by

$$\|f\| \equiv \sqrt{(f,f)} = \left[\int_a^b f^*(x) f(x)\, dx\right]^{\frac{1}{2}}, \tag{1.1.3}$$

provided that f is *square-integrable*, i.e., $\|f\| < \infty$. Equation (1.1.3) constitutes a proper definition for a norm since it satisfies the following conditions,

$$\begin{array}{llll} \text{(i)} & \textit{scalar multiplication} & \|\alpha f\| = |\alpha| \cdot \|f\|, & \text{for all complex } \alpha, \\ \text{(ii)} & \textit{positivity} & \|f\| > 0, & \text{for all } f \neq 0, \\ & & \|f\| = 0, & \text{if and only if } f = 0, \\ \text{(iii)} & \textit{triangular inequality} & \|f+g\| \leq \|f\| + \|g\|. & \end{array} \tag{1.1.4}$$

Applied Mathematics in Theoretical Physics. Michio Masujima

ISBN: 3-527-40534-8

A very important inequality satisfied by the *inner product* (1.1.1) is the so-called *Schwarz inequality* which says

$$|(f,g)| \leq \|f\| \cdot \|g\| . \tag{1.1.5}$$

To prove the latter, start with the trivial inequality $\|(f+\alpha g)\|^2 \geq 0$, which holds for any $f(x)$ and $g(x)$ and for any complex number α. With a little algebra, the left-hand side of this inequality may be expanded to yield

$$(f,f) + \alpha^*(g,f) + \alpha(f,g) + \alpha\alpha^*(g,g) \geq 0. \tag{1.1.6}$$

The latter inequality is true for any α, and is thus true for the value of α which minimizes the left-hand side. This value can be found by writing α as $a + ib$ and minimizing the left-hand side of Eq. (1.1.6) with respect to the real variables a and b. A quicker way would be to treat α and α^* as independent variables and requiring $\partial/\partial\alpha$ and $\partial/\partial\alpha^*$ of the left hand side of Eq. (1.1.6) to vanish. This immediately yields $\alpha = -(g,f)/(g,g)$ as the value of α at which the minimum occurs. Evaluating the left-hand side of Eq. (1.1.6) at this minimum then yields

$$\|f\|^2 \geq \frac{|(f,g)|^2}{\|g\|^2}, \tag{1.1.7}$$

which proves the Schwarz inequality (1.1.5).

Once the Schwarz inequality has been established, it is relatively easy to prove the *triangular inequality* (1.1.4)(iii). To do this, we simply begin from the definition

$$\|f+g\|^2 = (f+g, f+g) = (f,f) + (f,g) + (g,f) + (g,g). \tag{1.1.8}$$

Now the right-hand side of Eq. (1.1.8) is a sum of complex numbers. Applying the usual triangular inequality for complex numbers to the right-hand side of Eq. (1.1.8) yields

$$\begin{aligned} |\text{Right-hand side of Eq. (1.1.8)}| &\leq \|f\|^2 + |(f,g)| + |(g,f)| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2. \end{aligned} \tag{1.1.9}$$

Combining Eqs. (1.1.8) and (1.1.9) finally proves the triangular inequality (1.1.4)(iii).

We remark finally that the set of functions $f(x), g(x), \ldots$ is an example of a *linear vector space*, equipped with an inner product and a norm based on that inner product. A similar set of properties, including the Schwarz and triangular inequalities, can be established for other linear vector spaces. For instance, consider the set of all complex column vectors $\vec{u}, \vec{v}, \vec{w}, \ldots$ of finite dimension n. If we define the inner product

$$(\vec{u}, \vec{v}) \equiv (\vec{u}^*)^{\mathrm{T}} \vec{v} = \sum_{k=1}^{n} u_k^* v_k, \tag{1.1.10}$$

and the related norm

$$\|\vec{u}\| \equiv \sqrt{(\vec{u}, \vec{u})}, \tag{1.1.11}$$

then the corresponding Schwarz and triangular inequalities can be proven in an identical manner yielding

$$|(\vec{u}, \vec{v})| \leq \|\vec{u}\| \, \|\vec{v}\| \,, \tag{1.1.12}$$

and

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \,. \tag{1.1.13}$$

1.2 Orthonormal System of Functions

Two functions $f(x)$ and $g(x)$ are said to be *orthogonal* if their inner product vanishes, i.e.,

$$(f, g) = \int_a^b f^*(x) g(x) \; dx = 0. \tag{1.2.1}$$

A function is said to be *normalized* if its norm equals to unity, i.e.,

$$\|f\| = \sqrt{(f, f)} = 1. \tag{1.2.2}$$

Consider now a set of normalized functions $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ which are mutually orthogonal. Such a set is called an *orthonormal set of functions*, satisfying the orthonormality condition

$$(\phi_i, \phi_j) = \delta_{ij} = \begin{cases} 1, & \text{if } \; i = j, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2.3}$$

where δ_{ij} is the *Kronecker delta symbol* itself defined by Eq. (1.2.3).

An orthonormal set of functions $\{\phi_n(x)\}$is said to form a *basis for a function space*, or to be *complete*, if any function $f(x)$ in that space can be expanded in a series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{1.2.4}$$

(This is not the exact definition of a complete set but it will do for our purposes.) To find the coefficients of the expansion in Eq. (1.2.4), we take the inner product of both sides with $\phi_m(x)$ from the left to obtain

$$\begin{aligned} (\phi_m, f) &= \sum_{n=1}^{\infty} (\phi_m, a_n \phi_n) \\ &= \sum_{n=1}^{\infty} a_n \, (\phi_m, \phi_n) \\ &= \sum_{n=1}^{\infty} a_n \delta_{mn} = a_m. \end{aligned} \tag{1.2.5}$$

In other words, for any n,

$$a_n = (\phi_n, f) = \int_a^b \phi_n^* (x) f (x) \ dx. \tag{1.2.6}$$

An example of an orthonormal system of functions on the interval $(-l, l)$ is the infinite set

$$\phi_n (x) = \frac{1}{\sqrt{2l}} \exp\left[\frac{in\pi x}{l}\right], \quad n = 0, \pm 1, \pm 2, \ldots \tag{1.2.7}$$

with which the expansion of a *square-integrable* function $f(x)$ on $(-l, l)$ takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left[\frac{in\pi x}{l}\right], \tag{1.2.8a}$$

with

$$c_n = \frac{1}{2l} \int_{-l}^{+l} f(x) \exp\left[-\frac{in\pi x}{l}\right], \tag{1.2.8b}$$

which is the familiar complex form of the *Fourier series* of $f(x)$.

Finally the *Dirac delta function* $\delta\,(x - x')$, defined with x and x' in (a, b), can be expanded in terms of a complete set of orthonormal functions $\phi_n(x)$ in the form

$$\delta\,(x - x') = \sum_n a_n \phi_n(x)$$

with

$$a_n = \int_a^b \phi_n^*(x) \delta(x - x') \ dx = \phi_n^* \,(x')\,.$$

That is,

$$\delta(x - x') = \sum_n \phi_n^*(x') \phi_n(x). \tag{1.2.9}$$

The expression (1.2.9) is sometimes taken as the statement which implies the *completeness of an orthonormal system of functions*.

1.3 Linear Operators

An *operator* can be thought of as a mapping or a transformation which acts on a member of the function space (i.e., a function) to produce another member of that space (i.e., another function). The operator, typically denoted by a symbol such as L, is said to be *linear* if it satisfies

$$L(\alpha f + \beta g) = \alpha L f + \beta L g, \tag{1.3.1}$$

where α and β are complex numbers, and f and g are members of that function space.

Some trivial examples of linear operators L are

(i) multiplication by a constant scalar, i.e.,

$$L\phi = a\phi,$$

(ii) taking the third derivative of a function, i.e.,

$$L\phi = \frac{d^3}{dx^3}\phi \quad \text{or} \quad L = \frac{d^3}{dx^3},$$

which is a differential operator, or,

(iii) multiplying a function by the kernel, $K(x, x')$, and integrating over (a, b) with respect to x', i.e.,

$$L\phi(x) = \int_a^b K(x, x')\phi(x')\, dx',$$

which is an integral operator.

An important concept in the theory of the linear operator is that of the *adjoint* of the operator which is defined as follows. Given the operator L, together with an inner product defined on a vector space, the adjoint L^{adj} of the operator L is that operator for which

$$(\psi, L\phi) = (L^{\text{adj}}\psi, \phi), \tag{1.3.2}$$

is an identity for any two members ϕ and ψ of the vector space. Actually, as we shall see later, in the case of the differential operators, we frequently need to worry to some extent about the boundary conditions associated with the original and the adjoint problems. Indeed, there often arise additional terms on the right-hand side of Eq. (1.3.2) which involve the boundary points, and a prudent choice of the adjoint boundary conditions will need to be made in order to avoid unnecessary difficulties. These issues will be raised in connection with Green's functions for differential equations.

As our first example of the adjoint operator, consider the liner vector space of n-dimensional complex column vectors $\vec{u}$, $\vec{v}$, ... with their associated inner product (1.1.10). In this space, $n\times n$ square matrices A, B, ... with complex entries are linear operators when multiplied by the n-dimensional complex column vectors according to the usual rules of matrix multiplication. Consider now the problem of finding the adjoint A^{adj} of the matrix A. According to the definition (1.3.2) of the adjoint operator, we search for the matrix A^{adj} satisfying

$$(\vec{u}, A\vec{v}) = (A^{\text{adj}}\vec{u}, \vec{v}). \tag{1.3.3}$$

Now, from the definition of the inner product (1.1.10), we must have

$$\vec{u}^{*\text{T}}(A^{\text{adj}})^{*\text{T}}\vec{v} = \vec{u}^{*\text{T}}A\vec{v},$$

i.e.,

$$(A^{\text{adj}})^{*\text{T}} = A \quad \text{or} \quad A^{\text{adj}} = A^{*\text{T}}. \tag{1.3.4}$$

That is, the adjoint A^{adj} of a matrix A is equal to the complex conjugate of its transpose, which is also known as its *Hermitian transpose*,

$$A^{\text{adj}} = A^{*\text{T}} \equiv A^{\text{H}}. \tag{1.3.5}$$

As a second example, consider the problem of finding the adjoint of the linear integral operator

$$L = \int_a^b dx' K(x, x'), \tag{1.3.6}$$

on our function space. By definition, the adjoint L^{adj} of L is the operator which satisfies Eq. (1.3.2). Upon expressing the left-hand side of Eq. (1.3.2) explicitly with the operator L given by Eq. (1.3.6), we find

$$(\psi, L\phi) = \int_a^b dx\, \psi^*(x) L\phi(x) = \int_a^b dx' \left[\int_a^b dx K(x, x')\psi^*(x) \right] \phi(x'). \tag{1.3.7}$$

Requiring Eq. (1.3.7) to be equal to

$$(L^{\text{adj}}\psi, \phi) = \int_a^b dx (L^{\text{adj}}\psi(x))^* \phi(x)$$

necessitates defining

$$L^{\text{adj}}\psi(x) = \int_a^b d\xi K^*(\xi, x)\psi(\xi).$$

Hence the adjoint of integral operator (1.3.6) is found to be

$$L^{\text{adj}} = \int_a^b dx' K^*(x', x). \tag{1.3.8}$$

Note that, aside from the complex conjugation of the kernel $K(x, x')$, the integration in Eq. (1.3.6) is carried out with respect to the second argument of $K(x, x')$ while that in Eq. (1.3.8) is carried out with respect to the first argument of $K^*(x', x)$. Also, be careful to note which of the variables throughout the above is the dummy variable of integration.

Before we end this section, let us define what is meant by a *self-adjoint* operator. An operator L is said to be self-adjoint (or *Hermitian*) if it is equal to its own adjoint L^{adj}. Hermitian operators have very nice properties which will be discussed in Section 1.6. Not the least of these is that their eigenvalues are real. (Eigenvalue problems are discussed in the next section.)

Examples of self-adjoint operators are Hermitian matrices, i.e., matrices which satisfies

$$A = A^{\text{H}},$$

and linear integral operators of the type (1.3.6) whose kernel satisfy

$$K(x, x') = K^*(x', x),$$

each on their respective linear spaces and with their respective inner products.

1.4 Eigenvalues and Eigenfunctions

Given a linear operator L on a linear vector space, we can set up the following eigenvalue problem

$$L\phi_n = \lambda_n \phi_n \quad (n = 1, 2, 3, \ldots). \tag{1.4.1}$$

Obviously the trivial solution $\phi(x) = 0$ always satisfies this equation, but it also turns out that for some particular values of λ (called the *eigenvalues* and denoted by λ_n), nontrivial solutions to Eq. (1.4.1) also exist. Note that for the case of the differential operators on bounded domains, we must also specify an appropriate homogeneous boundary condition (such that $\phi = 0$ satisfies those boundary conditions) for the *eigenfunctions* $\phi_n(x)$. We have affixed the subscript n to the eigenvalues and eigenfunctions under the assumption that the eigenvalues are discrete and that they can be counted (i.e., with $n = 1, 2, 3, \ldots$). This is not always the case. The conditions which guarantee the existence of a discrete (and complete) set of eigenfunctions are beyond the scope of this introductory chapter and will not be discussed.

So, for the moment, let us tacitly assume that the eigenvalues λ_n of Eq. (1.4.1) are discrete and that their eigenfunctions ϕ_n form a basis (i.e., a complete set) for their space.

Similarly the adjoint L^{adj} of the operator L would posses a set of eigenvalues and eigenfunctions satisfying

$$L^{\text{adj}}\psi_m = \mu_m \psi_m \quad (m = 1, 2, 3, \ldots). \tag{1.4.2}$$

It can be shown that the eigenvalues μ_m of the adjoint problem are equal to complex conjugates of the eigenvalues λ_n of the original problem. (We will prove this only for matrices but it remains true for general operators.) That is, if λ_n is an eigenvalue of L, λ_n^* is an eigenvalue of L^{adj}. This prompts us to rewrite Eq. (1.4.2) as

$$L^{\text{adj}}\psi_m = \lambda_m^* \psi_m, \quad (m = 1, 2, 3, \ldots). \tag{1.4.3}$$

It is then a trivial matter to show that the eigenfunctions of the adjoint and original operators are all orthogonal, except those corresponding to the same index ($n = m$). To do this, take the inner product of Eq. (1.4.1) with ψ_m from the left, and the inner product of Eq. (1.4.3) with ϕ_n from the right, to find

$$(\psi_m, L\phi_n) = (\psi_m, \lambda_n \phi_n) = \lambda_n(\psi_m, \phi_n) \tag{1.4.4}$$

and

$$(L^{\text{adj}}\psi_m, \phi_n) = (\lambda_m^* \psi_m, \phi_n) = \lambda_m(\psi_m, \phi_n). \tag{1.4.5}$$

Subtract the latter two equations and note that their left-hand sides are equal because of the definition of the adjoint, to get

$$0 = (\lambda_n - \lambda_m)(\psi_m, \phi_n). \tag{1.4.6}$$

This implies

$$(\psi_m, \phi_n) = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m, \tag{1.4.7}$$

which proves the desired result. Also, since each ϕ_n and ψ_m is determined to within a multiplicative constant (e.g., if ϕ_n satisfies Eq. (1.4.1) so does $\alpha\phi_n$), the normalization for the latter can be chosen such that

$$(\psi_m, \phi_n) = \delta_{mn} = \begin{cases} 1, & \text{for } n = m, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4.8}$$

Now, if the set of eigenfunctions ϕ_n $(n = 1, 2, \ldots)$ forms a complete set, any arbitrary function $f(x)$ in the space may be expanded as

$$f(x) = \sum_n a_n \phi_n(x), \tag{1.4.9}$$

and to find the coefficients a_n, we simply take the inner product of both sides with ψ_k to get

$$\begin{aligned}(\psi_k, f) = \sum_n (\psi_k, a_n\phi_n) &= \sum_n a_n(\psi_k, \phi_n) \\ &= \sum_n a_n \delta_{kn} = a_k,\end{aligned}$$

i.e.,

$$a_n = (\psi_n, f), \quad (n = 1, 2, 3, \ldots). \tag{1.4.10}$$

Note the difference between Eqs. (1.4.9) and (1.4.10) and the corresponding formulas (1.2.4) and (1.2.6) for an orthonormal system of functions. In the present case, neither $\{\phi_n\}$ nor $\{\psi_n\}$ form an orthonormal system, but they are orthogonal to one another.

Proof that the eigenvalues of the adjoint matrix are complex conjugates of the eigenvalues of the original matrix.

Above, we claimed without justification that the eigenvalues of the adjoint of an operator are complex conjugates of those of the original operator. Here we show this for the matrix case. The eigenvalues of a matrix A are given by

$$\det(A - \lambda I) = 0. \tag{1.4.11}$$

The latter is the characteristic equation whose n solutions for λ are the desired eigenvalues. On the other hand, the eigenvalues of A^{adj} are determined by setting

$$\det(A^{\text{adj}} - \mu I) = 0. \tag{1.4.12}$$

Since the determinant of a matrix is equal to that of its transpose, we easily conclude that the eigenvalues of A^{adj} are the complex conjugates of λ_n. □

1.5 The Fredholm Alternative

The *Fredholm Alternative*, which may be also called the *Fredholm solvability condition*, is concerned with the existence of the solution $y(x)$ of the inhomogeneous problem

$$Ly(x) = f(x), \tag{1.5.1}$$

where L is a given linear operator and $f(x)$ a known forcing term. As usual, if L is a differential operator, additional boundary or initial conditions must also be specified.

The Fredholm Alternative *states that the unknown function* $y(x)$ *can be determined uniquely if the corresponding homogeneous problem*

$$L\phi_H(x) = 0 \tag{1.5.2}$$

with homogeneous boundary conditions, has no nontrivial solutions. On the other hand, if the homogeneous problem (1.5.2) *does possess a nontrivial solution, then the inhomogeneous problem* (1.5.1) *has either no solution or infinitely many solutions.*

What determines the latter is the homogeneous solution ψ_H to the adjoint problem

$$L^{\text{adj}}\psi_H = 0. \tag{1.5.3}$$

Taking the inner product of Eq. (1.5.1) with ψ_H from the left,

$$(\psi_H, Ly) = (\psi_H, f).$$

Then, by the definition of the adjoint operator (excluding the case wherein L is a differential operator, to be discussed in Section 1.7.), we have

$$(L^{\text{adj}}\psi_H, y) = (\psi_H, f).$$

The left-hand side of the equation above is zero by the definition of ψ_H, Eq. (1.5.3).

Thus the criteria for the solvability of the inhomogeneous problem Eq. (1.5.1) *is given by*

$$(\psi_H, f) = 0.$$

If these criteria are satisfied, there will be an infinity of solutions to Eq. (1.5.1)*, otherwise Eq.* (1.5.1) *will have no solution.*

To understand the above claims, let us suppose that L and L^{adj} possess complete sets of eigenfunctions satisfying

$$L\phi_n = \lambda_n\phi_n \quad (n = 0, 1, 2, \ldots), \tag{1.5.4a}$$

$$L^{\text{adj}}\psi_n = \lambda_n^*\psi_n \quad (n = 0, 1, 2, \ldots), \tag{1.5.4b}$$

with

$$(\psi_m, \phi_n) = \delta_{mn}. \tag{1.5.5}$$

The existence of a nontrivial homogeneous solution $\phi_H(x)$ to Eq. (1.5.2), as well as $\psi_H(x)$ to Eq. (1.5.3), is the same as having one of the eigenvalues λ_n in Eqs. (1.5.4a), (1.5.4b) to be zero. If this is the case, i.e., if zero is an eigenvalue of Eq. (1.5.4a) and hence Eq. (1.5.4b), we shall choose the subscript $n = 0$ to signify that eigenvalue ($\lambda_0 = 0$), and in that case

ϕ_0 and ψ_0 are the same as ϕ_H and ψ_H. The two circumstances in the Fredholm Alternative correspond to cases where zero is an eigenvalue of Eqs. (1.5.4a), (1.5.4b) and where it is not.

Let us proceed formally with the problem of solving the inhomogeneous problem Eq. (1.5.1). Since the set of eigenfunctions ϕ_n of Eq. (1.5.4a) is assumed to be complete, both the known function $f(x)$ and the unknown function $y(x)$ in Eq. (1.5.1) can presumably be expanded in terms of $\phi_n(x)$:

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \phi_n(x), \tag{1.5.6}$$

$$y(x) = \sum_{n=0}^{\infty} \beta_n \phi_n(x), \tag{1.5.7}$$

where the α_n are known (since $f(x)$ is known), i.e., according to Eq. (1.4.10)

$$\alpha_n = (\psi_n, f), \tag{1.5.8}$$

while the β_n are unknown. Thus, if all the β_n can be determined, then the solution $y(x)$ to Eq. (1.5.1) is regarded as having been found.

To try to determine the β_n, substitute both Eqs. (1.5.6) and (1.5.7) into Eq. (1.5.1) to find

$$\sum_{n=0}^{\infty} \lambda_n \beta_n \phi_n = \sum_{k=0}^{\infty} \alpha_k \phi_k, \tag{1.5.9}$$

where different summation indices have been used on the two sides to remind the reader that the latter are dummy indices of summation. Next, take the inner product of both sides with ψ_m (with an index which must be different from the two above) to get

$$\sum_{n=0}^{\infty} \lambda_n \beta_n (\psi_m, \phi_n) = \sum_{k=0}^{\infty} \alpha_k (\psi_m, \phi_k),$$

or

$$\sum_{n=0}^{\infty} \lambda_n \beta_n \delta_{mn} = \sum_{k=0}^{\infty} \alpha_k \delta_{mk},$$

i.e.,

$$\lambda_m \beta_m = \alpha_m. \tag{1.5.10}$$

Thus, for any $m = 0, 1, 2, \ldots$, we can solve Eq. (1.5.10) for the unknowns β_m to get

$$\beta_n = \alpha_n / \lambda_n \quad (n = 0, 1, 2, \ldots), \tag{1.5.11}$$

provided that λ_n is not equal to zero. Obviously the only possible difficulty occurs if one of the eigenvalues (which we take to be λ_0) is equal to zero. In that case, equation (1.5.10) with $m = 0$ reads

$$\lambda_0 \beta_0 = \alpha_0 \quad (\lambda_0 = 0). \tag{1.5.12}$$

Now if $\alpha_0 \neq 0$, then we cannot solve for β_0 and thus the problem $Ly = f$ has no solution. On the other hand if $\alpha_0 = 0$, i.e., if

$$(\psi_0, f) = (\psi_H, f) = 0, \tag{1.5.13}$$

implying that f is orthogonal to the homogeneous solution to the adjoint problem, then Eq. (1.5.12) is satisfied by any choice of β_0. All the other β_n ($n = 1, 2, \ldots$) are uniquely determined but there are infinitely many solutions $y(x)$ to Eq. (1.5.1) corresponding to the infinitely many values possible for β_0. The reader must make certain that he or she understands the equivalence of the above with the original statement of the Fredholm Alternative.

1.6 Self-adjoint Operators

Operators which are self-adjoint or Hermitian form a very useful class of operators. They possess a number of special properties, some of which are described in this section.

The first important property of self-adjoint operators is that their *eigenvalues are real*. To prove this, begin with

$$\begin{aligned} L\phi_n &= \lambda_n \phi_n, \\ L\phi_m &= \lambda_m \phi_m, \end{aligned} \tag{1.6.1}$$

and take the inner product of both sides of the former with ϕ_m from the left, and the latter with ϕ_n from the right, to obtain

$$\begin{aligned} (\phi_m, L\phi_n) &= \lambda_n (\phi_m, \phi_n), \\ (L\phi_m, \phi_n) &= \lambda_m^* (\phi_m, \phi_n). \end{aligned} \tag{1.6.2}$$

For a self-adjoint operator $L = L^{\text{adj}}$, the two left-hand sides of Eq. (1.6.2) are equal and hence, upon subtraction of the latter from the former, we find

$$0 = (\lambda_n - \lambda_m^*)(\phi_m, \phi_n). \tag{1.6.3}$$

Now, if $m = n$, the inner product $(\phi_n, \phi_n) = \|\phi_n\|^2$ is nonzero and Eq. (1.6.3) implies

$$\lambda_n = \lambda_n^*, \tag{1.6.4}$$

proving that all the eigenvalues are real. Thus Eq. (1.6.3) can be rewritten as

$$0 = (\lambda_n - \lambda_m)(\phi_m, \phi_n), \tag{1.6.5}$$

indicating that if $\lambda_n \neq \lambda_m$, then the eigenfunctions ϕ_m and ϕ_n are orthogonal. Thus, upon normalizing each ϕ_n, we verify a second important property of self-adjoint operators that (upon normalization) the *eigenfunctions of a self-adjoint operator form an orthonormal set*.

The Fredholm Alternative can also be restated for a self-adjoint operator L in the following form: The inhomogeneous problem $Ly = f$ (with L self-adjoint) is solvable for y, if f is orthogonal to all eigenfunctions ϕ_0 of L with eigenvalue zero (if indeed any exist). If zero is not an eigenvalue of L, the solution is unique. Otherwise, there is no solution if $(\phi_0, f) \neq 0$, and an infinite number of solutions if $(\phi_0, f) = 0$.

Diagonalization of Self-adjoint Operators: Any linear operator can be expanded in some sense in terms of any orthonormal basis set. To elaborate on this, suppose that the orthonormal system $\{e_i(x)\}_i$, with $(e_i, e_j) = \delta_{ij}$ forms a complete set. Any function $f(x)$ can be expanded as

$$f(x) = \sum_{j=1}^{\infty} \alpha_j e_j(x), \quad \alpha_j = (e_j, f). \tag{1.6.6}$$

Thus the function $f(x)$ can be thought of as an infinite dimensional vector with components α_j. Now consider the action of an arbitrary linear operator L on the function $f(x)$. Obviously

$$Lf(x) = \sum_{j=1}^{\infty} \alpha_j L e_j(x). \tag{1.6.7}$$

But L acting on $e_j(x)$ is itself a function of x which can be expanded in the orthonormal basis $\{e_i(x)\}_i$. Thus we write

$$Le_j(x) = \sum_{i=1}^{\infty} l_{ij} e_i(x), \tag{1.6.8}$$

wherein the coefficients l_{ij} of the expansion are found to be $l_{ij} = (e_i, Le_j)$. Substitution of Eq. (1.6.8) into Eq. (1.6.7) then shows

$$Lf(x) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} l_{ij} \alpha_j \right) e_i(x). \tag{1.6.9}$$

We discover that just as we can think of $f(x)$ as the infinite dimensional vector with components α_j, we can consider L to be equivalent to an infinite dimensional matrix with components l_{ij}, and we can regard Eq. (1.6.9) as a regular multiplication of the matrix L (components l_{ij}) with the vector f (components α_j). However, this equivalence of the operator L with the matrix whose components are l_{ij}, i.e., $L \Leftrightarrow l_{ij}$, depends on the choice of the orthonormal set.

For a self-adjoint operator $L = L^{\text{adj}}$, the most natural choice of the basis set is the set of eigenfunctions of L. Denoting these by $\{\phi_i(x)\}_i$, the components of the equivalent matrix for L take the form

$$l_{ij} = (\phi_i, L\phi_j) = (\phi_i, \lambda_j \phi_j) = \lambda_j (\phi_i, \phi_j) = \lambda_j \delta_{ij}. \tag{1.6.10}$$

1.7 Green's Functions for Differential Equations

In this section, we describe the conceptual basis of the theory of *Green's functions*. We do this by first outlining the abstract themes involved and then by presenting a simple example. More complicated examples will appear in later chapters.

Prior to discussing Green's functions, recall some of the elementary properties of the so-called Dirac delta function $\delta(x - x')$. In particular, remember that if x' is inside the domain of integration (a, b), for any well-behaved function $f(x)$, we have

$$\int_a^b \delta(x - x')f(x)\,dx = f(x'), \tag{1.7.1}$$

which can be written as

$$(\delta(x - x'), f(x)) = f(x'), \tag{1.7.2}$$

with the inner product taken with respect to x. Also remember that $\delta(x - x')$ is equal to zero for any $x \neq x'$.

Suppose now that we wish to solve a differential equation

$$Lu(x) = f(x), \tag{1.7.3}$$

on the domain $x \in (a, b)$ and subject to given boundary conditions, with L a differential operator. Consider what happens when a function $g(x, x')$ (which is as yet unknown but will end up being the Green's function) is multiplied on both sides of Eq. (1.7.3) followed by integration of both sides with respect to x from a to b. That is, consider taking the inner product of both sides of Eq. (1.7.3) with $g(x, x')$ with respect to x. (We suppose everything is real in this section so that no complex conjugation is necessary.) This yields

$$(g(x, x'), Lu(x)) = (g(x, x'), f(x)). \tag{1.7.4}$$

Now by definition of the adjoint L^{adj} of L, the left-hand side of Eq. (1.7.4) can be written as

$$(g(x, x'), Lu(x)) = (L^{\text{adj}}g(x, x'), u(x)) + \text{boundary terms}, \tag{1.7.5}$$

in which, for the first time, we explicitly recognize the terms involving the boundary points which arise when L is a differential operator. The *boundary terms* on the right-hand side of Eq. (1.7.5) emerge when we integrate by parts. It is difficult to be more specific than this when we work in the abstract, but our example should clarify what we mean shortly. If Eq. (1.7.5) is substituted back into Eq. (1.7.4), it provides

$$(L^{\text{adj}}g(x, x'), u(x)) = (g(x, x'), f(x)) + \text{boundary terms}. \tag{1.7.6}$$

So far we have not discussed what function $g(x, x')$ to choose. Suppose we choose that $g(x, x')$ which satisfies

$$L^{\text{adj}}g(x, x') = \delta(x - x'), \tag{1.7.7}$$

subject to appropriately selected boundary conditions which eliminate all the unknown terms within the boundary terms. This function $g(x, x')$ is known as Green's function. Substituting Eq. (1.7.7) into Eq. (1.7.6) and using property (1.7.2) then yields

$$u(x') = (g(x, x'), f(x)) + \text{known boundary terms}, \tag{1.7.8}$$

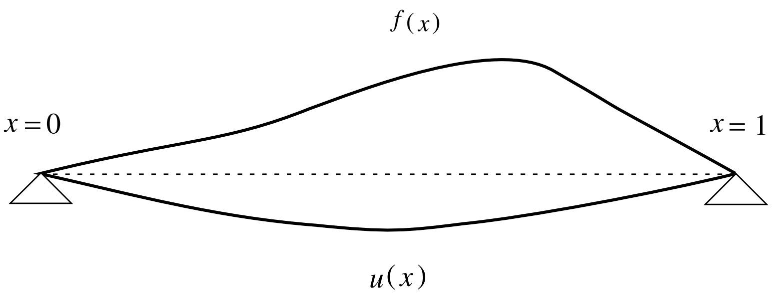

Fig. 1.1: Displacement $u(x)$ of a taut string under the distributed load $f(x)$ with $x \in (0,1)$.

which is the solution to the differential equation since everything on the right-hand side is known once $g(x,x')$ has been found. More accurately, if we change x' to x in the above and use a different dummy variable ξ of integration in the inner product, we have

$$u(x) = \int_a^b g(\xi,x) f(\xi)\, d\xi + \text{known boundary terms.} \tag{1.7.9}$$

In summary, to solve the linear inhomogeneous differential equation

$$Lu(x) = f(x)$$

using Green's function, we first solve the equation

$$L^{\text{adj}} g(x,x') = \delta(x - x')$$

for Green's function $g(x,x')$, subject to the appropriately selected boundary conditions, and immediately obtain the solution to our differential equation given by Eq. (1.7.9).

The above will we hope become more clear in the context of the following simple example.

❑ **Example 1.1.** Consider the problem of finding the displacement $u(x)$ of a taut string under the *distributed load* $f(x)$ as in Figure 1.1.

Solution. The governing ordinary differential equation for the vertical displacement $u(x)$ has the form

$$\frac{d^2u}{dx^2} = f(x) \quad \text{for} \quad x \in (0,1) \tag{1.7.10}$$

subject to boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0. \tag{1.7.11}$$

To proceed formally, multiply both sides of Eq. (1.7.10) by $g(x,x')$ and integrate from 0 to 1 with respect to x to find

$$\int_0^1 g(x,x') \frac{d^2u}{dx^2}\, dx = \int_0^1 g(x,x') f(x)\, dx.$$

Integrate the left-hand side by parts twice to obtain

$$\int_0^1 \frac{d^2}{dx^2} g(x,x')u(x)\,dx$$
$$+ \left[g(1,x')\frac{du}{dx}\bigg|_{x=1} - g(0,x')\frac{du}{dx}\bigg|_{x=0} - u(1)\frac{dg(1,x')}{dx} + u(0)\frac{dg(0,x')}{dx} \right]$$
$$= \int_0^1 g(x,x')f(x)\,dx. \tag{1.7.12}$$

The terms contained within the square brackets on the left-hand side of (1.7.12) are the boundary terms. In consequence of the boundary conditions (1.7.11), the last two terms therein vanish. Hence a prudent choice of boundary conditions for $g(x,x')$ would be to set

$$g(0,x') = 0 \quad \text{and} \quad g(1,x') = 0. \tag{1.7.13}$$

With that choice, all the boundary terms vanish (this does not necessarily happen for other problems). Now suppose that $g(x,x')$ satisfies

$$\frac{d^2 g(x,x')}{dx^2} = \delta(x - x'), \tag{1.7.14}$$

subject to the boundary conditions (1.7.13). Use of Eqs. (1.7.14) and (1.7.13) in Eq. (1.7.12) yields

$$u(x') = \int_0^1 g(x,x')f(x)\,dx, \tag{1.7.15}$$

as our solution, once $g(x,x')$ has been obtained. Note that, if the original differential operator d^2/dx^2 is denoted by L, its adjoint L^{adj} is also d^2/dx^2 as found by twice integrating by parts. Hence the latter operator is indeed self-adjoint.

The last step involves the actual solution of (1.7.14) subject to (1.7.13). The variable x' plays the role of a parameter throughout. With x' somewhere between 0 and 1, Eq. (1.7.14) can actually be solved separately in each domain $0 < x < x'$ and $x' < x < 1$. For each of these, we have

$$\frac{d^2 g(x,x')}{dx^2} = 0 \quad \text{for} \quad 0 < x < x', \tag{1.7.16a}$$

$$\frac{d^2 g(x,x')}{dx^2} = 0 \quad \text{for} \quad x' < x < 1. \tag{1.7.16b}$$

The general solution in each subdomain is easily written down as

$$g(x,x') = Ax + B \quad \text{for} \quad 0 < x < x', \tag{1.7.17a}$$

$$g(x,x') = Cx + D \quad \text{for} \quad x' < x < 1, \tag{1.7.17b}$$

involving the four unknown constants A, B, C and D. Two relations for the constants are found using the two boundary conditions (1.7.13). In particular, we have

$$g(0,x') = 0 \to B = 0, \tag{1.7.18a}$$

$$g(1,x') = 0 \to C + D = 0. \tag{1.7.18b}$$

To provide two more relations which are needed to permit all four of the constants to be determined, we return to the governing equation (1.7.14). Integrate both sides of the latter with respect to x from $x' - \varepsilon$ to $x' + \varepsilon$ and take the limit as $\varepsilon \to 0$ to find

$$\lim_{\varepsilon\to 0}\int_{x'-\varepsilon}^{x'+\varepsilon} \frac{d^2 g(x,x')}{dx^2}\, dx = \lim_{\varepsilon\to 0}\int_{x'-\varepsilon}^{x'+\varepsilon} \delta(x-x')\, dx,$$

from which, we obtain

$$\left.\frac{dg(x,x')}{dx}\right|_{x=x'^{+}} - \left.\frac{dg(x,x')}{dx}\right|_{x=x'^{-}} = 1. \tag{1.7.19}$$

Thus the first derivative of $g(x,x')$ undergoes a jump discontinuity as x passes through x'. But we can expect $g(x,x')$ itself to be continuous across x', i.e.,

$$\left. g(x,x')\right|_{x=x'^{+}} = \left. g(x,x')\right|_{x=x'^{-}}. \tag{1.7.20}$$

In the above, x'^{+} and x'^{-} denote points infinitesimally to the right and the left of x', respectively. Using solutions (1.7.17a) and (1.7.17b) for $g(x,x')$ in each subdomain, we find that Eqs. (1.7.19) and (1.7.20), respectively, imply

$$C - A = 1, \tag{1.7.21a}$$

$$Cx' + D = Ax' + B. \tag{1.7.21b}$$

Equations (1.7.18a), (1.7.18b) and (1.7.21a), (1.7.21b) can be used to solve for the four constants A, B, C and D to yield

$$A = x' - 1, \quad B = 0, \quad C = x', \quad D = -x',$$

from whence our solution (1.7.17) takes the form

$$g(x,x') = \begin{cases} (x'-1)x & \text{for} \quad x < x', \\ x'(x-1) & \text{for} \quad x > x', \end{cases} \tag{1.7.22a}$$

$$= x_<(x_> - 1) \quad \text{for} \quad \begin{cases} x_< &= \frac{(x+x')}{2} - \frac{\left|x-x'\right|}{2}, \\ x_> &= \frac{(x+x')}{2} + \frac{\left|x-x'\right|}{2}. \end{cases} \tag{1.7.22b}$$

Physically the Green's function (1.7.22) represents the displacement of the string subject to a *concentrated load* $\delta(x-x')$ at $x = x'$ as in Figure 1.2. For this reason, it is also called the *influence function*.

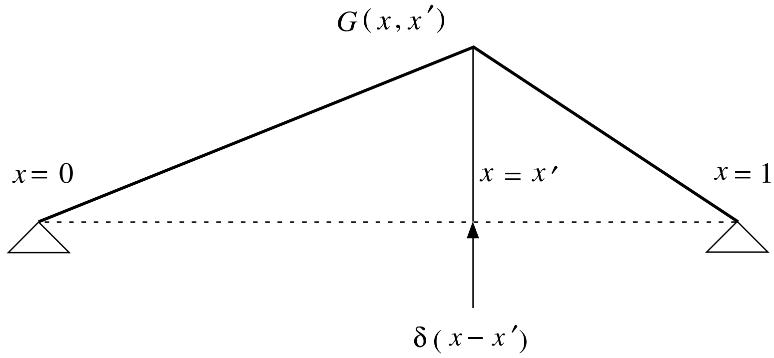

Fig. 1.2: Displacement $u(x)$ of a taut string under the concentrated load $\delta(x - x')$ at $x = x'$.

Having found the influence function above for a concentrated load, the solution with any given distributed load $f(x)$ is given by Eq. (1.7.15) as

$$\begin{aligned} u(x) &= \int_0^1 g(\xi, x) f(\xi)\, d\xi \\ &= \int_0^x (x-1)\xi f(\xi)\, d\xi + \int_x^1 x(\xi - 1) f(\xi)\, d\xi \\ &= (x-1)\int_0^x \xi f(\xi)\, d\xi + x\int_x^1 (\xi-1) f(\xi)\, d\xi. \end{aligned} \tag{1.7.23}$$

Although this example has been rather elementary, we hope it has provided the reader with a basic understanding of what the Green's function is. More complex, and hence more interesting examples, are encountered in later chapters.

1.8 Review of Complex Analysis

Let us review some important results from complex analysis.

Cauchy Integral Formula. *Let $f(z)$ be analytic on and inside the closed, positively oriented contour C. Then we have*

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.8.1}$$

Differentiate this formula with respect to z to obtain

$$\frac{d}{dz} f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta, \quad \text{and} \quad \left(\frac{d}{dz}\right)^n f(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta. \tag{1.8.2}$$

Liouville's theorem. *The only entire functions which are bounded (at infinity) are constants.*

Proof. Suppose that $f(z)$ is entire. Then it can be represented by the Taylor series,

$$f(z) = f(0) + f^{(1)}(0)z + \frac{1}{2!}f^{(2)}(0)z^2 + \cdots .$$

Now consider $f^{(n)}(0)$. By the Cauchy Integral Formula, we have

$$f^{(n)}(0) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta^{n+1}}d\zeta.$$

Since $f(\zeta)$ is bounded, we have

$$|f(\zeta)| \leq M.$$

Consider C to be a circle of radius R, centered at the origin. Then we have

$$\left|f^{(n)}(0)\right| \leq \frac{n!}{2\pi}\cdot\frac{2\pi RM}{R^{n+1}} = n!\cdot\frac{M}{R^n} \to 0 \quad \text{as} \quad R\to\infty.$$

Thus

$$f^{(n)}(0) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots .$$

Hence

$$f(z) = \text{constant},$$

completing the proof. □

More generally,

(i) *Suppose that $f(z)$ is entire and we know $|f(z)| \leq |z|^a$ as $R\to\infty$, with $0 < a < 1$. We still find $f(z) = $ constant.*

(ii) *Suppose that $f(z)$ is entire and we know $|f(z)| \leq |z|^a$ as $R\to\infty$, with $n-1 \leq a < n$. Then $f(z)$ is at most a polynomial of degree $n-1$.*

Discontinuity theorem. *Suppose that $f(z)$ has a branch cut on the real axis from a to b. It has no other singularities and it vanishes at infinity. If we know the difference between the value of $f(z)$ above and below the cut,*

$$D(x) \equiv f(x+i\varepsilon) - f(x-i\varepsilon), \quad (a \leq x \leq b), \tag{1.8.3}$$

with ε positive infinitesimal, then

$$f(z) = \frac{1}{2\pi i}\int_a^b \frac{D(x)}{(x-z)}\,dx. \tag{1.8.4}$$

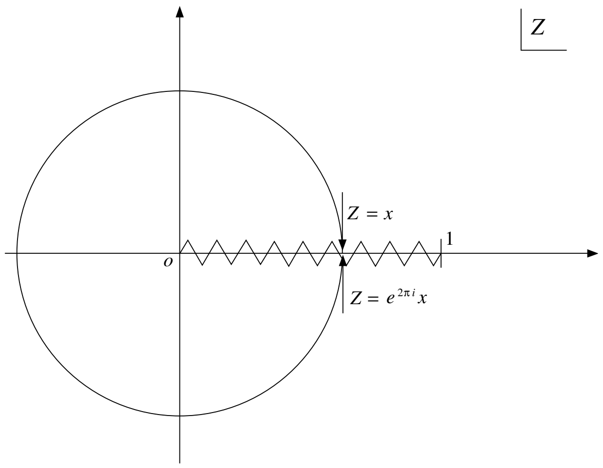

Fig. 1.3: The contours of the integration for $f(z)$. C_R is the circle of radius R centered at the origin.

Proof. By the Cauchy Integral Formula, we know

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where Γ consists of the following pieces (see Figure 1.3),

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + C_R.$$

The contribution from C_R vanishes since $|f(z)| \to 0$ as $R \to \infty$. Contributions from Γ_3 and Γ_4 cancel each other. Hence we have

$$f(z) = \frac{1}{2\pi i}\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right) \frac{f(\zeta)}{\zeta - z} d\zeta.$$

On Γ_1, we have

$$\zeta = x + i\varepsilon \quad \text{with} \quad x : a \to b, \quad f(\zeta) = f(x + i\varepsilon),$$

$$\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_a^b \frac{f(x + i\varepsilon)}{x - z + i\varepsilon}\, dx \to \int_a^b \frac{f(x + i\varepsilon)}{x - z}\, dx \quad \text{as} \quad \varepsilon \to 0^+.$$

On Γ_2, we have

$$\zeta = x - i\varepsilon \quad \text{with} \quad x : b \to a, \quad f(\zeta) = f(x - i\varepsilon),$$

$$\int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_b^a \frac{f(x - i\varepsilon)}{x - z - i\varepsilon}\, dx \to -\int_a^b \frac{f(x - i\varepsilon)}{x - z}\, dx \quad \text{as} \quad \varepsilon \to 0^+.$$

Thus we obtain

$$f(z) = \frac{1}{2\pi i}\int_a^b \frac{f(x + i\varepsilon) - f(x - i\varepsilon)}{x - z}\, dx = \frac{1}{2\pi i}\int_a^b \frac{D(x)}{x - z}\, dx,$$

completing the proof. □

If, in addition, $f(z)$ is known to have other singularities elsewhere, or may possibly be nonzero as $|z| \to \infty$, then it is of the form

$$f(z) = \frac{1}{2\pi i} \int_a^b \frac{D(x)}{x-z}\, dx + g(z), \tag{1.8.5}$$

with $g(z)$ free of cut on $[a, b]$. This is a very important result. Memorizing it will give a better understanding of the subsequent sections.

Behavior near the endpoints. *Consider the case when z is in the vicinity of the endpoint a. The behavior of $f(z)$ as $z \to a$ is related to the form of $D(x)$ as $x \to a$. Suppose that $D(x)$ is finite at $x = a$, say $D(a)$. Then we have*

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_a^b \frac{D(a) + D(x) - D(a)}{x-z}\, dx \\ &= \frac{D(a)}{2\pi i} \ln\left(\frac{b-z}{a-z}\right) + \frac{1}{2\pi i} \int_a^b \frac{D(x) - D(a)}{x-z}\, dx. \end{aligned} \tag{1.8.6}$$

The second integral above converges as $z \to a$ as long as $D(x)$ satisfies a *Hölder condition* (which is implicitly assumed) requiring

$$|D(x) - D(a)| < A\,|x-a|^{\mu}, \quad A, \mu > 0. \tag{1.8.7}$$

Thus the endpoint behavior of $f(z)$ as $z \to a$ is of the form

$$f(z) = O\left(\ln(a-z)\right) \quad \text{as} \quad z \to a, \tag{1.8.8}$$

if

$$D(x) \quad \text{finite} \quad \text{as} \quad x \to a. \tag{1.8.9}$$

Another possibility is for $D(x)$ to be of the form

$$D(x) \to 1/(x-a)^{\alpha} \quad \text{with} \quad \alpha < 1 \quad \text{as} \quad x \to a, \tag{1.8.10}$$

since, even with such a singularity in $D(x)$, the integral defining $f(z)$ is well-defined. We claim that in that case, $f(z)$ also behaves as

$$f(z) = O\left(1/(z-a)^{\alpha}\right) \quad \text{as} \quad z \to a, \quad \text{with} \quad \alpha < 1, \tag{1.8.11}$$

that is, $f(z)$ is less singular than a simple pole.

Proof of the claim. Using the Cauchy Integral Formula, we have

$$1/\left(z-a\right)^{\alpha} = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\zeta}{\left(\zeta-a\right)^{\alpha}\left(\zeta-z\right)},$$

where Γ consists of the following paths (see Figure 1.4)

$$\Gamma = \Gamma_1 + \Gamma_2 + C_R.$$

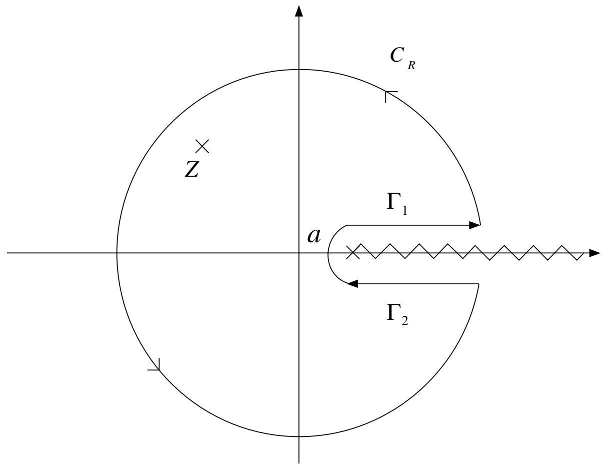

Fig. 1.4: The contour Γ of the integration for $1/(z-a)^\alpha$.

The contribution from C_R vanishes as $R \to \infty$.
On Γ_1, we set

$$\zeta - a = r, \quad \text{and} \quad (\zeta - a)^\alpha = r^\alpha,$$

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{d\zeta}{(\zeta - a)^\alpha(\zeta - z)} = \frac{1}{2\pi i}\int_0^{+\infty} \frac{dr}{r^\alpha(r + a - z)}.$$

On Γ_2, we set

$$\zeta - a = re^{2\pi i}, \quad \text{and} \quad (\zeta - a)^\alpha = r^\alpha e^{2\pi i\alpha},$$

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{d\zeta}{(\zeta - a)^\alpha(\zeta - z)} = \frac{e^{-2\pi i\alpha}}{2\pi i}\int_{+\infty}^0 \frac{dr}{r^\alpha(r + a - z)}.$$

Thus we obtain

$$1/(z-a)^\alpha = \frac{1 - e^{-2\pi i\alpha}}{2\pi i}\int_a^{+\infty} \frac{dx}{(x-a)^\alpha(x-z)},$$

which may be written as

$$1/(z-a)^\alpha = \frac{1 - e^{-2\pi i\alpha}}{2\pi i}\left[\int_a^b \frac{dx}{(x-a)^\alpha(x-z)} + \int_b^{+\infty} \frac{dx}{(x-a)^\alpha(x-z)}\right].$$

The second integral above is convergent for z close to a. Obviously then, we have

$$\frac{1}{2\pi i}\int_a^b \frac{dx}{(x-a)^\alpha(x-z)} = O\left(\frac{1}{(z-a)^\alpha}\right) \quad \text{as} \quad z \to a.$$

A similar analysis can be done as $z \to b$, completing the proof. □

Summary of behavior near the endpoints

$$f(z) = \frac{1}{2\pi i}\int_a^b \frac{D(x)dx}{x-z},$$

$$\begin{cases} \text{if } D(x\to a) = D(a), & & \text{then} \quad f(z) = O(\ln(a-z)), \\ \text{if } D(x\to a) = \frac{1}{(x-a)^\alpha}, & (0<\alpha<1), & \text{then} \quad f(z) = O\left(\frac{1}{(z-a)^\alpha}\right), \end{cases} \tag{1.8.12a}$$

$$\begin{cases} \text{if } D(x\to b) = D(b), & & \text{then} \quad f(z) = O(\ln(b-z)), \\ \text{if } D(x\to b) = \frac{1}{(x-b)^\beta}, & (0<\beta<1), & \text{then} \quad f(z) = O\left(\frac{1}{(z-b)^\beta}\right). \end{cases} \tag{1.8.12b}$$

Principal Value Integrals. We define the principal value integral by

$$\mathrm{P}\int_a^b \frac{f(x)}{x-y}\,dx \equiv \lim_{\varepsilon\to 0^+}\left[\int_a^{y-\varepsilon}\frac{f(x)}{x-y}\,dx + \int_{y+\varepsilon}^b \frac{f(x)}{x-y}\,dx\right]. \tag{1.8.13}$$

Graphically expressed, the principal value integral contour is as in Figure 1.5. As such, to evaluate a principal value integral by doing complex integration, we usually make use of either of the two contours as in Figure 1.6.

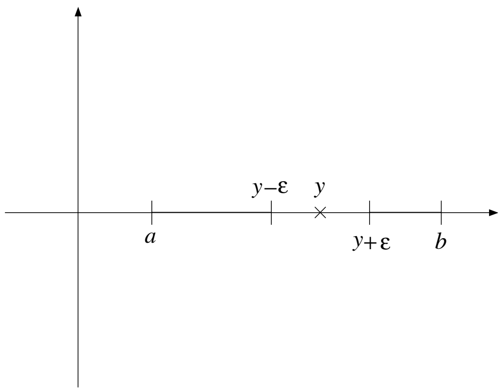

Fig. 1.5: The principal value integral contour.

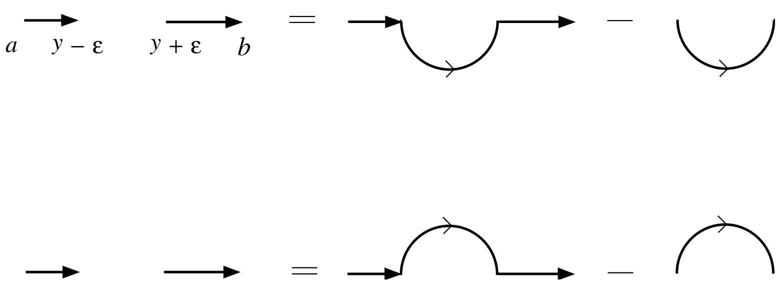

Fig. 1.6: Two contours for the principal value integral (1.8.13).

Now, the contour integrals on the right of Figure 1.6 are usually possible and hence the principal value integral can be evaluated. Also, the contributions from the lower semicircle C_- and the upper semicircle C_+ take the forms,

$$\int_{C_-} \frac{f(z)}{z-y}\, dz = i\pi f(y), \qquad \int_{C_+} \frac{f(z)}{z-y}\, dz = -i\pi f(y),$$

as $\varepsilon \to 0^+$, as long as $f(z)$ is not singular at y. Mathematically expressed, the principal value integral is given by either of the following two formulae, known as the *Plemelj formula*,

$$\frac{1}{2\pi i} \mathrm{P} \int_a^b \frac{f(x)}{x-y}\, dx = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \frac{f(x)}{x-y-i\varepsilon}\, dx - \frac{1}{2} f(y), \tag{1.8.14a}$$

$$\frac{1}{2\pi i} \mathrm{P} \int_a^b \frac{f(x)}{x-y}\, dx = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \frac{f(x)}{x-y+i\varepsilon}\, dx + \frac{1}{2} f(y). \tag{1.8.14b}$$

These are customarily written as

$$\lim_{\varepsilon \to 0^+} \frac{1}{x-y \mp i\varepsilon} = \mathrm{P}\left(\frac{1}{x-y}\right) \pm i\pi\delta(x-y), \tag{1.8.15a}$$

or may equivalently be written as

$$\mathrm{P}\left(\frac{1}{x-y}\right) = \lim_{\varepsilon \to 0^+} \frac{1}{x-y \mp i\varepsilon} \mp i\pi\delta(x-y). \tag{1.8.15b}$$

Then we interchange the order of the limit $\varepsilon \to 0^+$ and the integration over x. The principal value integrand seems to diverge at $x = y$, but it is actually finite at $x = y$ as long as $f(x)$ is not singular at $x = y$. This comes about as follows;

$$\begin{aligned} \frac{1}{x-y \mp i\varepsilon} &= \frac{(x-y) \pm i\varepsilon}{(x-y)^2 + \varepsilon^2} = \frac{(x-y)}{(x-y)^2 + \varepsilon^2} \pm i\pi \cdot \frac{1}{\pi} \frac{\varepsilon}{(x-y)^2 + \varepsilon^2} \\ &= \frac{(x-y)}{(x-y)^2 + \varepsilon^2} \pm i\pi\delta_\varepsilon(x-y), \end{aligned} \tag{1.8.16}$$

where $\delta_\varepsilon(x-y)$ is defined by

$$\delta_\varepsilon(x-y) \equiv \frac{1}{\pi} \frac{\varepsilon}{(x-y)^2 + \varepsilon^2}, \tag{1.8.17}$$

with the following properties,

$$\delta_\varepsilon(x \neq y) \to 0^+ \quad \text{as} \quad \varepsilon \to 0^+,$$

$$\delta_\varepsilon(x = y) = \frac{1}{\pi}\frac{1}{\varepsilon} \to +\infty \quad \text{as} \quad \varepsilon \to 0^+,$$

and

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(x-y)\, dx = 1.$$

The first term on the right-hand side of Eq. (1.8.16) vanishes at $x = y$ before we take the limit $\varepsilon \to 0^+$, while the second term $\delta_\varepsilon(x-y)$ approaches the Dirac delta function, $\delta(x-y)$, as $\varepsilon \to 0^+$. This is the content of Eq. (1.8.15a).

1.9 Review of Fourier Transform

The Fourier transform of a function $f(x)$, where $-\infty < x < \infty$, is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \exp[-ikx] f(x). \tag{1.9.1}$$

There are two distinct theories of the Fourier transforms.

(I) Fourier transform of square-integrable functions

It is assumed that

$$\int_{-\infty}^{\infty} dx \left| f(x) \right|^2 < \infty. \tag{1.9.2}$$

The inverse Fourier transform is given by

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp[ikx] \tilde{f}(k). \tag{1.9.3}$$

We note that, in this case, $\tilde{f}(k)$ is defined for real k. Accordingly, the inversion path in Eq. (1.9.3) coincides with the entire real axis. It should be borne in mind that Eq. (1.9.1) is meaningful in the sense of the convergence in the mean, namely, Eq. (1.9.1) means that there exists $\tilde{f}(k)$ for all real k such that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} dk \left| \tilde{f}(k) - \int_{-R}^{R} dx \exp[-ikx] f(x) \right|^2 = 0. \tag{1.9.4}$$

Symbolically we write

$$\tilde{f}(k) = \text{l.i.m.}_{R\to\infty} \int_{-R}^{R} dx \exp[-ikx] f(x). \tag{1.9.5}$$

Similarly in Eq. (1.9.3), we mean that, given $\tilde{f}(k)$, there exists an $f(x)$ such that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} dx \left| f(f) - \int_{-R}^{R} \frac{dk}{2\pi} \exp[ikx] \tilde{f}(k) \right|^2 = 0. \tag{1.9.6}$$

We can then prove that

$$\int_{-\infty}^{\infty} dk \left| \tilde{f}(k) \right|^2 = 2\pi \int_{-\infty}^{\infty} dx \left| f(x) \right|^2, \tag{1.9.7}$$

which is Parseval's identity for the *square-integrable* functions. We see that the pair $(f(x), \tilde{f}(k))$ defined in this way, consists of two functions with very similar properties. We shall find that this situation may change drastically if the condition (1.9.2) is relaxed.

(II) Fourier transform of integrable functions

We relax the condition on the function $f(x)$ as

$$\int_{-\infty}^{\infty} dx\, |f(x)| < \infty. \tag{1.9.8}$$

Then we can still define $\tilde{f}(k)$ for real k. Indeed, from Eq. (1.9.1), we obtain

$$\begin{aligned} \left|\tilde{f}(k\text{: real})\right| &= \left|\int_{-\infty}^{\infty} dx \exp[-ikx] f(x)\right| \\ &\leq \int_{-\infty}^{\infty} dx\, |\exp[-ikx] f(x)| = \int_{-\infty}^{\infty} dx\, |f(x)| < \infty. \end{aligned} \tag{1.9.9}$$

We can further show that the function defined by

$$\tilde{f}_{+}(k) = \int_{-\infty}^{0} dx \exp[-ikx] f(x) \tag{1.9.10}$$

is analytic in the *upper* half-plane of the complex k plane, and

$$\tilde{f}_{+}(k) \to 0 \quad \text{as} \quad |k| \to \infty \quad \text{with} \quad \operatorname{Im} k > 0. \tag{1.9.11}$$

Similarly, we can show that the function defined by

$$\tilde{f}_{-}(k) = \int_{0}^{\infty} dx \exp[-ikx] f(x) \tag{1.9.12}$$

is analytic in the *lower* half-plane of the complex k plane, and

$$\tilde{f}_{-}(k) \to 0 \quad \text{as} \quad |k| \to \infty \quad \text{with} \quad \operatorname{Im} k < 0. \tag{1.9.13}$$

Clearly we have

$$\tilde{f}(k) = \tilde{f}_{+}(k) + \tilde{f}_{-}(k), \quad k\text{: real}. \tag{1.9.14}$$

We can show that

$$\tilde{f}(k) \to 0 \quad \text{as} \quad k \to \pm\infty, \quad k\text{: real}. \tag{1.9.15}$$

This is a property in common with the Fourier transform of the *square-integrable* functions.

❑ **Example 1.2.** Find the Fourier transform of the following function,

$$f(x) = \frac{\sin(ax)}{x}, \quad a > 0, \quad -\infty < x < \infty. \tag{1.9.16}$$

Solution. The Fourier transform $\tilde{f}(k)$ is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \exp[ikx] \frac{\sin(ax)}{x} = \int_{-\infty}^{\infty} dx \exp[ikx] \frac{\exp[iax] - \exp[-iax]}{2ix}$$
$$= \int_{-\infty}^{\infty} dx \frac{\exp[i(k+a)x] - \exp[i(k-a)x]}{2ix} = I(k+a) - I(k-a),$$

where we define the integral $I(b)$ by

$$I(b) \equiv \int_{-\infty}^{\infty} dx \frac{\exp[ibx]}{2ix} = \int_{\Gamma} dx \frac{\exp[ibx]}{2ix}.$$

The contour Γ extends from $x = -\infty$ to $x = \infty$ with the infinitesimal indent below the real x-axis at the pole $x = 0$. Noting $x = \mathrm{Re}\, x + i\, \mathrm{Im}\, x$ for the complex x, we have

$$I(b) = \begin{cases} 2\pi i \cdot \mathrm{Res} \left[\frac{\exp[ibx]}{2ix} \right]_{x=0} = \pi, & b > 0, \\ 0, & b < 0. \end{cases}$$

Thus we have

$$\tilde{f}(k) = I(k+a) - I(k-a)$$
$$= \int_{-\infty}^{\infty} dx \exp[ikx] \frac{\sin(ax)}{x} = \begin{cases} \pi & \text{for} \quad |k| < a, \\ 0 & \text{for} \quad |k| > a, \end{cases} \tag{1.9.17}$$

while at $k = \pm a$, we have

$$\tilde{f}(k = \pm a) = \frac{\pi}{2},$$

which is equal to

$$\frac{1}{2}[\tilde{f}(k = \pm a^+) + \tilde{f}(k = \pm a^-)].$$

❑ **Example 1.3.** Find the Fourier transform of the following function,

$$f(x) = \frac{\sin(ax)}{x(x^2 + b^2)}, \quad a, b > 0, \quad -\infty < x < \infty. \tag{1.9.18}$$

Solution. The Fourier transform $\tilde{f}(k)$ is given by

$$\tilde{f}(k) = \int_{\Gamma} dz \frac{\exp[i(k+a)z]}{2iz(z^2 + b^2)} - \int_{\Gamma} dz \frac{\exp[i(k-a)z]}{2iz(z^2 + b^2)} = I(k+a) - I(k-a), \tag{1.9.19}$$

where we define the integral $I(c)$ by

$$I(c) \equiv \int_{-\infty}^{\infty} dz \frac{\exp[icz]}{2iz(z^2 + b^2)} = \int_{\Gamma} dz \frac{\exp[icz]}{2iz(z^2 + b^2)}, \tag{1.9.20}$$

where the contour Γ is the same as in Example 1.2. The integrand has simple poles at

$$z = 0 \quad \text{and} \quad z = \pm ib.$$

Noting $z = \operatorname{Re} z + i \operatorname{Im} z$, we have

$$I(c) = \begin{cases} 2\pi i \cdot \operatorname{Res}\left[\frac{\exp[icz]}{2iz(z^2+b^2)}\right]_{z=0} + 2\pi i \cdot \operatorname{Res}\left[\frac{\exp[icz]}{2iz(z^2+b^2)}\right]_{z=ib}, & c > 0, \\ -2\pi i \cdot \operatorname{Res}\left[\frac{\exp[icz]}{2iz(z^2+b^2)}\right]_{z=-ib}, & c < 0, \end{cases}$$

or

$$I(c) = \begin{cases} (\pi/2b^2)(2 - \exp[-bc]), & c > 0, \\ (\pi/2b^2)\exp[bc], & c < 0. \end{cases}$$

Thus we have

$$\begin{aligned} \tilde{f}(k) &= I(k+a) - I(k-a) \\ &= \begin{cases} (\pi/b^2)\sinh(ab)\exp[bk], & k < -a, \\ (\pi/b^2)\{1 - \exp[-ab]\cosh(bk)\}, & |k| < a, \\ (\pi/b^2)\sinh(ab)\exp[-bk], & k > a. \end{cases} \end{aligned} \tag{1.9.21}$$

We note that $\tilde{f}(k)$ is *step-discontinuous* at $k = \pm a$ in Example 1.2. We also note that $\tilde{f}(k)$ and $\tilde{f}'(k)$ are *continuous* for real k, while $\tilde{f}''(k)$ is *step-discontinuous* at $k = \pm a$ in Example 1.3.

We note the rate with which

$$f(x) \to 0 \quad \text{as} \quad |x| \to +\infty$$

affects the degree of smoothness of $\tilde{f}(k)$. For the square-integrable functions, we usually have

$$f(x) = O\left(\frac{1}{x}\right) \text{ as } |x| \to +\infty \Rightarrow \tilde{f}(k) \text{ step-discontinuous},$$

$$f(x) = O\left(\frac{1}{x^2}\right) \text{ as } |x| \to +\infty \Rightarrow \begin{cases} \tilde{f}(k) \text{ continuous}, \\ \tilde{f}'(k) \text{ step-discontinuous}, \end{cases}$$

$$f(x) = O\left(\frac{1}{x^3}\right) \text{ as } |x| \to +\infty \Rightarrow \begin{cases} \tilde{f}(k), \tilde{f}'(k) \text{ continuous}, \\ \tilde{f}''(k) \text{ step-discontinuous}, \end{cases}$$

and so on.

* * *

Having learned in the above the abstract notions relating to linear space, inner product, the operator and its adjoint, eigenvalue and eigenfunction, Green's function, and having reviewed the Fourier transform and complex analysis, we are now ready to embark on our study of integral equations. We encourage the reader to make an effort to connect the concrete example that will follow with the abstract idea of linear function space and the linear operator. This will not be possible in all circumstances.

The abstract idea of function space is also useful in the discussion of the calculus of variations where a piecewise continuous, but nowhere differentiable, function and a discontinuous function appear as the solution of the problem.

We present the applications of the calculus of variations to theoretical physics specifically, classical mechanics, canonical transformation theory, the Hamilton–Jacobi equation, classical electrodynamics, quantum mechanics, quantum field theory and quantum statistical mechanics.

The mathematically oriented reader is referred to the monographs by R. Kress, and I. M. Gelfand and S. V. Fomin for details of the theories of integral equations and the calculus of variations.