1.1

1

Preview and Rationale

This introductory chapter has two main purposes. The first is to review Lagrangian mechanics. Some of this material takes the form of worked examples, chosen both to be appropriate as examples and to serve as bases for topics in later chapters.

The second purpose is to introduce the mechanics of classical strings. This topic is timely, being introductory to the modern subject of (quantum field theoretical) string theory. But, also, the Lagrangian theory of strings is an appropriate area in which to practice using supposedly well-known concepts and methods in a context that is encountered (if at all) toward the end of a traditional course in intermediate mechanics. This introduces the topic of Lagrangian field theory in a well-motivated and elementary way. Classical strings have the happy properties of being the simplest system for which Lagrangian field theory is appropriate.

The motivation for emphasizing strings from the start comes from the daring, and apparently successful, introduction by Barton Zwiebach, of string theory into the M.I.T. undergraduate curriculum. This program is fleshed out in his book *A First Course in String Theory*. The present chapter, and especially Chapter 12 on relativistic strings, borrows extensively from that text. Unlike Zwiebach though, the present text stops well short of quantum field theory.

An eventual aim of this text is to *unify* "all" of classical physics within suitably generalized Lagrangian mechanics. Here "all" will be taken to be adequately represented by the following topics: mechanics of particles, special relativity, electromagnetic theory, classical (and, eventually, relativistic) string theory, and general relativity. This list, which is to be regarded as defining by example what constitutes "classical physics," is indeed ambitious, though it leaves out many other important fields of classical physics, such as elasticity and fluid dynamics.¹ The list also includes enough varieties of geometry

 By referring to a text such as *Theoretical Mechanics of Particles and Continua*, by Fetter and Walecka, which covers fluids and elastic solids in very much the same spirit as in the present text, it should be clear that these two topics can also be included in the list of fields *unified* by Lagrangian mechanics.

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to support another aim of the text, which is to illuminate the important role played by geometry in physics.

An introductory textbook on Lagrangian mechanics (which this is not) might be expected to begin by announcing that the reader is assumed to be familiar with Newtonian mechanics - kinematics, force, momentum and energy and their conservation, simple harmonic motion, moments of inertia, and so on. In all likelihood such a text would then proceed to review these very same topics before advancing to its main topic of Lagrangian mechanics. This would not, of course, contradict the original assumption since, apart from the simple pedagogical value of review, it makes no sense to study Lagrangian mechanics without anchoring it firmly in a Newtonian foundation. The student who had not learned this material previously would be well advised to start by studying a less advanced, purely Newtonian mechanics textbook. So many of the most important problems of physics can be solved cleanly without the power of Lagrangian mechanics; it is uneconomical to begin with an abstract formulation of mechanics before developing intuition better acquired from a concrete treatment. One might say that Newtonian methods give better "value" than Lagrangian mechanics because, though ultimately less powerful, Newtonian methods can solve the most important problems and are easier to learn. Of course this would only be true in the sort of foolish system of accounting that might attempt to rate the relative contributions of Newton and Einstein. One (but not the only) purpose of this textbook, is to go beyond Lagrange's equations. By the same foolish system of accounting just mentioned, these methods could be rated less valuable than Lagrangian methods since, though more powerful, they are more abstract and harder to learn.

It is assumed the reader has had some (not necessarily much) experience with Lagrangian mechanics.² Naturally this presupposes familiarity with the above-mentioned elementary concepts of Newtonian mechanics. Nevertheless, for the same reasons as were described in the previous paragraph, we start by reviewing material that is, in principle, already known. It is assumed the reader can define a Lagrangian, can write it down for a simple mechanical system, can write down (or copy knowledgeably) the Euler–Lagrange equations and from them derive the equations of motion of the system, and finally (and most important of all) trust these equations to the same extent that she or he trusts Newton's law itself. A certain (even if grudging) acknowledgement of the method's power to make complicated systems appear simple is also helpful. Any reader unfamiliar with these ideas would be well advised

by Goldstein, *Classical Mechanics*) it is my intention that the level of difficulty and the anticipated level of preparation be much the same as is appropriate for Goldstein.

²⁾ Though "prerequisites" have been mentioned, this text still attempts to be "not too advanced." Though the subject matter deviates greatly from the traditional curriculum at this level (as represented, say,

to begin by repairing the defect with the aid of one of the numerous excellent textbooks explaining Lagrangian mechanics.

Since a systematic review of Newtonian and Lagrangian mechanics would be too lengthy, this chapter starts with worked examples that illustrate the important concepts. To the extent possible, examples in later chapters are based on these examples. This is especially appropriate for describing the evolution of systems that are close to solvable systems.

1.2

Review of Lagrangians and Hamiltonians

Recall the formulas of Lagrangian mechanics. For the next few equations, for mnemonic purposes, each equation will be specialized (sometimes in parenthesis) to the simplest prototype, mass and spring. The kinetic and potential energies for this system are given by

$$T = \frac{1}{2}m\dot{x}^2, \quad V = \frac{1}{2}kx^2, \tag{1.1}$$

where $\dot{x} \equiv dx/dt \equiv v$. The Lagrangian, a function of *x* and \dot{x} (and, in general though not in this special case, *t*) is given by

$$L(x, \dot{x}, t) = T - V \quad \left(= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right).$$
(1.2)

The Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \quad \left(\text{or} \quad m\ddot{x} = -kx\right). \tag{1.3}$$

The momentum p, "canonically conjugate to x," is defined by

$$p = \frac{\partial L}{\partial \dot{x}} \quad (= m\dot{x}). \tag{1.4}$$

The Hamiltonian is derived from the Lagrangian by a transformation in which both independent and dependent variables are changed. This transformation is known as a "Legendre transformation." Such a transformation has a geometric interpretation,³ but there is no harm in thinking of it as purely a formal calculus manipulation. Similar manipulations are common in thermodynamics to define quantities that are constant under special circumstances. For a function L(x, v, t), one defines a new independent variable $p = \partial L/\partial v$ and a new function H(x, p, t) = vp - L(x, v, t), in which v has to be expressed in

³) The geometric interpretation of a Legendre transformation is discussed in Arnold, *Mathematical Methods of Classical Mechanics*, and Lanczos, *The Variational Principles of Mechanics*.

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terms of x and p by inverting $p = \partial L / \partial v$. The motivation behind this definition is to produce cancellation of second and fourth terms in the differential

$$dH = v \, dp + p \, dv - \frac{\partial L}{\partial x} \, dx - \frac{\partial L}{\partial v} \, dv$$

= $v \, dp - \frac{\partial L}{\partial x} \, dx.$ (1.5)

Applying these substitutions to our Lagrangian, with v being \dot{x} , one obtains the "Hamiltonian" function,

$$H(x, p, t) = p \dot{x}(x, p) - L(x, \dot{x}(x, p), t).$$
(1.6)

With (1.5) being the differential of this function, using Eq. (1.4), one obtains Hamilton's equations;

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$
 (1.7)

The third equation here, obvious from Eq. (1.6), has been included for convenience, especially in light of the following argument. As well as its formal role, as a function to be differentiated to obtain the equations of motion, the Hamiltonian H(x, p, t) can be evaluated for the actually evolving values of its arguments. This evolution of H is governed by

$$\dot{H} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t},$$
(1.8)

where Eqs. (1.7) were used in the final step. This equation implies that the absence of explicit dependence on *t* implies the constancy of *H*.

To be able to apply Hamiltonian mechanics it is necessary to be able to express \dot{x} as a function of p – trivial in our example;

$$\dot{x} = \frac{p}{m},\tag{1.9}$$

and to express the combination $\dot{x}p - L(x, \dot{x})$ in terms of *x* and *p*, thereby defining the Hamiltonian;

$$H(x,p) = \frac{p^2}{m} - L = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \mathcal{E}.$$
(1.10)

Since H(x, p) does not depend explicitly on time (in this example) H(x, p) is a constant of the motion, equal to the "energy" \mathcal{E} .

1.2.1

Hamilton's Equations in Multiple Dimensions

Given coordinates **q** and Lagrangian *L*, "canonical momenta" are defined by

$$p_j = \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}^j}; \tag{1.11}$$

 p_j is said to be "conjugate" to q^j . To make partial differentiation like this meaningful it is necessary to specify what variables are being held fixed. We mean implicitly that variables q^i for all i, \dot{q}^i for $i \neq j$, and t are being held fixed. Having established variables \mathbf{p} it is required in all that follows that velocities $\dot{\mathbf{q}}$ be explicitly expressible in terms of the \mathbf{q} and \mathbf{p} , as in

$$\dot{q}^i = f^i(\mathbf{q}, \mathbf{p}, t), \quad \text{or} \quad \dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \mathbf{p}, t).$$
 (1.12)

Hamilton's equations can be derived using the properties of differentials. Define the "Hamiltonian" by

$$H(\mathbf{q}, \mathbf{p}, t) = p_i f^i(\mathbf{q}, \mathbf{p}, t) - L(\mathbf{q}, \mathbf{f}(\mathbf{q}, \mathbf{p}, t), t), \qquad (1.13)$$

where the functions f^i were defined in Eq. (1.12). If these functions are, for any reason, unavailable, the procedure cannot continue; the velocity variables *must* be eliminated in this way. Furthermore, as indicated on the left-hand side of Eq. (1.13), it is essential for the formal arguments of H to be **q**, **p** and t. Then, when writing partial derivatives of H, it will be implicit that the variables being held constant are all but one of the **q**, **p**, and t. If all independent variables of the Lagrangian are varied independently the result is

$$dL = \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i + \frac{\partial L}{\partial t} dt.$$
(1.14)

(It is important to appreciate that the q^i and the \dot{q}^i are being treated as formally independent at this point. Any temptation toward thinking of \dot{q}^i as some sort of derivative of q^i must be fought off.) The purpose of the additive term $p_i f^i$ in the definition of *H* is to cancel terms proportional to $d\dot{q}^i$ in the expression for dH;

$$dH = f^{i}dp_{i} + p_{i}df^{i} - \frac{\partial L}{\partial q^{i}}dq^{i} - \frac{\partial L}{\partial \dot{q}^{i}}d\dot{q}^{i} - \frac{\partial L}{\partial t}dt$$

$$= -\frac{\partial L}{\partial q^{i}}dq^{i} + f^{i}dp_{i} - \frac{\partial L}{\partial t}dt$$

$$= -\dot{p}_{i}dq^{i} + \dot{q}^{i}dp_{i} - \frac{\partial L}{\partial t}dt,$$
(1.15)

where the Lagrange equations as well as Eq. (1.12) have been used. Hamilton's first-order equations follow from Eq. (1.15);

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$
 (1.16)

Remember that in the partial derivatives of *H* the variables **p** are held constant but in $\partial L/\partial t$ the variables **q** are held constant.

Example 1.2.1. Charged Particle in Electromagnetic Field. To exercise the Hamiltonian formalism consider a nonrelativistic particle in an electromagnetic field. In Chapter 11 it is shown that the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + e(A_x\dot{x} + A_y\dot{y} + A_z\dot{z}) - e\Phi(x, y, z),$$
(1.17)

where $\mathbf{A}(\mathbf{x})$ is the vector potential and $\Phi(\mathbf{x})$ is the electric potential. The middle terms, linear in velocities, cannot be regarded naturally as either kinetic or potential energies. Nevertheless, their presence does not impede the formalism. In fact, consider an even more general situation,

$$L = \frac{1}{2} A_{rs}(\mathbf{q}) \, \dot{q}^r \dot{q}^s + A_r(\mathbf{q}) \, \dot{q}^r - V(\mathbf{q}).$$
(1.18)

Then

$$p_r = A_{rs}\dot{q}^s + A_r$$
, and $\dot{q}^r = B_{rs}(p_s - A_r)$. (1.19)

It can be seen in this case that the momentum and velocity components are inhomogeneously, though still linearly, related. The Hamiltonian is

$$H = \frac{1}{2} B_{rs}(p_r - A_r)(p_s - A_s) + V, \qquad (1.20)$$

and Hamilton's equations follow easily.

1.3 Derivation of the Lagrange Equation from Hamilton's Principle

The Lagrange equation is derivable from the "principle of least action" (or Hamilton's principle) according to which the actual trajectory taken by a particle as it moves from point P_0 to P between times t_0 and t, is that trajectory that minimizes the "action" function S defined by

$$S = \int_{t_0}^t L(x, \dot{x}, t) \, dt.$$
(1.21)

As shown in Fig. 1.1, a possible small deviation from the true orbit x(t) is symbolized by $\delta x(t)$. Except for being infinitesimal and vanishing at the end points, the function $\delta x(t)$ is an arbitrary function of time. Note that the express-

sions $(d/dt)\delta x(t)$, $\delta \dot{x}(t)$, and $\delta x(t)$ all mean the same thing. The second form might be considered ambiguous but, for the sake of brevity, it is the symbol we will use.



Fig. 1.1 Graph showing the extremal trajectory x(t) and a nearby nontrajectory $x(t) + \delta x(t)$.

Using elementary calculus, the variation in *S* that accompanies the replacement $x(t) \Rightarrow x(t) + \delta x(t)$ is

$$\delta S = \int_{t_0}^t dt \left(\frac{\partial L}{\partial x} \delta x(t) + \frac{\partial L}{\partial \dot{x}} \delta \dot{x}(t) \right).$$
(1.22)

Preparing for integration by parts, one substitutes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}(t),$$
(1.23)

to obtain

$$\delta S = \int_{t_0}^t dt \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \delta x \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) \delta x \right).$$
(1.24)

The first term, being a total derivative, can be integrated directly, and then be expressed in terms of initial and final values. For now we require δx to vanish in both cases. The motivation for performing this manipulation was to make δx be a common factor of the remaining integrand.

Since δx is an arbitrary function, the vanishing of δS implies the vanishing of the other factor in the integrand. The result is the Lagrange equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$
(1.25)

The very meaning of the Lagrange equations requires a clear understanding of the difference between d/dt and $\partial/\partial t$. The former refers to the time rate of change along the actual particle trajectory, while the latter refers to a formal derivative with respect to time with the other independent variables (called

out in the argument list of the function being differentiated) held constant. When operating on an arbitrary function $F(\mathbf{x}(t), t)$ these derivatives are related by

$$\frac{d}{dt}F = \frac{\partial}{\partial t}F + (\mathbf{v} \cdot \nabla)F.$$
(1.26)

The first term gives the change in *F* at a fixed point, while the second gives the change due to the particle's motion.

This derivation has been restricted to a single Cartesian coordinate x, and the corresponding velocity \dot{x} , but the same derivation also applies to y and z and, for that matter to any generalized coordinates and their corresponding velocities. With this greater generality the Lagrange equations can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{r}} \equiv \nabla L. \tag{1.27}$$

Figure 1.1 shows the dependence of just one of the coordinates, x, on time t. Similar figures for other independent variables need not be drawn since we need only one variable, say x(t), to obtain its Lagrange equation.

Problem 1.3.1. *The action S in mechanics is the analog of the optical path length, O.P.L., of physical optics. The basic integral like (1.21) in optics has the form*

$$\frac{1}{c}O.P.L. = \frac{1}{c}\int_{z_1}^{z_2} L\left(x, y, \frac{dx}{dz}, \frac{dy}{dz}, z\right) dz = \frac{1}{c}\int_{z_1}^{z_2} n(\mathbf{r})\sqrt{x'^2 + y'^2 + 1} \, dz.$$
(1.28)

Here x, y, and z are Cartesian coordinates with x and y "transverse" and z defining a longitudinal axis relative to which x and y are measured. The optical path length is the path length weighted by the local index of refraction n. O.P.L./c, is the "time" in "principle of least time." Though it is not entirely valid in physical optics to say that the "speed of light" in a medium is c / n, acting as if it were, the formula gives the time of flight of a particle (photon) following the given trajectory with this velocity.

The calculus of variations can be used to minimize O.P.L. Show that the differential equation (which will reappear as Eq. (7.18)) satisfied by an optical ray is

$$\frac{d}{ds}\left(n\frac{\mathbf{d}\mathbf{r}}{ds}\right) = \nabla\mathbf{n},\tag{1.29}$$

where $n(\mathbf{r})$ is index of refraction, \mathbf{r} is a radius vector from an arbitrary origin to a point on a ray, and s is arc length s along the ray.

1.4

Linear, Multiparticle Systems

The approximate Lagrangian for an *n*-dimensional system with coordinates $(q_1, q_2, ..., q_n)$, valid in the vicinity of a stable equilibrium point (that can be

taken to be $(0, 0, \ldots, 0)$), has the form

$$L(\mathbf{q}, \dot{\mathbf{q}}) = T - V, \quad \text{where} \quad T = \frac{1}{2} \sum_{r,s=1}^{n} m_{(rs)} \dot{q}_{(r)} \dot{q}_{(s)},$$

$$V = \frac{1}{2} \sum_{r,s=1}^{n} k_{(rs)} q_{(r)} q_{(s)}.$$
(1.30)

It is common to use the summation convention for summations like this, but in this text the summation convention is reserved for tensor summations. When subscripts are placed in parenthesis (as here) it indicates they refer to different variables or parameters (as here) rather than different components of the same vector or tensor. Not to be obsessive about it however, for the rest of this discussion the parentheses will be left off, but the summation signs will be left explicit. It is known from algebra that a linear transformation $q_i \rightarrow y_j$ can be found such that *T* takes the form

$$T = \frac{1}{2} \sum_{r=1}^{n} m_r \dot{y}_r^2, \tag{1.31}$$

where, in this case each "mass" m_r is necessarily positive because T is positive definite. By judicious choice of the scale of the y_r each "mass" can be adjusted to 1. We will assume this has already been done.

$$T = \frac{1}{2} \sum_{r=1}^{n} \dot{y}_{r}^{2}.$$
(1.32)

For these coordinates y_r the equation

$$\sum_{r=1}^{n} y_r^2 = 1 \tag{1.33}$$

defines a surface (to be called a hypersphere). From now on we will consider only points $\mathbf{y} = (y_1, \dots, y_n)$ lying on this sphere. Also two points \mathbf{u} and \mathbf{v} will be said to be "orthogonal" if the "quadratic form" $\mathcal{I}(\mathbf{u}, \mathbf{v})$ defined by

$$\mathcal{I}(\mathbf{u}, \mathbf{v}) \equiv \sum_{r=1}^{n} u_r v_r \tag{1.34}$$

vanishes. Being linear in both arguments $\mathcal{I}(u, v)$ is said to be "bilinear." We also define a bilinear form $\mathcal{V}(u, v)$ by

$$\mathcal{V}(\mathbf{u},\mathbf{v}) \equiv \sum_{r,s=1}^{n} k_{rs} u_r v_s, \tag{1.35}$$

where coefficients k_{rs} have been redefined from the values given above to correspond to the new coordinates y_r so that

$$V(\mathbf{y}) = \frac{1}{2} \mathcal{V}(\mathbf{y}, \mathbf{y}). \tag{1.36}$$

The following series of problems (adapted from Courant and Hilbert, Vol. 1, p. 37) will lead to the conclusion that a further linear transformation $y_i \rightarrow z_j$ can be found that, on the one hand, enables the equation for the sphere in Eq. (1.33) to retain the same form,

$$\sum_{r=1}^{n} z_r^2 = 1, \tag{1.37}$$

and, on the other, enables *V* to be expressible as a sum of squares with positive coefficients;

$$V = \frac{1}{2} \sum_{r=1}^{n} \kappa_r z_r^2, \quad \text{where} \quad 0 < \kappa_n \le \kappa_{n-1} \le \dots \le \kappa_1 < \infty.$$
(1.38)

Pictorially the strategy is, having deformed the scales so that surfaces of constant *T* are spherical and surfaces of constant *V* ellipsoidal, to orient the axes to make these ellipsoids erect. In the jargon of mechanics this process is known as "normal mode" analysis.

The "minimax" properties of the "eigenvalues" to be found have important physical implications, but we will not go into them here.

Problem 1.4.1.

- (a) Argue, for small oscillations to be stable, that V must also be positive definite.
- (b) Let \mathbf{z}_1 be the point on sphere (1.33) for which $V\left(\begin{array}{c} \text{def.} \\ = \\ 1 \\ 2 \\ \kappa_1 \end{array}\right)$ is maximum. (If there is more than one such point pick any one arbitrarily.) Then argue that

$$0 < \kappa_1 < \infty. \tag{1.39}$$

- (c) Among all the points that are both on sphere (1.33) and orthogonal to \mathbf{z}_1 , let \mathbf{z}_2 be the one for which $V(\stackrel{\text{def.}}{=} \frac{1}{2}\kappa_2)$ is maximum. Continuing in this way, show that a series of points $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$, each maximizing V consistent with being orthogonal to its predecessors, is determined, and that the sequence of values, $V(\mathbf{z}_r) = \frac{1}{2}\kappa_r, \quad r = 1, 2, \dots, n$, is monotonically nonincreasing.
- (d) Consider a point $\mathbf{z}_1 + \epsilon \boldsymbol{\zeta}$ which is assumed to lie on surface (1.33) but with $\boldsymbol{\zeta}$ otherwise arbitrary. Next assume this point is "close to" \mathbf{z}_1 in the sense that ϵ is arbitrarily small (and not necessarily positive). Since \mathbf{z}_1 maximizes V it follows that

$$\mathcal{V}(\mathbf{z}_1 + \epsilon \boldsymbol{\zeta}, \mathbf{z}_1 + \epsilon \boldsymbol{\zeta}) \le 0. \tag{1.40}$$

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Show therefore that

$$\mathcal{V}(\mathbf{z}_1, \boldsymbol{\zeta}) = 0. \tag{1.41}$$

This implies that

$$\mathcal{V}(\mathbf{z}_1, \mathbf{z}_r) = 0 \quad \text{for} \quad r > 1, \tag{1.42}$$

because, other than being orthogonal to z_1 , ζ is arbitrary.

Finally, extend the argument to show that

$$\mathcal{V}(\mathbf{z}_r, \mathbf{z}_s) = \kappa_r \delta_{rs},\tag{1.43}$$

where the coefficients κ_r have been shown to satisfy the monotonic conditions of Eq. (1.38) and δ_{rs} is the usual Kronecker- δ symbol.

(e) Taking these \mathbf{z}_r as basis vectors, an arbitrary vector \mathbf{z} can be expressed as

$$\mathbf{z} = \sum_{r=1}^{n} z_r \mathbf{z}_r.$$
(1.44)

In these new coordinates, show that Eqs. (1.30) become

$$L(\mathbf{z}, \dot{\mathbf{z}}) = T - V, \quad T = \frac{1}{2} \sum_{r=1}^{n} \dot{z}_{r}^{2}, \quad V = \frac{1}{2} \sum_{r=1}^{n} \kappa_{r} z_{r}^{2}.$$
(1.45)

Write and solve the Lagrange equations for coordinates z_r *.*

Problem 1.4.2. *Proceeding from the previous formula, the Lagrange equations resulting from Eq.* (1.30) *are*

$$\sum_{s=1}^{n} m_{rs} \ddot{q}_s + \sum_{s=1}^{n} k_{rs} q_s = 0.$$
(1.46)

These equations can be expressed compactly in matrix form;

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0; \tag{1.47}$$

or, assuming the existence of M^{-1} , as

$$\ddot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{K}\mathbf{q} = 0. \tag{1.48}$$

Seeking a solution of the form

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$$q_r = A_r e^{i\omega t}$$
 $r = 1, 2, \dots, n,$ (1.49)

the result of substitution into Eq. (1.46) is

$$(\mathbf{M}^{-1}\mathbf{K} - \omega^2 \mathbf{1})\mathbf{A} = 0.$$
(1.50)



Fig. 1.2 Three beads on a stretched string. The transverse displacements are much exaggerated. Gravity and string mass are negligible.

These equations have nontrivial solutions for values of ω that cause the determinant of the coefficients to vanish;

$$|\mathbf{M}^{-1}\mathbf{K} - \omega^2 \mathbf{1}| = 0. \tag{1.51}$$

Correlate these ω *"eigenvalues" with the constants* κ_r *defined in the previous problem.*

Problem 1.4.3. As shown in Fig. 1.2, particles of mass 3m, 4m, and 3m, are spaced at uniform intervals λ along a light string of total length 4λ , stretched with tension T, and rigidly fixed at both ends. To legitimize ignoring gravity, the system is assumed to lie on a smooth horizontal table so the masses can oscillate only horizontally. Let the transverse displacements be x_1 , x_2 , and x_3 . Find the normal modes frequencies and find and sketch the corresponding normal mode oscillation "shapes." Discuss the "symmetry" of the shapes, their "wavelengths," and the (monotonic) relation between mode frequency and number of nodes (axis crossings) in each mode.

Already with just three degrees of freedom the eigenmode calculations are sufficiently tedious to make some efforts at simplifying the work worthwhile. In this problem, with the system symmetric about its midpoint it is clear that the modes will be either symmetric or antisymmetric and, since the antisymmetric mode vanishes at the center point, it is characterized by a single amplitude, say $y = x_1 = -x_3$. Introducing "effective mass" and "effective strength coefficient" the kinetic energy of the mode, necessarily proportional to \dot{y} , can be written as $T_2 = \frac{1}{2}m_{\text{eff}}\dot{y}^2$ and the potential energy can be written as $V_2 = \frac{1}{2}k_{\text{eff}}y^2$. The frequency of this mode is then given by $\omega_2 = \sqrt{k_{\text{eff}}/m_{\text{eff}}}$ which, by dimensional analysis, has to be proportional to $\eta = \sqrt{T/(m\lambda)}$. (The quantities T_2 , V_2 , and ω_2 have been given subscript 2 because this mode has the second highest frequency.) Factoring this expression out of Eq. (1.51), the dimensionless eigenvalues are the eigenfrequencies in units of η .

Problem 1.4.4. Complete the analysis to show that the normal mode frequencies are $(\omega_1, \omega_2, \omega_3) = (1, \sqrt{2/3}, \sqrt{1/6})$, and find the corresponding normal mode "shapes."

1.4.1 The Laplace Transform Method

Though the eigenmode/eigenvalue solution method employed in solving the previous problem is the traditional method used in classical mechanics, equations of the same form, when they arise in circuit analysis and other engineering fields, are traditionally solved using Laplace transforms – a more robust method, it seems to me. Let us continue the solution of the previous problem using this method. Individuals already familiar with this method or not wishing to become so should skip this section. Here we use the notation

$$\overline{x}(s) = \int_0^\infty e^{-st} x(t) \, dt, \tag{1.52}$$

as the formula giving the Laplace transform $\overline{x}(s)$, of the function of time x(t). $\overline{x}(s)$ is a function of the "transform variable" s (which is a complex number with positive real part.) With this definition the Laplace transform satisfies many formulas but, for present purposes we use only

$$\frac{\overline{dx}}{dt} = s\overline{x} - x(0), \tag{1.53}$$

which is easily demonstrated. Repeated application of this formula converts time derivatives into functions of *s* and therefore converts (linear) differential equations into (linear) algebraic equations. This will now be applied to the system described in the previous problem.

The Lagrange equations for the beaded string shown in Fig. 1.2 are

$$\begin{aligned} 3\ddot{x}_1 + \eta^2(2x_1 - x_2) &= 0, \\ 4\ddot{x}_2 + \eta^2(2x_2 - x_1 - x_3) &= 0, \\ 3\ddot{x}_3 + \eta^2(2x_3 - x_2) &= 0. \end{aligned} \tag{1.54}$$

Suppose the string is initially at rest but that a transverse impulse *I* is administered to the first mass at t = 0; as a consequence it acquires initial velocity $v_{10} \equiv \dot{x}(0) = I/(3m)$. Transforming all three equations and applying the initial conditions (the only nonvanishing initial quantity, v_{10} , enters via Eq. (1.53))

$$(3s^{2} + 2\eta^{2})\overline{x}_{1} - \eta^{2}\overline{x}_{2} = I/m,$$

$$-\eta^{2}\overline{x}_{1} + (4s^{2} + 2\eta^{2})\overline{x}_{2} - \eta^{2}\overline{x}_{3} = 0,$$

$$-\eta^{2}\overline{x}_{2} + (3s^{2} + 2\eta^{2})\overline{x}_{3} = 0.$$
(1.55)

Solving these equations yields

$$\overline{x}_{1} = \frac{I}{10m} \left(\frac{2/3}{s^{2} + \eta^{2}/6} + \frac{1}{s^{2} + \eta^{2}} + \frac{5/3}{s^{2} + 2\eta^{2}/3} \right),$$

$$\overline{x}_{2} = \frac{I}{10m} \left(\frac{1}{s^{2} + \eta^{2}/6} - \frac{1}{s^{2} + \eta^{2}} \right),$$

$$\overline{x}_{3} = \frac{I}{10m} \left(\frac{2/3}{s^{2} + \eta^{2}/6} + \frac{1}{s^{2} + \eta^{2}} - \frac{5/3}{s^{2} + 2\eta^{2}/3} \right).$$
(1.56)

It can be seen, except for factors $\pm i$, that the poles (as a function of *s*) of the transforms of the variables, are the normal mode frequencies. This is not surprising since the determinant of the coefficients in Eq. (1.55) is the same as the determinant entering the normal mode solution, but with ω^2 replaced with $-s^2$. Remember then, from Cramer's rule for the solution of linear equations, that this determinant appears in the denominators of the solutions. For "inverting" Eq. (1.56) it is sufficient to know just one inverse Laplace transformation,

$$\mathcal{L}^{-1}\frac{1}{s-\alpha} = e^{\alpha t},\tag{1.57}$$

but it is easier to look in a table of inverse transforms to find that the terms in Eq. (1.56) yield sinusoids that oscillate with the normal mode frequencies. Furthermore, the "shapes" asked for in the previous problem can be read off directly from (1.56) to be (2:3:2), (1:0:1), and (1:-1:1).

When the first mass is struck at t = 0, all three modes are excited and they proceed to oscillate at their own natural frequencies, so the motion of each individual particle is a superposition of these frequencies. Since there is no damping the system will continue to oscillate in this superficially complicated way forever. In practice there is always some damping and, in general, it is different for the different modes; commonly damping increases with frequency. In this case, after a while, the motion will be primarily in the lowest frequency mode; if the vibrating string emits audible sound, an increasingly pure, low-pitched tone will be heard as time goes on.

1.4.2

Damped and Driven Simple Harmonic Motion

The equation of motion of mass *m*, subject to restoring force $-\omega_0^2 mx$, damping force $-2\lambda m\dot{x}$, and external drive force $f \cos \gamma t$ is

$$\ddot{x} + 2\lambda \dot{x} + \omega_0^2 = \frac{f}{m} \cos \gamma t.$$
(1.58)

Problem 1.4.5.

(a) Show that the general solution of this equation when f = 0 is

$$x(t) = ae^{-\lambda t}\cos(\omega t + \phi), \tag{1.59}$$

where a and ϕ depend on initial conditions and $\omega = \sqrt{\omega^2 - \lambda^2}$. This "solution of the homogeneous equation" is also known as "transient" since when it is superimposed on the "driven" or "steady state" motion caused by f it will eventually become negligible.

- (b) Correlate the stability or instability of the transient solution with the sign of λ . Equivalently, after writing the solution (1.59) as the sum of two complex exponential terms, Laplace transform them, and correlate the stability or instability of the transient with the locations in the complex s-plane of the poles of the Laplace transform.
- (c) Assuming $x(0) = \dot{x}(0) = 0$, show that Laplace transforming equation (1.58) yields

$$\overline{x}(s) = f \frac{s}{s^2 + \gamma^2} \frac{1}{s^2 + 2\lambda s + \omega_0^2}.$$
(1.60)

This expression has four poles, each of which leads to a complex exponential term in the time response. To neglect transients we need to only drop the terms for which the poles are off the imaginary axis. (By part (b) they must be in the left half-plane for stability.) To "drop" these terms it is necessary first to isolate them by partial fraction decomposition of Eq. (1.60). Performing these operations, show that the steady-state solution of Eq. (1.58), is

$$x(t) = \frac{f}{m} \sqrt{\frac{1}{(\omega_0^2 - \gamma^2)^2 + 4\lambda^2 \gamma^2}} \cos(\gamma t + \delta),$$
 (1.61)

where

$$\omega_0^2 - \gamma^2 - 2\lambda\gamma i = \sqrt{(\omega_0^2 - \gamma^2)^2 + 4\lambda^2\gamma^2} e^{i\delta}.$$
(1.62)

(*d*) The response is large only for γ close to ω_0 . To exploit this, defining the "small" "frequency deviation from the natural frequency"

$$\epsilon = \gamma - \omega_0, \tag{1.63}$$

show that $\gamma^2 - \omega^2 \approx 2\epsilon\omega$ and that the approximate response is

$$x(t) = \frac{f}{2m\omega_0} \sqrt{\frac{1}{\epsilon^2 + \lambda^2}} \cos(\gamma t + \delta).$$
(1.64)

Find the value of ϵ for which the amplitude of the response is reduced from its maximum value by the factor $1/\sqrt{2}$.

1.4.3 Conservation of Momentum and Energy

It has been shown previously that the application of energy conservation in one-dimensional problems permits the system evolution to be expressed in terms of a single integral – this is "reduction to quadrature." The following problem exhibits the use of momentum conservation to reduce a two-dimensional problem to quadratures, or rather, because of the simplicity of the configuration in this case, to a closed-form solution.

Problem 1.4.6. *A point mass m with total energy* E*, starting in the left half-plane, moves in the* (x, y) *plane subject to potential energy function*

$$U(x,y) = \begin{cases} U_1, & \text{for } x < 0, \\ U_2, & \text{for } 0 < x. \end{cases}$$
(1.65)

The "angle of incidence" to the interface at x = 0 is θ_i and the outgoing angle is θ . Specify the qualitatively different cases that are possible, depending on the relative values of the energies, and in each case find θ in terms of θ_i . Show that all results can be cast in the form of "Snell's law" of geometric optics if one introduces a factor $\sqrt{E - U(r)}$, analogous to index of refraction.

1.5

Effective Potential and the Kepler Problem

Since one-dimensional motion is subject to such simple and satisfactory analysis, anything that can reduce the dimensionality from two to one has great value. The "effective potential" is one such device. No physics problem has received more attention over the centuries than the problem of planetary orbits. In later chapters of this text the analytic solution of this so-called "Kepler problem" will be the foundation on which perturbative solution of more complicated problems will be based. Though this problem is now regarded as "elementary" one is well-advised to stick to traditional manipulations as the problem can otherwise get seriously messy.

The problem of two masses m_1 and m_2 moving in each other's gravitational field is easily converted into the problem of a single particle of mass m moving in the gravitational field of a mass m_0 assumed very large compared to m; that is $\mathbf{F} = -K\hat{\mathbf{r}}/r^2$, where $K = Gm_0m$ and r is the distance to m from m_0 . Anticipating that the orbit lies in a plane (as it must) let χ be the angle of the radius vector from a line through the center of m_0 ; this line will be later taken as the major axis of the elliptical orbit. The potential energy function is given by

$$U(r) = -\frac{K}{r},\tag{1.66}$$

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Fig. 1.3 Geometric construction defining the "true anomaly" χ and "eccentric anomaly" u in terms of other orbit parameters.

and the orbit geometry is illustrated in Fig. 1.3. Two conserved quantities can be identified immediately: energy E and angular momentum M. Show that they are given by

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\chi}^2) - \frac{K}{r},$$

$$M = mr^2\dot{\chi}.$$
(1.67)

One can exploit the constancy of *M* to eliminate $\dot{\chi}$ from the expression for *E*,

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff.}}(r), \text{ where } U_{\text{eff.}}(r) = \frac{M^2}{2mr^2} - \frac{K}{r}.$$
 (1.68)

The function $U_{\text{eff.}}(r)$, known as the "effective potential," is plotted in Fig. 1.4. Solving the expressions for *E* and *M* individually for differential dt

$$dt = \frac{mr^2}{M} d\chi, \quad dt = \left(\frac{2}{m} \left(E + \frac{K}{r}\right) - \frac{M^2}{m^2 r^2}\right)^{-1/2} dr.$$
(1.69)



Fig. 1.4 The effective potential $U_{\rm eff.}$ for the Kepler problem.

Equating the two expressions yields a differential equation that can be solved by "separation of variables." This has permitted the problem to be "reduced to quadratures,"

$$\chi(r) = \int^{r} \frac{Mdr'/r'^{2}}{\sqrt{2m(E+K/r') - M^{2}/r'^{2}}}.$$
(1.70)

Note that this procedure yields only an "orbit equation," the dependence of χ on r (which is equivalent to, if less convenient than, the dependence of r on χ .) Though *a priori* one should have had the more ambitious goal of finding a solution in the form r(t) and $\chi(t)$, no information whatsoever is given yet about time dependence by Eq. (1.70).

Problem 1.5.1.

- (a) Show that all computations so far can be carried out for any central force that is radially directed with magnitude dependent only on r. At worst the integral analogous to (1.70) can be performed numerically.
- (b) Specializing again to the Kepler problem, perform the integral (1.70) and show that the orbit equation can be written as

$$\epsilon \cos \chi + 1 = \frac{M^2}{mK} \frac{1}{r}.$$
(1.71)
where $\epsilon \equiv \sqrt{1 + \frac{2EM^2}{m^2K^2}}.$

- (c) Show that (1.71) is the equation of an ellipse if $\epsilon < 1$ and that this condition is equivalent to E < 0.
- (d) It is traditional to write the orbit equation purely in terms of "orbit elements" which can be identified as the "eccentricity" ϵ , and the "semimajor axis" a;

$$a = \frac{r_{\max} + r_{\min}}{2} = \frac{M^2}{mK} \frac{1}{1 - \epsilon^2}.$$
(1.72)

The reason *a* and ϵ are special is that they are intrinsic properties of the orbit unlike, for example, the orientations of the semimajor axis and the direction of the perpendicular to the orbit plane, both of which can be altered at will and still leave a "congruent" system. Derive the relations

$$E = -\frac{K}{2a}, \quad M^2 = (1 - \epsilon^2) \text{ mKa},$$
 (1.73)

so the orbit equation is

$$\frac{a}{r} = \frac{1 + \epsilon \cos \chi}{1 - \epsilon^2}.$$
(1.74)

(e) Finally derive the relation between r and t;

$$t(r) = \sqrt{\frac{ma}{K}} \int^{r} \frac{r'dr'}{\sqrt{a^{2}\epsilon^{2} - (r'-a)^{2}}}.$$
(1.75)

An "intermediate" variable u that leads to worthwhile simplification is defined by

$$r = a(1 - \epsilon \cos u). \tag{1.76}$$

The geometric interpretation of u is indicated in Fig. 1.3. If (x, z) are Cartesian coordinates of the planet along the major and an axis parallel to the minor axis through the central mass, they are given in terms of u by

$$x = a\cos u - a\epsilon, \quad z = a\sqrt{1 - \epsilon^2}\sin u,$$
 (1.77)

since the semimajor axis is $a\sqrt{1-\epsilon^2}$ and the circumscribed circle is related to the ellipse by a *z*-axis scale factor $\sqrt{1-\epsilon^2}$. The coordinate *u*, known as the "eccentric anomaly" is a kind of distorted angular coordinate of the planet, and is related fairly simply to *t*;

$$t = \sqrt{\frac{ma^3}{K}(u - \epsilon \sin u)}.$$
(1.78)

This is especially useful for nearly circular orbits, since then u is nearly proportional to t. Because the second term is periodic, the full secular time accumulation is described by the first term.

Analysis of this Keplerian system is continued using Hamilton–Jacobi theory in Section 8.3, and then again in Section 14.6.3 to illustrate action/angle variables, and then again as a system subject to perturbation and analyzed by "variation of constants" in Section 16.1.1.

Problem 1.5.2. The effective potential formalism has reduced the dimensionality of the Kepler problem from two to one. In one dimension, the linearization (to simple harmonic motion) procedure, can then be used to describe motion that remains close to the minimum of the effective potential (see Fig. 1.4). The radius $r_0 = M^2/(mK)$ is the radius of the circular orbit with angular momentum M. Consider an initial situation for which M has this same value and $\dot{r}(0) = 0$, but $r(0) \neq r_0$, though r(0) is in the region of good parabolic fit to U_{eff} . Find the frequency of small oscillations and express r(t) by its appropriate simple harmonic motion. Then find the orbit elements a and ϵ , as defined in Problem 1.5.1, that give the matching two-dimensional orbit.

1.6

Multiparticle Systems

Solving multiparticle problems in mechanics is notoriously difficult; for more than two particles it is usually impossible to get solutions in closed form. But

the equations of motion can be made simpler by the appropriate choice of coordinates as the next problem illustrates. Such coordinate choices exploit exact relations such as momentum conservation and thereby simplify subsequent approximate solutions. For example, this is a good pre-quantum starting point for molecular spectroscopy.

Problem 1.6.1. The position vectors of three point masses, m_1 , m_2 , and m_3 , are \mathbf{r}_1 , \mathbf{r}_2 , and \mathbf{r}_3 . Express these vectors in terms of the alternative configuration vectors \mathbf{s}_C , \mathbf{s}_3' , and \mathbf{s}_{12} shown in the figure. Define "reduced masses" by

$$m_{12} = m_1 + m_2, \quad M = m_1 + m_2 + m_3, \quad \mu_{12} = \frac{m_1 m_2}{m_{12}}, \quad \mu_3 = \frac{m_3 m_{12}}{M}.$$
 (1.79)

Calculate the total kinetic energy in terms of \dot{s} , $\dot{s_3}'$, and \dot{s}_{12} and interpret the result. Defining corresponding partial angular momenta l, l_3' , and l_{12} , show that the total angular momentum of the system is the sum of three analogous terms.



Fig. 1.5 Coordinates describing three particles. *C* is the center of mass and s_C its position vector relative to origin *O*. C_{12} is the center of mass of m_1 and m_2 and s_3' is the position of m_3 relative to C_{12} .

In Fig. 1.5, relative to origin O, the center of mass C is located by radius vector \mathbf{s}_{C} . Relative to particle 1, particle 2 is located by vector \mathbf{s}_{12} . Relative to the center of mass at C₁₂ mass 3 is located by vector \mathbf{s}_{3} '. In terms of these quantities the position vectors of the three masses are

$$\mathbf{r}_1 = \mathbf{s}_C - \frac{m_3}{M} \mathbf{s}_3' + \frac{m_2}{m_{12}} \mathbf{s}_{12}, \tag{1.80}$$

$$\mathbf{r}_{2} = \mathbf{s}_{C} - \frac{m_{3}}{M} \mathbf{s}_{3}' + \frac{m_{1}}{m_{12}} \mathbf{s}_{12}, \tag{1.81}$$

$$\mathbf{r}_3 = \mathbf{s}_C + \frac{m_{12}}{M} \mathbf{s}_3'. \tag{1.82}$$

Substituting these into the kinetic energy of the system

$$T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 + \frac{1}{2}m_3\dot{\mathbf{r}}_3^2, \tag{1.83}$$

the "cross terms" proportional to $s_C\cdot s_3',\,s_C\cdot s_{12}$, and $s_3'\cdot s_{12}$ all cancel out, leaving the result

$$T = \frac{1}{2}M v_{\rm C}^2 + \frac{1}{2}\mu_3 v_{3'}^2 + \frac{1}{2}\mu_{12} v_{12'}^2, \tag{1.84}$$

where $v_C = |\dot{\mathbf{s}}_C|$, $v'_3 = |\dot{\mathbf{s}}'_3|$, and $v_{12} = |\dot{\mathbf{s}}_{12}|$. The angular momentum (about O) is given by

$$\mathbf{L} = \mathbf{r}_1 \times (m_1 \dot{\mathbf{r}}_1) + \mathbf{r}_2 \times (m_2 \dot{\mathbf{r}}_2) + \mathbf{r}_3 \times (m_3 \dot{\mathbf{r}}_3).$$
(1.85)

Upon expansion the same simplifications occur, yielding

$$\mathbf{L} = \frac{1}{2} M \mathbf{r}_{C} \times \mathbf{v}_{C} + \frac{1}{2} \mu_{3} \mathbf{r}_{3}' \times \mathbf{v}_{3}' + \frac{1}{2} \mu_{12} \mathbf{r}_{12} \times \mathbf{v}_{12}.$$
 (1.86)

Problem 1.6.2. Determine the moment of inertia tensor about center of mass C for the system described in the previous problem. Choose axes to simplify the problem initially and give a formula for transforming from these axes to arbitrary (orthonormal) axes. For the case $m_3 = 0$ find the principal axes and the principal moments of inertia.

Setting $\mathbf{s}_{C} = 0$, the particle positions are given by

$$\mathbf{r}_1 = -\frac{m_3}{M}\mathbf{s}_3' + \frac{m_2}{m_{12}}\mathbf{s}_{12}, \quad \mathbf{r}_2 = -\frac{m_3}{M}\mathbf{s}_3' + \frac{m_1}{m_{12}}\mathbf{s}_{12}, \quad \mathbf{r}_3 = \frac{m_{12}}{M}\mathbf{s}_3'.$$
 (1.87)

Since the masses lie in a single plane it is convenient to take the *z*-axis normal to that plane. Let us orient the axes such that the unit vectors satisfy

$$\mathbf{s_3}' = \hat{\mathbf{x}}, \quad \mathbf{s_{12}} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}}, \tag{1.88}$$

and hence $a = \mathbf{s_3}' \cdot \mathbf{s_{12}}$. So the particle coordinates are

$$x_1 = -\frac{m_3}{M} + \frac{m_2}{m_{12}}a, \quad y_1 = \frac{m_2}{m_{12}}b, \tag{1.89}$$

$$x_2 = -\frac{m_3}{M} + \frac{m_1}{m_{12}}a, \quad y_2 = \frac{m_1}{m_{12}}b, \tag{1.90}$$

$$x_3 = \frac{m_{12}}{M}, \quad y_3 = 0. \tag{1.91}$$

In terms of these, the moment of inertia tensor I is given by

$$\begin{pmatrix} \sum m_i y_i^2 & -\sum m_i x_i y_i & 0\\ -\sum m_i x_i y_i & \sum m_i x_i^2 & 0\\ 0 & 0 & \sum m_i (x_i^2 + y_i^2) \end{pmatrix}.$$
(1.92)

For the special case $m_3 = 0$ these formulas reduce to

$$\mathbf{I} = \mu_{12} \begin{pmatrix} b^2 & -ab & 0\\ -ab & a^2 & 0\\ 0 & 0 & a^2 + b^2 \end{pmatrix}.$$
 (1.93)

Problem 1.6.3. A uniform solid cube can be supported by a thread from the center of a face, from the center of an edge, or from a corner. In each of the three cases the system acts as a torsional pendulum, with the thread providing all the restoring torque and the cube providing all the inertia. In which configuration is the oscillation period the longest? [If your answer involves complicated integrals you are not utilizing properties of the inertia tensor in the intended way.]

1.7

Longitudinal Oscillation of a Beaded String

A short length of a stretched string, modeled as point "beads" joined by light stretched springs, is shown in Fig. 1.6. With *a* being the distance between beads in the stretched, but undisturbed, condition and, using the fact that the spring constant of a section of a uniform spring is inversely proportional to the length of the section, the parameters of this system are:

unstretched string length =
$$L_0$$
,
stretched string length = $L_0 + \Delta L$,
extension, $\Delta L \times$ string constant of full string, K = tension, τ_0 ,
number of springs, $N = \frac{L_0 + \Delta L}{a}$
spring constant of each spring, $k = NK$,
mass per unit length, $\mu_0 = m/a$. (1.94)

The kinetic energy of this system is

$$T = \frac{m}{2} \left(\dots + \dot{\eta}_{i-1}^2 + \dot{\eta}_i^2 + \dot{\eta}_{i+1}^2 + \dots \right), \tag{1.95}$$



Fig. 1.6 A string under tension modeled as point "beads" joined by light stretched springs.

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and the potential energy is

$$V = \frac{k}{2} \left(\dots + (\eta_i - \eta_{i-1})^2 + (\eta_{i+1} - \eta_i)^2 + \dots \right).$$
(1.96)

The Lagrangian being L = T - V, the momentum conjugate to η_i is $p_i = \partial L / \partial \dot{\eta}_i = m \dot{\eta}_i$, and the Lagrange equations are

$$m\ddot{\eta}_{i} = \frac{\partial L}{\partial \dot{\eta}_{i}} = k(\eta_{i-1} - 2\eta_{i} + \eta_{i+1}), \quad i = 1, 2, \dots, N.$$
(1.97)

1.7.1

Monofrequency Excitation

Suppose that the beads of the spring are jiggling in response to sinusoidal excitation at frequency ω . Let us conjecture that the response can be expressed in the form

$$\eta_i(t) = \begin{pmatrix} \sin\\\cos \end{pmatrix} (\omega t + \Delta \psi i), \tag{1.98}$$

where $\Delta \psi$ is a phase advance per section that remains to be determined, and where "in phase" and "out of phase" responses are shown as the two rows of a matrix – their possibly different amplitudes are not yet shown. For substitution into Eq. (1.97) one needs

$$\eta_{i+1} = \begin{pmatrix} \sin(\omega t + \Delta \psi \, i) \, \cos \Delta \psi + \cos(\omega t + \Delta \psi \, i) \, \sin \Delta \psi \\ \cos(\omega t + \Delta \psi \, i) \, \cos \Delta \psi + -\sin(\omega t + \Delta \psi \, i) \, \sin \Delta \psi \end{pmatrix},\tag{1.99}$$

along with a similar equation for η_{i-1} . Then one obtains

$$\eta_{i-1} - 2\eta_i + \eta_{i+1} = (2\cos\Delta\psi - 2)\eta_i, \tag{1.100}$$

and then, from the Lagrange equation,

$$-m\omega^2 = 2\cos\Delta\psi - 2. \tag{1.101}$$

Solving this, one obtains

$$\Delta\psi(\omega) = \pm\cos^{-1}\left(1 - \frac{m\omega^2}{2k}\right),\tag{1.102}$$

as the phase shift per cell of a wave having frequency ω . The sign ambiguity corresponds to the possibility of waves traveling in either direction, and the absence of real solution $\Delta \psi$ above a "cut-off" frequency $\omega_{co} = \sqrt{4k/m}$ corresponds to the absence of propagating waves above that frequency. At low frequencies,

$$m\omega^2/k \ll 1, \tag{1.103}$$

which we assume from here on, Eq. (1.102) reduces to

$$\Delta \psi \approx \pm \sqrt{\frac{m}{k}} \,\omega. \tag{1.104}$$

Our assumed solution (1.98) also depends sinusoidally on the longitudinal coordinate *x*, which is related to the index *i* by x = ia. At fixed time, after the phase $i\Delta\psi$ has increases by 2π , the wave returns to its initial value. In other words, the wavelength of a wave on the string is given by

$$\lambda = \frac{2\pi}{\Delta\psi} a \approx \frac{2\pi}{\omega} \sqrt{\frac{k}{m}} a, \qquad (1.105)$$

and the wave speed is given by

$$v = \lambda \frac{\omega}{2\pi} = \sqrt{\frac{k}{m}} a. \tag{1.106}$$

(In this low frequency approximation) since this speed is independent of ω , low frequency pulses will propagate undistorted on the beaded string. Replacing the index *i* by a continuous variable x = ia, our conjectured solution therefore takes the form

$$\eta(x,t) = {\sin \atop \cos} \omega \left(t \pm \frac{x}{v} \right). \tag{1.107}$$

These equations form the basis for the so-called "lumped constant delay line," especially when masses and springs are replaced by inductors and capacitors.

1.7.2 The Continuum Limit

Propagation on a continuous string can be described by appropriately taking a limit $N \rightarrow \infty$, $a \rightarrow 0$, while holding $Na = L_0 + \Delta L$. Clearly, in this limit, the low frequency approximations just described become progressively more accurate and, eventually, exact. One can approximate the terms in the Lagrangian by the relations

$$\frac{\eta_{i+1} - \eta_i}{a} \approx \frac{\partial \eta}{\partial x}\Big|_{i+1/2},$$

$$\frac{\eta_i - \eta_{i-1/2}}{a} \approx \frac{\partial \eta}{\partial x}\Big|_{i-1/2},$$

$$\frac{\eta_{i+1} - 2\eta_i + \eta_{i-1}}{a^2} \approx \frac{\partial^2 \eta}{\partial x^2}\Big|_{i},$$
(1.108)

and, substituting from Eqs. (1.94), the Lagrange equation (1.97) becomes

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{k}{m} a^2 \frac{\partial^2 \eta}{\partial x^2} = \frac{N\tau_0/\Delta L}{\mu_0(L+\Delta L)/N} \frac{(L_0+\Delta L)^2}{N^2} \frac{\partial^2 \eta}{\partial x^2}$$
$$= \frac{\tau_0}{\mu_0} \frac{L_0+\Delta L}{\Delta L} \frac{\partial^2 \eta}{\partial x^2}.$$
(1.109)

In this form there is no longer any reference to the (artificially introduced) beads and springs, and the wave speed is given by

$$v^{2} = \frac{\tau_{0}}{\mu_{0}} \frac{L_{0} + \Delta L}{\Delta L}.$$
(1.110)

Though no physically realistic string could behave this way, it is convenient to imagine that the string is *ideal*, in the sense that with zero tension its length vanishes, $L_0 = 0$, in which case the wave equation becomes

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\tau_0}{\mu_0} \frac{\partial^2 \eta}{\partial x^2}.$$
(1.11)

1.7.2.1 Sound Waves in a Long Solid Rod

It is a bit of a digression, but a similar setup can be used to describe sound waves in a long solid rod. Superficially, Eq. (1.110) seems to give the troubling result that the wave speed is infinite since, the rod not being stretched at all, $\Delta L = 0$. A reason for this "paradox" is that the dependence of string length on tension is nonlinear at the point where the tension vanishes. (You can't "push on a string.") The formulation in the previous section only makes sense for motions per bead small compared to the extension per bead $\Delta L/N$. Stated differently, the instantaneous tension τ must remain small compared to the standing tension τ_0 .

A solid, on the other hand, resists both stretching and compression and, if there is no standing tension, the previously assumed approximations are invalid. To repair the analysis one has to bring in Young's modulus *Y*, in terms of which the length change ΔL of a rod of length L_0 and cross sectional area *A*, subject to tension τ , is given by

$$\Delta L = L_0 \, \frac{\tau/A}{Y}.\tag{1.112}$$

This relation can be used to eliminate ΔL from Eq. (1.109). Also neglecting ΔL relative to L_0 , and using the relation $\mu_0 = \rho_0 A$ between mass density and line density, the wave speed is given by

$$v^{2} = \frac{\tau}{\mu_{0}} \frac{L_{0}}{L_{0} \frac{1}{Y} \frac{\tau}{A}} = \frac{Y}{\rho_{0}}.$$
(1.113)

This formula for the speed of sound meets the natural requirement of depending only on intrinsic properties of the solid.

Prescription (1.112) can also be applied to evaluate the coefficient in Eq. (1.109) in the (realistic) stretched string case;

$$ka = YA$$
, and $\frac{a}{m} = \frac{1}{\mu}$, give $v^2 = \frac{YA}{\mu}$. (1.114)

Here *Y* is the "effective Young's modulus" in the particular stretched condition.

1.8 Field Theoretical Treatment and Lagrangian Density

It was somewhat artificial to treat a continuous string as a limiting case of a beaded string. The fact is that the string configuration can be better described by a continuous function $\eta(x, t)$ rather than by a finite number of discrete generalized coordinates $\eta_i(t)$. It is then natural to express the kinetic and potential energies by the integrals

$$T = \frac{\mu}{2} \int_0^L \left(\frac{\partial \eta}{\partial t}\right)^2 dx, \quad V = \frac{\tau}{2} \int_0^L \left(\frac{\partial \eta}{\partial x}\right)^2 dx.$$
(1.115)

In working out *V* here, the string has been taken to be ideal and Eq. (1.96) was expressed in continuous terms. The Lagrangian L = T - V can therefore be expressed as

$$L = \int_0^L \mathcal{L} \, dx,\tag{1.116}$$

where the "Lagrangian density" \mathcal{L} is given by

$$\mathcal{L} = \frac{\mu}{2} \left(\frac{\partial\eta}{\partial t}\right)^2 - \frac{\tau}{2} \left(\frac{\partial\eta}{\partial x}\right)^2.$$
(1.117)

Then the action *S* is given by

$$S = \int_{t_0}^{t_1} \int_0^L \mathcal{L}(\eta, \eta_{,x}, \eta_{,t}, x, t) \, dx \, dt.$$
(1.118)

For \mathcal{L} as given by Eq. (1.117) not all the Lagrangian arguments shown in Eq. (1.118) are, in fact, present. Only the partial derivative of η with respect to x, which is indicated by $\eta_{,x}$ and $\eta_{,t}$, which similarly stands for $\partial \eta / \partial t$, are present. In general, \mathcal{L} could also depend on η , because of nonlinear restoring force, or on x, for example because the string is nonuniform, or on t, for example because the string (slowly) with time.



Fig. 1.7 Appropriate slicing of the integration domain for integrating the term $\partial/\partial x (\delta \eta (\partial \mathcal{L}/\partial \eta_{,x}))$ in Eq. (1.119).

The variation of \mathcal{L} needed as the integrand of δS is given by

$$\delta \mathcal{L} = \mathcal{L}(\eta_{,x} + \delta \eta_{,x}, \eta_{,t} + \delta \eta_{,t}) - \mathcal{L}(\eta_{,x})$$

$$\approx \frac{\partial \mathcal{L}}{\partial \eta_{,x}} \,\delta \eta_{,x} + \frac{\partial \mathcal{L}}{\partial \eta_{,t}} \,\delta \eta_{,t}$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \eta_{,x}} \,\delta \eta \right) + \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \eta_{,t}} \,\delta \eta \right) - \delta \eta \left(\frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \eta_{,x}} \right) + \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \eta_{,t}} \right) \right).$$
(1.119)

This is expressed as an approximation but, in the limit in which Hamilton's principle applies, the approximation will have become exact. The purpose of the final manipulation, as always in the calculus of variations, has been to re-express the integrand of δS as the sum of two terms, one of which is proportional to $\delta \eta$ and the other of which depends only on values of the functions on the boundaries.

In the present case the boundary is a rectangle bounded by $t = t_0$, t = t, x = 0, and x = L, as shown in Fig. 1.7. The region can, if one wishes, be broken into strips parallel to the *x*-axis, as shown. When integrated over any one of these strips, the first term on the right-hand side in the final form of Eq. (1.119) can be written immediately as the difference of the function in parenthesis evaluated at the end points of the strip. The integral over the second term can be evaluated similarly, working with strips parallel to the *t*-axis. In this way the integral over the first two terms can be evaluated as a line integral around the boundary. There is a form of Green's theorem that permits this line integral to be expressed explicitly but, for simplicity, we simply assume that this boundary integral vanishes, for example because $\delta\eta$ vanishes everywhere on the boundary.

Finally δS can be expressed as an integral over the remaining term of Eq. (1.119) and required to be zero. Because $\delta \eta$ is arbitrary, the quantity in

parenthesis must therefore vanish;

$$\frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \eta_{,x}} \right) + \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \eta_{,t}} \right) = 0.$$
(1.120)

This is the form taken by the Lagrange equations in this (simplest possible) continuum example. When applied to Lagrangian density (1.117), the result is a wave equation identical to Eq. (1.111).

For comparison with relativistic string theory in Chapter 12, one can introduce generalized momentum densities

$$\mathcal{P}^{(x)} = \frac{\partial \mathcal{L}}{\partial \eta_{,x}} = -\tau \frac{\partial \eta}{\partial x},$$
(1.121)

$$\mathcal{P}^{(t)} = \frac{\partial \mathcal{L}}{\partial \eta_{,t}} = \mu \, \frac{\partial \eta}{\partial t}.$$
(1.122)

In terms of these quantities the wave equation is

$$\frac{\partial \mathcal{P}^{(x)}}{\partial x} + \frac{\partial \mathcal{P}^{(t)}}{\partial t} = 0.$$
(1.123)

Boundary conditions at the ends of the string are referred to as Dirichlet (fixed ends) or Neumann (free ends). The Dirichlet end condition can be expressed by $\mathcal{P}^{(t)}(t, x = 0, L) = 0$; the Neumann end condition by $\mathcal{P}^{(x)}(t, x = 0, L) = 0$.

A closer analogy with relativistic string theory is produced by generalizing the disturbance $\eta \rightarrow \eta^{\mu}$ in order to represent the disturbance as just one of the three components of a vector – transverse-horizontal, or transverse-vertical, or longitudinal. Also we introduce the abbreviations of overhead dot for $\partial/\partial t$ and prime for $\partial/\partial x$. With these changes the momentum densities become

$$\mathcal{P}_{\mu}^{(t)} = \frac{\partial \mathcal{L}}{\partial \dot{\eta}^{\mu}}, \quad \mathcal{P}_{\mu}^{(x)} = \frac{\partial \mathcal{L}}{\partial \eta^{\mu'}}.$$
(1.124)

An unattractive aspect of the dot and prime notation is that the indices on the two sides of these equations seem not to match. The parentheses on (*t*) and (*x*) are intended to mask this defect. In this case the Lagrange equation (1.123) also acquires an index μ , one value for each possible component of displacement;

$$\frac{\partial \mathcal{P}_{\mu}^{(x)}}{\partial x} + \frac{\partial \mathcal{P}_{\mu}^{(t)}}{\partial t} = 0, \quad \mu = x, y, z.$$
(1.125)

If η corresponds to, say, *y*-displacement, in Eq. (1.123), then that equation is reproduced by Eq. (1.125) by setting μ to *y*.

1.9 Hamiltonian Density for Transverse String Motion

The generalization from discrete to continuous mass distributions is less straightforward for Hamiltonian analysis than for Lagrangian analysis. In defining the Lagrangian density the spatial and time coordinates were treated symmetrically, but the Hamiltonian density has to single out time for special treatment. Nevertheless, starting from Eq. (1.122), suppressing the (t) superscript, and mimicking the discrete treatment, the Hamiltonian density is defined by

$$\mathcal{H} = \mathcal{P}\dot{\eta} - \mathcal{L}(\dot{\eta}(x, \mathcal{P}), \eta'). \tag{1.126}$$

In terms of this equation the arguments are shown only for \mathcal{L} , and only to make the points that \mathcal{L} is independent of η and t and that, as usual, $\dot{\eta}$ has to be eliminated. Exploiting these features, $\partial \mathcal{H}/\partial t$ is given by

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{P}}{\partial t} \dot{\eta} - \frac{\partial L}{\partial \eta'} \dot{\eta}' = -\frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \eta'} \right) \dot{\eta} - \frac{\partial L}{\partial \eta'} \dot{\eta}'.$$
(1.127)

In the first form here, the usual cancellation on which the Hamiltonian formalism is based has been performed and, in the second the Lagrange equation has been used. The equation can be further simplified to

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \eta'} \, \dot{\eta} \right). \tag{1.128}$$

The Hamiltonian for the total system is defined by

$$H = \int_0^L \mathcal{H} \, dx. \tag{1.129}$$

Because energy can "slosh around" internally, one cannot expect ${\cal H}$ to be conserved, but one can reasonably evaluate

$$\frac{dH}{dt} = \int_0^L \frac{\partial \mathcal{H}}{\partial t} \, dx = -\int_0^L \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \eta'} \, \dot{\eta}\right) dx = -\left[\frac{\partial \mathcal{L}}{\partial \eta'} \, \dot{\eta}\right]_0^L. \tag{1.130}$$

where, under the integral, because x is fixed, only the partial derivative of \mathcal{H} is needed. In this form one sees that any change in total energy H is ascribable to external influence exerted on the string at its ends.

Problem 1.9.1. For a string the Lagrangian density can be expressed in terms of T and V given in Eq. (1.115). Define kinetic energy density T and potential energy density V and show that

$$\mathcal{H} = \mathcal{T} + \mathcal{V}. \tag{1.131}$$

Problem 1.9.2. Show that, for a string with passive (fixed or free) connections at its ends, the total energy is conserved.

Problem 1.9.3. For a nonuniform string the mass density $\mu(x)$ depends on position *x*, though not on time. The tension $\tau(x)$ may also depend on *x*, perhaps because its own weight causes tension as the string hangs vertically. Repeat the steps in the discussion of Hamiltonian density and show how all equations can be generalized so that the same conclusions can be drawn.

1.10

String Motion Expressed as Propagating and Reflecting Waves

(Following Zwiebach) the general motion of a string can be represented as a superposition of traveling waves, with reflections at the ends dependent on the boundary conditions there. For simplicity here, let us assume the boundaries are free – so-called Neumann boundary conditions. Such boundary conditions can be achieved, in principle, by attaching the ends of the string to rings that are free to ride frictionlessly on rigid, transverse posts. The slope of the string at a free end has to vanish since there can be no transverse external force capable of balancing a transverse component of the force of tension.

The general solution for transverse displacement of a string stretched on the range $0 \le x \le L$, for which the wave speed is $v = \sqrt{T/\mu}$, is

$$y = \frac{1}{2} \Big(f(vt + x) + g(vt - x) \Big). \tag{1.132}$$

Here *f* and *g* are arbitrary functions. Because of the free end at x = 0, one has

$$0 = \left. \frac{\partial y}{\partial x} \right|_{x=0} = \frac{1}{2} \left(f'(vt) - g'(vt) \right). \tag{1.133}$$

As time evolves, since the argument *vt* takes on all possible values, this equation can be expressed as

$$f'(u) = g'(u), (1.134)$$

for arbitrary argument *u*. One therefore has

$$f(u) = g(u) + \text{constant.} \tag{1.135}$$

Since the "constant" can be suppressed by redefinition of f, this can be expressed, without loss of generality, by f(u) = g(u) and the general solution written as

$$y = \frac{1}{2} \Big(f(vt + x) + f(vt - x) \Big).$$
(1.136)



Fig. 1.8 The shape of a 2L-periodic function f(u) which can produce general string motion as a superposition of the form (1.136).

Because there is also a free end at x = L we have

$$0 = \left. \frac{\partial y}{\partial x} \right|_{x=L} = \frac{1}{2} \Big(f'(vt+L) - f'(vt-L) \Big). \tag{1.137}$$

Again using the variable *u* to express a general argument, it follows that f'(u) is a periodic function of *u* with period 2*L*;

$$f'(u+2L) = f'(u). (1.138)$$

This relation is consistent with a term in f(u) proportional to u, but if one or both of the string ends are fixed such an inexorable growth would be excluded and f(u) would be a function such as shown in Fig. 1.8. Any function satisfying Eq. (1.138) can be expressed as a Fourier series;

$$f'(u) = f_1 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi}{L} nu + b_n \sin \frac{\pi}{L} nu \right).$$
(1.139)

This can be integrated and, for simplicity, new coefficients introduced to swallow the multiplicative factors;

$$f(u) = f_0 + f_1 u + \sum_{n=1}^{\infty} \left(A_n \cos \frac{\pi}{L} n u + B_n \sin \frac{\pi}{L} n u \right).$$
(1.140)

The general solution can then be written by inserting this sum into Eq. (1.136). Restoring the explicit dependences on x and t, and using well-known trigonometric identities yields

$$y = f_0 + f_1 v t + \sum_{n=1}^{\infty} \left(A_n \cos n \frac{\pi v t}{L} + B_n \sin n \frac{\pi v t}{L} \right) \cos n \frac{\pi x}{L}.$$
 (1.141)

Stated as an initial value problem, the initial displacements and velocities would be given functions $y|_0$ and $\partial y/\partial t|_0$, which are necessarily expressible as

$$y|_{0}(x) = f_{0} + \sum_{n=1}^{\infty} \left(A_{n} \cos n \frac{\pi x}{L} \right),$$

$$\frac{\partial y}{\partial t}\Big|_{0}(x) = f_{1}v + \frac{\pi v}{L} \sum_{n=1}^{\infty} \left(n B_{n} \cos n \frac{\pi x}{L} \right).$$
(1.142)

This permits the coefficients to be determined:

$$\int_{0}^{L} y|_{0}(x) dx = f_{0}L,$$

$$\int_{0}^{L} \cos\left(\frac{m\pi x}{L}\right) y|_{0}(x) dx = A_{m}\frac{L}{2},$$

$$\int_{0}^{L} \sin\left(\frac{m\pi x}{L}\right) \frac{\partial y}{\partial t}\Big|_{0}(x) dx = \frac{m\pi v}{L} B_{m}\frac{L}{2}.$$
(1.143)

Motion always "back and forth" between two limits, say *a* and *b*, in one dimension, due to a force derivable from a potential energy function V(x), is known as "libration." Conservation of energy then requires the dependence of velocity on position to have the form

$$\dot{x}^2 = (x-a)(b-x)\psi(x), \text{ or } \dot{x} = \pm\sqrt{(x-a)(b-x)\psi(x)},$$
 (1.144)

where $\psi(x) > 0$ through the range $a \le x \le b$, but is otherwise an arbitrary function of x (derived, of course, from the actual potential function). It is necessary to toggle between the two \pm choices depending on whether the particle is moving to the right or to the left. Consider the change of variable $x \rightarrow \theta$ defined by

$$x = \alpha - \beta \cos \theta$$
, where $\alpha - \beta = a$, $\alpha + \beta = b$. (1.145)

1.11 Problems

Problem 1.11.1. Show that $(x - a)(b - x) = \beta^2 \sin^2 \theta$ and that energy conservation is expressed by

$$\dot{\theta} = \sqrt{\psi(\alpha - \beta \cos \theta)},$$
 (1.146)

where there is no longer a sign ambiguity because $\dot{\theta}$ is always positive. The variable θ is known as an "angle variable." One-dimensional libration motion can always be expressed in terms of an angle variable in this way, and then can be "reduced to quadrature" as

$$t = \int^{\theta} \frac{d\theta'}{\sqrt{\psi(\alpha - \beta \cos \theta')}}.$$
(1.147)

This type of motion is especially important in the conditionally periodic motion of multidimensional oscillatory systems. This topic is studied in Section 14.6.

Problem 1.11.2. The Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\omega^2 x^2 + \omega_2^2 y^2) + \alpha xy, \qquad (1.148)$$

with $|\alpha| \ll \omega^2$ and $|\alpha| \ll \omega_2^2$, describes two oscillators that are weakly coupled.

- (a) Find normal coordinates and normal mode frequencies Ω_1 and Ω_2 .
- (b) For the case $\omega = \omega_2$, describe free motion of the oscillator.
- (c) Holding α and ω_2 fixed, make a plot of Ω versus ω showing a branch for each of Ω_1 and Ω_2 . Do it numerically or with a programming language if you wish. Describe the essential qualitative features exhibited? Note that the branches do not cross each other.

Problem 1.11.3. In integral calculus the vanishing of a definite integral does not, in general, imply the vanishing of the integrand; there can be cancellation of negative and positive contributions to the integral. Yet, in deriving Eqs. (1.25) and (1.120), just such an inference was drawn. Without aspiring to mathematical rigor, explain why the presence of an arbitrary multiplicative factor in the integrand makes the inference valid.

Problem 1.11.4. Transverse oscillations on a string with just three beads, shown in Fig. 1.2, has been analyzed in Problem 1.4.3. The infinite beaded string shown in Fig. 1.6 is similarly capable of transverse oscillation, with transverse bead locations being \ldots , y_{i-1} , y_i , y_{i+1} , \ldots Using the parameters k and m of the longitudinal model, replicate, for transverse oscillations, all the steps that have been made in analyzing longitudinal oscillations of the beaded string. Start by finding the kinetic and potential energies and the Lagrangian and deriving the Lagrange equations of the discrete system, and finding the propagation speed. Then proceed to the continuum limit, deriving the wave equation and the Lagrangian density.

Problem 1.11.5. Struck string. To obtain a short pulse on a stretched string it is realistic to visualize the string being struck with a hammer, as in a piano, rather than being released from rest in a distorted configuration. Consider an infinite string with tension T_0 and mass density μ_0 . An impulse I (which is force times time) is administered at position $x = x_0$ to a short length Δx of the string. Immediately after being struck, while the string is still purely horizontal, the string's transverse velocity can be expressed (in terms of unit step function U) by a square pulse

$$\frac{\partial y}{\partial t}(0+,x) = K\left(U(x-x_0) - U\left(x - (x_0 + \Delta x)\right)\right). \tag{1.149}$$

(a) Express the constant K in terms of the impulse I and establish initial traveling waves on the string which match the given initial excitation. Sketch the shape of the string for a few later times.

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(b) Consider a length of the same string stretched between smooth posts at x = 0and x = a. (i.e., Neumann boundary conditions). Describe (by words or sketches) the subsequent motion of the string. Does the motion satisfy the conservation of momentum?

Problem 1.11.6. *In the same configuration as in Problem 1.11.5, with the string stretched between smooth posts, let the initial transverse velocity distribution be given by*

$$y(0+,x) = 0, \quad \frac{1}{v_0} \frac{\partial y}{\partial t}(0+,x) = \frac{x-a/2}{a} - 4\left(\frac{x-a/2}{a}\right)^3. \tag{1.150}$$

Find the subsequent motion of the string.

Problem 1.11.7. On a graph having x as abscissa and vt as ordinate the physical condition at t = 0 of a string with wave speed v stretched between 0 and a can be specified by values of the string displacement at all points on the horizontal axis between 0 and a. An "area of influence" on this graph can be specified by a condition such as "the area containing points at which a positive-traveling wave could have been launched that affects the t = 0 condition of the string." Other areas of influence can be specified by replacing "positive-traveling" by "negative-traveling" or by reversing cause and effect. From these areas find the region on the plot containing points at which the nature of the end connection cannot be inferred from observation of the string.

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