7 Composite Systems

We turn now to composite systems, and begin by providing the necessary mathematical tools. We then generalise the postulates and discuss the special case of measurements on subsystems in detail. The consequences of entanglement will become clear in this discussion. We will demonstrate a conjuring trick which cannot be explained by classical physics. The unitary dynamics can once again be formulated with the aid of Liouville operators. The action of simple quantum gates on multiple qubit systems will be introduced.

7.1 Subsystems

We are accustomed from classical physics to the fact that *composite systems* (or compound systems) can be decomposed into their *subsystems* and that conversely, individual systems can be combined to give overall composite systems. The classical total system is completely describable in terms of the states of its subsystems and their mutual dynamic interactions. The solar system with the sun, the planets and the gravitational field is an example. In quantum physics, however, it is found that composite systems can have in addition completely different and surprisingly unified properties. These come to light when the composite quantum systems are in *entangled states* In such cases, it is indeed true in a certain sense that “the whole is more than the sum of its parts”. We will present the details in a similar fashion as in Sect. 1.2 and begin with a discussion of preparation and measurements.

But first: what are composite systems? There are particular quantum systems which exhibit an internal structure. *One can distinguish in them two or more subsystems which can be accessed separately.* With this we mean that subsystems can be experimentally identified on which individually (and in this sense locally) interventions can be carried out. The corresponding operations are referred to as *local operations* These can be for example preparations or measurements.

We list some *bipartite systems* consisting of two subsystems. One can prepare quantum systems for which, in a measurement at two different locations, a photon can be registered at each location. There are analogous systems involving electrons. There are systems in which at one location a photon and at another an atom are detected. Subsystems are in general termed *local*, but they need not in fact be spatially separated. A composite system can be composed of e. g. an orbit (an external degree of freedom) and the polarisation (an internal degree of

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freedom) of a single quantum object. Of course two separate systems, which are completely independent of one another, can also be considered formally as a total system.

It is essential not to assume e. g. for a 2-photon system that the photons involved are themselves distinguishable (which they are *not*, as is well-known). The *locations* at which for example the photon polarisation is measured *are* distinguishable. We know that in measurements on this system, always exactly two photons are prepared together and therefore the overall system is a bipartite system. The corresponding subsystems S^A and S^B are in this case associated with the locations of the detectors, A and B (a photon at the location A or a photon at the location B). In general, apparatus which carry out operations are classical objects and thus have an individual identity. In contrast, owing to the indistinguishability of the photons, the question of *which* photon was detected in a particular measurement, e.g. by the detector at the location A, makes no sense. We will return to this point in Sect. 7.9.

Alice and Bob In order to make it especially clear that measurements or manipulations are carried out on different subsystems S^A and S^B of the composite system S^{AB}, one often introduces the experimentalists *Alice* and *Bob*, who carry out local operations on the subsystem S^A or S^B (often, but not necessarily, at different locations). By referring to Alice and Bob, we emphasize once more that many quantum-mechanical statements are to be understood *operationally* (i. e. as instructions for carrying out an action); e.g. of the type: “If Alice does *this* to subsystem S^A, then Bob will measure *that* on subsystem S^B”.

Existence We will once again assume, in agreement with the standard interpretation from Sect. 1.2, that such subsystems are not just abstract auxiliary constructions like the quantum systems in the minimal interpretation, but rather that they exist in reality. With this, we do not mean to imply that a state can be ascribed to an individual subsystem which is independent of the state of the other subsystem. In entangled systems, precisely this independence does not exist. This is the cause of many startling quantum-physical effects. It is furthermore not meant by our assumption of existence in reality that similar elementary particles of the same type, such as two photons, have individual identities and are therefore distinguishable. The assumption that the photons exist cannot lead us to such conclusions. *The possibility of separate manipulations, and not the individuality of quantum objects, defines the subsystem (compare Sect. 7.9).*

7.2 The Product Hilbert Space

We first wish to supply the mathematical formalism which we need to formulate the physics of composite systems. We require for this purpose the *product Hilbert space*.

7.2.1 Vectors

The *tensor product* $\mathcal{H}^{AB}$ of two Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$, whose dimensions need not be the same,

$$\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B \tag{7.1}$$

is itself a Hilbert space. We call $\mathcal{H}^A$ and $\mathcal{H}^B$ the *factor spaces*. For each pair of vectors $|\varphi^A\rangle \in \mathcal{H}^A$ and $|\chi^B\rangle \in \mathcal{H}^B$, there is a *product vector* in $\mathcal{H}^{AB}$, which can be written in different ways

$$|\varphi^A\rangle \otimes |\chi^B\rangle =: |\varphi^A\rangle|\chi^B\rangle =: |\varphi^A, \chi^B\rangle =: |\varphi, \chi\rangle \ . \tag{7.2}$$

It is linear in each argument with respect to multiplication by complex numbers.
With $\lambda, \mu \in \mathbb{C}$

$$|\varphi^A\rangle \otimes (\lambda|\chi_1^B\rangle + \mu|\chi_2^B\rangle) = \lambda|\varphi^A\rangle \otimes |\chi_1^B\rangle + \mu|\varphi^A\rangle \otimes |\chi_2^B\rangle \ , \tag{7.3}$$

and

$$(\lambda|\varphi_1^A\rangle + \mu|\varphi_2^A\rangle) \otimes |\chi^B\rangle = \lambda|\varphi_1^A\rangle \otimes |\chi^B\rangle + \mu|\varphi_2^A\rangle \otimes |\chi^B\rangle \ . \tag{7.4}$$

Entangled vectors If $\{|n^A\rangle\}$ is a basis of $\mathcal{H}^A$ and $\{|i^B\rangle\}$ is a basis of $\mathcal{H}^B$, then

$$\{|n^A\rangle \otimes |i^B\rangle\} \tag{7.5}$$

is a basis of $\mathcal{H}^{AB}$. For the dimension of $\mathcal{H}^{AB}$, we have $\dim\mathcal{H}^{AB} = (\dim\mathcal{H}^A)\cdot(\dim\mathcal{H}^B)$. Every vector $|\psi^{AB}\rangle$ in $\mathcal{H}^{AB}$ can be expanded in terms of the basis

$$|\psi^{AB}\rangle = \sum_{n,i} \alpha_{ni}|n^A, i^B\rangle \ . \tag{7.6}$$

All the definitions and statements can be directly applied to the product of a finite number of Hilbert spaces $\mathcal{H}^{AB\ldots M} = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \cdots \otimes \mathcal{H}^M$. We introduce also the abbreviations:

$$\mathcal{H}^{\otimes n} := \mathcal{H} \otimes \mathcal{H} \otimes \cdots \mathcal{H}, \quad |\phi\rangle^{\otimes n} := |\phi\rangle|\phi\rangle \cdots |\phi\rangle \ . \tag{7.7}$$

Vectors in $\mathcal{H}^{AB}$ which are not product vectors are called *entangled*. They can be written only as a *superposition* of product vectors. We will represent entangled pure states with such vectors; they will play an important role in the following sections. The superposition is an important reason for this. It can usually not be read directly off the decomposition in terms of the basis (7.6) whether or not a vector $|\psi^{AB}\rangle$ is entangled. Later, we will develop a criterion (Sect. 8.3.1) and also extend the concept of entanglement to density operators (Sect. 8.1.1).

The scalar product The bra vector of the product vector $|\varphi^A\rangle \otimes |\chi^B\rangle$ has the form

$$(|\varphi^A\rangle \otimes |\chi^B\rangle)^\dagger = \langle\varphi^A| \otimes \langle\chi^B| =: \langle\varphi^A|\langle\chi^B| =: \langle\varphi^A, \chi^B| =: \langle\varphi, \chi| \ . \tag{7.8}$$

It follows from this for the dual corresponding vector of $|\psi^{AB}\rangle$ as in Eq. (7.6)

$$\langle\psi^{AB}| = \sum_{n,i} \alpha_{ni}^* \langle n^A, i^B| \ . \tag{7.9}$$

The scalar product is formed in a "space by space" manner:

$$\langle\varphi^A, \chi^B|\xi^A, \zeta^B\rangle = \langle\varphi^A|\xi^A\rangle\langle\chi^B|\zeta^B\rangle \ . \tag{7.10}$$

A basis $\{|n^A, i^B\rangle\}$ of $\mathcal{H}^{AB}$ is orthonormal if

$$\langle n^A, i^B|n'^A, i'^B\rangle = \delta_{nn'}\delta_{ii'} \tag{7.11}$$

holds, i. e. when $\{|n^A\rangle\}$ and $\{|i^B\rangle\}$ are an ONB.

The Bell basis As can readily be verified, the following four vectors make up a particular ONB in the space $\mathcal{H}^{AB} = \mathcal{H}_2^A \otimes \mathcal{H}_2^B$ of 2-qubit vectors:

$$|\Phi_\pm^{AB}\rangle := \frac{1}{\sqrt{2}}(|0^A, 0^B\rangle \pm |1^A, 1^B\rangle) \; , \quad |\Psi_\pm^{AB}\rangle := \frac{1}{\sqrt{2}}(|0^A, 1^B\rangle \pm |1^A, 0^B\rangle) \; . \tag{7.12}$$

This basis plays a special role in many investigations. We shall show later that these frequently-used *Bell states* are maximally entangled. With reference to an implementation in terms of spin polarisation states, $|\Psi_-^{AB}\rangle$ is often called a *singlet state*.

7.2.2 Operators

Product operators Let C^A be a linear operator on the space $\mathcal{H}^A$ and D^B a linear operator on $\mathcal{H}^B$. The tensor product

$$C^A \otimes D^B =: C^A D^B \tag{7.13}$$

refers to a *product operator*, which acts "space by space",

$$[C^A \otimes D^B]|\varphi^A, \chi^B\rangle = |C^A\varphi^A, D^B\chi^B\rangle \; . \tag{7.14}$$

The product operator is a linear operator on $\mathcal{H}^{AB}$

$$[C^A \otimes D^B] \sum_{n,i} \alpha_{ni} |n^A, i^B\rangle = \sum_{n,i} \alpha_{ni} |C^A n^A, D^B i^B\rangle \; . \tag{7.15}$$

The dyadic operator $|\psi^{AB}\rangle\langle\theta^{AB}|$ formed from the product vectors $|\psi^{AB}\rangle = |\varphi^A, \chi^B\rangle$ and $|\theta^{AB}\rangle = |\xi^A, \zeta^B\rangle$ is likewise a product operator

$$|\psi^{AB}\rangle\langle\theta^{AB}| = |\varphi^A, \chi^B\rangle\langle\xi^A, \zeta^B| = (|\varphi^A\rangle\langle\xi^A|) \otimes (|\chi^B\rangle\langle\zeta^B|) \; . \tag{7.16}$$

The round brackets can also be left off. The identity operator on $\mathcal{H}^{AB}$ can be dyadically expanded in terms of an ONB:

$$\mathbb{1}^{AB} = \sum_{n,i} |n^A, i^B\rangle\langle n^A, i^B| = \mathbb{1}^A \otimes \mathbb{1}^B \; . \tag{7.17}$$

With the identity operator of a factor space, product operators can be constructed which are particularly important for the physical applications. The *extended operators (subsystem operators)* which are indicated by a symbol with a hat

$$\hat{C}^A := \hat{C}^{AB} := C^A \otimes \mathbb{1}^B ; \quad \hat{D}^B := \hat{D}^{AB} := \mathbb{1}^A \otimes D^B \tag{7.18}$$

are defined within $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B$, but they act only in the individual factor Hilbert spaces in a nontrivial way. They are also called *local operators*. $\hat{C}^{AB}$ and $\hat{D}^{AB}$ commute within $\mathcal{H}^{AB}$ and they obey the relations

$$\hat{C}^{AB}\hat{D}^{AB} = \hat{D}^{AB}\hat{C}^{AB} = C^A \otimes D^B \; . \tag{7.19}$$

Generalised operators Referring to the dyadic decomposition (7.17) of $\mathbb{1}^{AB}$, we can write the generalised operator Z^{AB} on $\mathcal{H}^{AB}$ in the form

$$Z^{AB} = \mathbb{1}^{AB} Z^{AB} \mathbb{1}^{AB} = \sum_{n,m} \sum_{i,j} \langle n^A, i^B | Z^{AB} | m^A, j^B \rangle (|n^A\rangle\langle m^A| \otimes |i^B\rangle\langle j^B|) \ . \quad (7.20)$$

It is determined by its matrix elements in the orthonormal basis (7.5).

The trace and partial trace The *trace* is also defined in the usual way in terms of an orthonormal basis of $\mathcal{H}^{AB}$

$$\text{tr}[Z^{AB}] := \text{tr}_{AB}[Z^{AB}] := \sum_{n,i} \langle n^A, i^B | Z^{AB} | n^A, i^B \rangle \ . \quad (7.21)$$

For product operators, it follows from this that

$$\text{tr}_{AB}[C^A \otimes D^B] = \sum_{n,i} C^A_{nn} D^B_{ii} = \text{tr}_A[C^A] \, \text{tr}_B[C^B] \quad (7.22)$$

with the matrix elements C^A_{nn} and D^B_{ii}. The trace is constituted "space by space".

The computation of the *partial trace* over one of the factor spaces, for example the space $\mathcal{H}^A$, is particularly important for physical results. It is defined by

$$\text{tr}_A[Z^{AB}] := \sum_n \langle n^A | Z^{AB} | n^A \rangle \ . \quad (7.23)$$

We can read off from Eq. (7.20) that an operator on $\mathcal{H}^B$ is generated in the process. For product operators, it follows that

$$\text{tr}_A[C^A \otimes D^B] = \text{tr}_A[C^A] D^B \ . \quad (7.24)$$

The overall trace is found to be a series of partial traces

$$\text{tr}_{AB}[Z^{AB}] = \text{tr}_B[\text{tr}_A[Z^{AB}]] = \text{tr}_A[\text{tr}_B[Z^{AB}]] \ . \quad (7.25)$$

Here, the order in which the partial traces are taken is irrelevant.

The operator basis This concept also, which we have already encountered in Sect. 1.2, can be directly applied to the product space $\mathcal{H}^{AB}$. If $\{Q^A_\alpha, \ \alpha = 1, \ldots, (\dim \mathcal{H}^A)^2\}$ represents an operator basis of $\mathcal{H}^A$ and $\{R^B_\kappa, \ \kappa = 1, \ldots, (\dim \mathcal{H}^B)^2\}$ an operator basis of $\mathcal{H}^B$, then the product operators

$$T^{AB}_{\alpha\kappa} := Q^A_\alpha \otimes R^B_\kappa \quad (7.26)$$

form an orthonormal basis of the product space $\mathcal{H}^{AB}$, owing to

$$\text{tr}_{AB}[T^{AB\dagger}_{\alpha\kappa} T^{AB}_{\beta\lambda}] = \delta_{\alpha\beta}\delta_{\kappa\lambda} \ . \quad (7.27)$$

Every operator Z^{AB}, which acts within $\mathcal{H}^{AB}$, can be expanded in terms of this basis:

$$Z^{AB} = \sum_{\alpha,\kappa} T^{AB}_{\alpha\kappa} \mathrm{tr}_{AB}[T^{AB\dagger}_{\alpha\kappa} Z^{AB}] \; . \tag{7.28}$$

There are operators on $\mathcal{H}^{AB}$ which cannot be written as products of two operators in the form $C^A \otimes D^B$ (cf. Sect. 7.8). But all the operators on $\mathcal{H}^{AB}$ can be written as the sum of product operators.

The product Liouville space We apply the concepts from Sect. 1.2 and form the *product Liouville space*

$$\mathbb{L}^{AB} = \mathbb{L}^A \otimes \mathbb{L}^B \; . \tag{7.29}$$

Its elements are the operators

$$C^{AB} = \sum_{\alpha,\beta} c_{\alpha\beta} Q^A_\alpha \otimes R^B_\beta \tag{7.30}$$

on $\mathcal{H}^{AB}$. The *Liouville operator* is defined through a generalisation of Eq. (1.87) with the Hamiltonian H^{AB} on $\mathcal{H}^{AB}$:

$$\mathcal{L}^{AB} Z^{AB} := \mathcal{L}^{AB}(Z^{AB}) := \frac{1}{\hbar}[H^{AB}, Z^{AB}]_- \; . \tag{7.31}$$

7.3 The Fundamentals of the Physics of Composite Quantum Systems

7.3.1 Postulates for Composite Systems and Outlook

We consider a *composite quantum system*, which itself is assumed to be isolated. Therefore, we can take over all the postulates from Chaps. 2 and 4 directly. In particular, the state of the composite system is described by a density operator in a Hilbert space. The operational interpretation of the concept "state" of a quantum system as "the system has been generated by a particular preparation procedure" holds here as well. The composite system $S^{AB\cdots}$ is supposed to consist of *subsystems* $S^A, S^B, \ldots$. *Since we wish to consider subsystems which are themselves quantum systems, it suggests itself that we associate each of them with a particular Hilbert space* $\mathcal{H}^A, \mathcal{H}^B, \ldots$ Then the only open question is what structure has the Hilbert space of the composite system, i.e. how is it composed from the $\mathcal{H}^A, \mathcal{H}^B, \ldots$. Here, there are in principle many mathematical possibilities. One is for example the direct sum $\mathcal{H}^{AB\cdots} = \mathcal{H}^A \oplus \mathcal{H}^B \oplus \ldots$. However, one in fact postulates the tensor product as described in Sect. 7.2.1, in order to obtain agreement with experiments. This specification has far-reaching consequences for all physical statements about composite quantum systems. We shall be interested in precisely these statements in the following sections.

The postulate *The states of an isolated composite system $S^{AB\cdots}$ which is composed of the subsystems $S^A, S^B, \ldots$ are described by density operators $\rho^{AB\cdots}$ in the product Hilbert space*

$$\mathcal{H}^{AB\cdots} = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \ldots \tag{7.32}$$

The postulates for isolated systems from Sect. 2.1 and Sect. 4.2 can be applied to the overall system $S^{AB\cdots}$. If a system is not isolated, it can be made into an isolated system by including the "rest of the world". It then becomes itself a subsystem.

Outlook We can immediately read off a series of special properties of composite systems from this postulate. The mathematical product structure (7.32) defines an organisation scheme. We demonstrate it using the example of a bipartite system S^{AB}.

(i) States: a pure state can be a product state $|\psi^{AB}\rangle = |\phi^A\rangle \otimes |\chi^B\rangle$ or an entangled state $|\psi^{AB}\rangle \neq |\phi^A\rangle \otimes |\chi^B\rangle$ (compare Sect. 7.2.1). The unusual properties of entangled states, in particular the appearance of non-classical correlations and their applications, will be discussed in the rest of this chapter and in all the remaining chapters in detail. We consider correlated density operators $\rho^{AB} \neq \rho^A \otimes \rho^B$ in Sect. 8.1.

(ii) Observables: there is a special case of the extended observable operators, such as $\hat{C}^{AB} = C^A \otimes \mathbb{1}^B$, which is generated from an observable operator which acts on only one of the product spaces. These describe *local measurements* which are carried out on only one of the subsystems (e. g. a measurement of the observable C^A on the subsystem S^A). There are however more general Hermitian operators on $\mathcal{H}^{AB}$ (e.g. $Z^{AB} = C^A \otimes D^B + E^A \otimes F^B$), which cannot be expressed as extended operators. They also correspond to projective measurements of physical observables Z^{AB}. These latter observables are called *non-local observables* or *collective observables*. The corresponding measurements are *non-local measurements*, which in general cannot be carried out directly as local measurements on S^A and S^B. This holds also for the special case of the observables which correspond mathematically to operator products (e. g. $Z^{AB} = C^A \otimes D^B$), but cannot be implemented physically as local measurements of the extended observables ($C^A \otimes \mathbb{1}^B$ and $\mathbb{1}^A \otimes D^B$). Non-local measurements are important in connection with quantum correlations and non-local information storage. We will therefore discuss them only in Sect. 9.2.

(iii) Unitary evolution: the unitary evolutions also need not have the structure $U^{AB} = U^A \otimes U^B$. There can be for example an interaction between the systems S^A and S^B. We discuss this in Sect. 7.6. Non-local unitary evolution can act to entangle and to disentangle states. In order for a composite system to be in an entangled state, dynamic interactions between the subsystems must not exist at the same time.

(iv) The postulate (7.32) provides the required possibility of separate interventions and therefore the resolution of the composite system into subsystems. Not only local observable operators, but rather all local operators which act on a subsystem commute with all the local operators which act on some other subsystem (cf. Eq. (7.19)). This does not depend on the order in which the corresponding actions occur. Thus, in measurements on subsystems, the correlations between the measured values obtained become an important quantity. They are characterised by the joint probabilities for the occurrence of the measured values.

7.3.2 The State of a Subsystem, the Reduced Density Operator, and General Mixtures

Via the postulate, the details of the projective measurement of an observable of the composite systems are determined. This measurement on the composite system is described by an Hermitian operator on $\mathcal{H}^{AB\cdots}$. The measurement of an observable on a subsystem, e. g. on S^A, is included as a special case. It is associated with an observable operator C^A which acts on $\mathcal{H}^A$. This *local measurement* corresponds in $\mathcal{H}^{AB\cdots}$ to a *local observable*

$$\hat{C}^{AB\ldots E} = C^A \otimes \mathbb{1}^B \otimes \ldots \otimes \mathbb{1}^E \ . \tag{7.33}$$

In this chapter, we shall restrict ourselves to composite systems which are composed of two subsystems. The extension to a greater number of subsystems is straightforward.

Probability statements According to the postulate, the rules for the measurement dynamics apply also to the states ρ^{AB} of the composite system S^{AB}. We investigate the resulting consequences for local measurements. To this end, it is expedient to associate to each subsystem a *reduced density operator* by taking the partial trace over the other subsystem

$$\rho^A := \mathrm{tr}_B\left[\rho^{AB}\right], \quad \rho^B := \mathrm{tr}_A\left[\rho^{AB}\right] \ . \tag{7.34}$$

Since ρ^{AB} is a density operator, ρ^A and ρ^B likewise fulfill the conditions for being density operators. The eigenvalue equation of the observable C^A,

$$C^A|c_n^{(r)A}\rangle = c_n|c_n^{(r)A}\rangle, \quad r = 1, \ldots, g_n \tag{7.35}$$

leads to the ONB $\{|c_n^{(r)A}\rangle\}$ of $\mathcal{H}^A$ and the eigenvalues $\{c_n\}$ with the degeneracies g_n. The probability of obtaining the measured value c_n from a measurement of C on the system S^A is then given by the *local projection operator*

$$\hat{P}_n^A = P_n^A \otimes \mathbb{1}^B, \quad P_n^A := \sum_{r=1}^{g_n} |c_n^{(r)A}\rangle\langle c_n^{(r)A}| \tag{7.36}$$

through the mean value

$$p(c_n) = \mathrm{tr}_{AB}[\hat{P}_n^A \rho^{AB}] = \mathrm{tr}_A[\mathrm{tr}_B\{\hat{P}_n^A \rho^{AB}\}] = \mathrm{tr}_A[P_n^A \rho^A] \ . \tag{7.37}$$

In a similar manner, for the expectation value of the observables C, we obtain

$$\langle \hat{C}^A\rangle = \mathrm{tr}_{AB}[\rho^{AB}\hat{C}^A] = \mathrm{tr}_A[\rho^A C^A] \ . \tag{7.38}$$

To summarise, we can conclude that: *all probability statements about local measurements on a subsystem S^A are obtained by associating the reduced density operator ρ^A from Eq. (7.34) to the system S^A and applying the rules postulated for the density operators of isolated systems.*

The state of a subsystem Since all probability statements for measurements on S^A are unambiguously determined by the reduced density operator ρ^A, it is tempting to say that the subsystem S^A *is in the state* ρ^A. Thus, in Chap. 2, we introduced the concept of a state. The composite system S^{AB} passes through a preparation procedure which leads to the state ρ^{AB}. Together with it, the state $\rho^A = \text{tr}_B\left[\rho^{AB}\right]$ is prepared.

General mixtures If the composite system S^{AB} is in the product state $|\alpha_k^A, \beta_k^B\rangle$, then the subsystem S^A is in the pure state $|\alpha_k^A\rangle$. If the state of S^{AB} is, in particular, a statistical mixture (blend or *proper mixture*) of such product states (cf. Chap. 4),

$$\rho^{AB} = \sum_s p_s |\alpha_s^A, \beta_s^B\rangle\langle\alpha_s^A, \beta_s^B| = \sum_s p_s |\alpha_s^A\rangle\langle\alpha_s^A| \otimes |\beta_s^B\rangle\langle\beta_s^B|, \quad \sum_s p_s = 1\;, \tag{7.39}$$

then S^A or S^B are likewise statistical mixtures

$$\rho^A = \text{tr}_B[\rho^{AB}] = \sum_s p_s |\alpha_s^A\rangle\langle\alpha_s^A|\,, \quad \rho^B = \sum_s p_s |\beta_s^B\rangle\langle\beta_s^B| \tag{7.40}$$

of the states $|\alpha_k^A\rangle$ or $|\beta_S^B\rangle$. They were produced by the preparation procedure and are present as real states. An ignorance interpretation (compare Sect. 4.3) is possible. Equation (7.40) is obtained from (7.24) and the dyadic decomposition of $\mathbb{1}^A$ or $\mathbb{1}^B$.

In general, the state of a quantum systems S^{AB} will not be a statistical mixture as in Eq. (7.39). The state of the subsystem S^A is then also not a statistical mixture. An ignorance interpretation is not possible. Nevertheless, the state is described by a reduced density operator ρ^A. We therefore employ the concept *mixture* to this state ρ^A of S^A also, although – as already mentioned in Sect. 4.2 – no "mixing" has occurred; and we simply leave off the adjective "statistical" for clarity. In this case, one also speaks of the state as an *improper mixture* in contrast to a *proper mixture*. "Mixture" is thus the umbrella term.

To make this clear, we can consider for example a system S^{AB} which is in a Bell state. In this case, the states of the subsystems are maximally mixed as a result of the entanglement

$$\rho^A = \text{tr}_B[\Phi_\pm^{AB}] = \frac{1}{2}\mathbb{1}^A\,, \quad \rho^A = \text{tr}_B[\Psi_\pm^{AB}] = \frac{1}{2}\mathbb{1}^A\,. \tag{7.41}$$

A corresponding relation holds for S^B. However, S^{AB} was prepared in a pure state.

In quantum systems, the states of subsystems can be mixtures which – with respect to their preparation – are not statistical mixtures and therefore do not permit an ignorance interpretation. For their density operators, there are formally arbitrarily many ensemble decompositions. There are therefore arbitrarily many statistical mixtures of an isolated individual system, with which they can be indistinguishably *simulated* with respect to all probability statements for local measurements. By means of local measurements, one cannot determine whether a density operator ρ^A belongs to an individual system S^A or is rather a reduced density operator of a subsystem S^A which is part of a larger system. This again justifies the application of the term "mixture" to all reduced density operators. We mention finally that mixtures in classical physics are always statistical mixtures. We shall return to the connection with entanglement later in Sect. 8.1.

7.4 Manipulations on a Subsystem

7.4.1 Relative States and Local Unitary Transformations

Relative states Making use of the ONB $\{|c_n^A\rangle\}$ and $\{|d_i^B\rangle\}$ of $\mathcal{H}^A$ or $\mathcal{H}^B$, we can write the pure state $|\psi^{AB}\rangle$ of the composite system S^{AB} as a decomposition

$$|\psi^{AB}\rangle = \sum_{n,i} \alpha_{ni} |c_n^A, d_i^B\rangle \tag{7.42}$$

in terms of the basis vectors. It proves expedient to split up the double sum in the form

$$|\psi^{AB}\rangle = \sum_n |c_n^A, \tilde{w}_n^B\rangle \tag{7.43}$$

with

$$|\tilde{w}_n^B\rangle := \sum_i \alpha_{ni} |d_i^B\rangle; \quad |w_n^B\rangle = \frac{|\tilde{w}_n^B\rangle}{\sqrt{\langle \tilde{w}_n^B | \tilde{w}_n^B\rangle}} \; . \tag{7.44}$$

The vector $|w_n^B\rangle$ describes the *relative state* belonging to $|c_n^A\rangle$. Non-normalised states are again denoted by a tilde. The relative vectors $|w_n^B\rangle$ in general do not make up an orthonormal system. Their number need not be the same as the dimension of the Hilbert state $\mathcal{H}^B$. $|\psi^{AB}\rangle$ can, in analogy to Eq. (7.43), also be decomposed in terms of the relative states $|\tilde{v}_i^A\rangle$ belonging to the $|d_i^B\rangle$:

$$|\psi^{AB}\rangle = \sum_i |\tilde{v}_i^A, d_i^B\rangle \; . \tag{7.45}$$

Local unitary manipulations We now allow a unitary dynamics to act upon the subsystem S^A

$$\hat{U}^{AB} = U^A \otimes \mathbb{1} \; . \tag{7.46}$$

It produces the transition

$$|\psi^{AB}\rangle \to |\psi'^{AB}\rangle = \sum_n |U^A c_n^A\rangle |\tilde{w}_n^B\rangle \; . \tag{7.47}$$

Here, in general the state of S^A is changed, and in particular that of S^{AB}. The vectors $|U^A c_n^A\rangle$ again represent an ONB of $\mathcal{H}^A$; thus, for the state of S^B we have the unchanged result

$$\rho^B \to \rho'^B = \rho^B = \sum_n |\tilde{w}_n^B\rangle\langle \tilde{w}_n^B| \; . \tag{7.48}$$

We take as an example of this a transition between two vectors of the Bell basis (cf. Eq. (7.12)) to which we shall return later:

$$(\sigma_1^A \otimes \mathbb{1}^B)|\Psi_+^{AB}\rangle = |\Phi_+^{AB}\rangle \; . \tag{7.49}$$

In this special case, not only the reduced density operator of S^B remains unchanged, but also that of S^A: $\rho'^A = \rho'^B = \rho^A = \rho^B = \frac{1}{2}\mathbb{1}$.

A dynamic manipulation which effects a unitary transformation of the subsystem S^A has no influence on the state of the other subsystem S^B (and vice versa). *Even when an entangled state is present, Bob can by no means determine via measurements on his subsystem S^B whether Alice has carried out a unitary manipulation on her subsystem.* One can readily convince oneself that this statement is still true if the state of S^{AB} is a mixture.

7.4.2 Selective Local Measurements

The resulting state of the composite systems We again consider a quantum system S^{AB} which is composed of the (sub) systems S^A and S^B. We wish to measure the observable C on the subsystem S^A and the observable D on the subsystem S^B (*local measurements*). The associated observable operators $\hat{C}^A = C^A \otimes \mathbb{1}^B$ and $\hat{D}^B = \mathbb{1}^A \otimes D^B$ commute

$$[\hat{C}^A, \hat{D}^B]_- = 0\,. \tag{7.50}$$

We note also the corresponding eigenvalue equations

$$C^A|c_n^A\rangle = c_n|c_n^A\rangle, \quad D^B|d_i^B\rangle = d_i|d_i^B\rangle\,. \tag{7.51}$$

The vectors $|c_n^A\rangle$ and $|d_i^B\rangle$ make up an ONB of $\mathcal{H}^A$ or $\mathcal{H}^B$. The possible measured values c_n and d_i resulting from the local measurements are assumed for simplicity not to be degenerate.

We first carry out measurements only on the subsystem S^A and apply the postulate from Sect. 7.3.1. A measurement of the observable C on the subsystem S^A, in which a selection among the results c_n of the measurements is made, transforms the state ρ^{AB} of the composite system S^{AB} into the (non-normalised) state $\tilde{\rho}'^{AB}$

$$\rho^{AB} \to \tilde{\rho}'^{AB}_n = \hat{P}_n^A \rho^{AB} \hat{P}_n^A\,. \tag{7.52}$$

The projection operator $\hat{P}_n^A$ is defined in Eq. (7.36). One can read off from Eq. (7.37) that the trace of the resulting non-normalised density operator $\tilde{\rho}'^{AB}_n$ again gives directly the probability $p(n)$ that a measurement will lead to the value c_n (cf. Eq. (4.23))

$$p(c_n) = \mathrm{tr}[\tilde{\rho}'^{AB}_n]\,. \tag{7.53}$$

If the composite system S^{AB} was previously in the pure state $|\psi^{AB}\rangle$ of Eq. (7.42), then the selective measurement causes the transition

$$|\psi^{AB}\rangle \to |\tilde{\psi}'^{AB}_n\rangle = \hat{P}_n^A|\psi^{AB}\rangle = |c_n^A\rangle \otimes \sum_i \alpha_{ni}|d_i^B\rangle = |c_n^A\rangle \otimes |\tilde{w}_n^B\rangle\,. \tag{7.54}$$

The subsystem S^B transforms into the relative state $|w_n^B\rangle$ of Eq. (7.44). For an entangled state $|\psi^{AB}\rangle$, a non-degenerate selective measurement on a subsystem breaks the entanglement.

Furthermore, we find as a special case of Eq. (7.53): *the probability of obtaining the measured value c_n is, from Eq. (7.37), given by the square of the norm of the non-normalised relative state vector $|\tilde{w}_n^B\rangle$:*

$$p(c_n) = \langle\psi^{AB}|\left(|c_n^A\rangle\langle c_n^A| \otimes \mathbb{1}^B\right)|\psi^{AB}\rangle = \langle\tilde{w}_n^B|\tilde{w}_n^B\rangle = ||\tilde{w}_n^B||^2\,. \tag{7.55}$$

$p(c_n)$ can also be written as a function of the expansion coefficients α_{ni} of Eq. 7.42:

$$p(c_n) = \sum_i |\alpha_{ni}|^2 \,. \tag{7.56}$$

The resulting state of the subsystem The reduced density operator ρ^A of the subsystem S^A is transformed into $\tilde{\rho}_n'^A$ by a selective measurement:

$$\rho^A \to \tilde{\rho}_n'^A = \mathrm{tr}_B[\tilde{\rho}_n'^{AB}] = \mathrm{tr}_B[\hat{P}_n^A \rho^{AB} \hat{P}_n^A] \,. \tag{7.57}$$

Insertion of $\hat{P}_n^A$ and normalisation leads with Eqs. (7.36) and (7.37) to

$$\rho^A \to \rho_n'^{\,A} = \frac{P_n^A \rho^A P_n^A}{\mathrm{tr}_A[P_n^A \rho^A]} = \frac{P_n^A \rho^A P_n^A}{p(c_n)} \,. \tag{7.58}$$

If the initial state is the pure state $|\psi^{AB}\rangle$, then we obtain

$$\rho_n'^A = |c_n^A\rangle\langle c_n^A| \,. \tag{7.59}$$

S^A is in the state $|c_n^A\rangle$ after the measurement. This also follows directly from Eq. (7.54).

Operational description It is helpful to make it clear on an operational level just how a selective measurement of Eq. (7.54) is carried out in practice and how the state $|\psi'^{AB}\rangle$ (cf. Eq. (7.54)) is produced. As we have seen in Sect. 2.1.2, state vectors are associated with preparation procedures. How is the corresponding preparation procedure for $|\psi'^{AB}\rangle$ carried out? Many individual bipartite systems have passed through the preparation device for $|\psi^{AB}\rangle$. The single system S^{AB} can for example consist of a photon moving to the left and one moving to the right in the state $|\psi^{AB}\rangle$. Alice measures (on the subsystem S^A, left photon) the observable C without annihilating the system. Those complete bipartite systems (photon pairs) from which Alice has obtained the measured value c_n are sorted out. Only they are used for further manipulations. This is the significance of Eq. (7.54). All the remaining bipartite systems are eliminated and no longer take part in future experiments.

To ensure that in fact complete bipartite systems (photon pairs) are sorted out, Bob must also act and eliminate his subsystem (right photon) when Alice has eliminated hers. In the example of the photons, he cannot simply let them all continue on their way. In order that he allow the correct ones to continue, Alice must give him the information for each photon pair as to whether she has sorted out her photon or not. Those photon pairs which then finally are allowed to continue are all in the state $|\psi'^{AB}\rangle = |c_n^A, w_n^B\rangle$. The overall procedure, which also includes an exchange of information, then prepares the subsystem S^B (photon at Bob's location) in the state $|w_n^B\rangle$. Bob can also number his photons, store them and later, following Alice's instructions, he can sort them. *A selective local measurement is a preparation procedure for the overall system, which is based on a selective measurement on a subsystem and on classical communication. It requires a selection process for both subsystems.*

7.4.3 A Non-Selective Local Measurement

The resulting state of the composite system Following a measurement of the observable C on the system S^A, in which no selection according to the measured values is carried out (cf. Fig. 7.1), for the composite system S^{AB}, the state $\tilde{\rho}'^{AB}_{n.s.}$ is present:

$$\rho^{AB} \xrightarrow{n.s.} \rho'^{AB}_{n.s.} = \sum_n p(c_n) \frac{\tilde{\rho}'^{AB}_n}{\mathrm{tr}[\tilde{\rho}'^{AB}_n]} = \sum_n \tilde{\rho}'^{AB}_n \ . \tag{7.60}$$

This follows immediately from equations (7.52) and (7.53). For the pure initial state $|\psi^{AB}\rangle$, we obtain, corresponding to Eq. (7.54)[1]:

$$|\psi^{AB}\rangle \xrightarrow{n.s.} \rho'^{AB}_{n.s.} = \sum_n |c_n^A, \tilde{w}_n^B\rangle\langle c_n^A, \tilde{w}_n^B| = \sum_n |c_n^A\rangle\langle c_n^A| \otimes |\tilde{w}_n^B\rangle\langle \tilde{w}_n^B| \ . \tag{7.61}$$

The superposition of Eq. (7.42) has been decomposed into the mixture of Eq. (7.61).

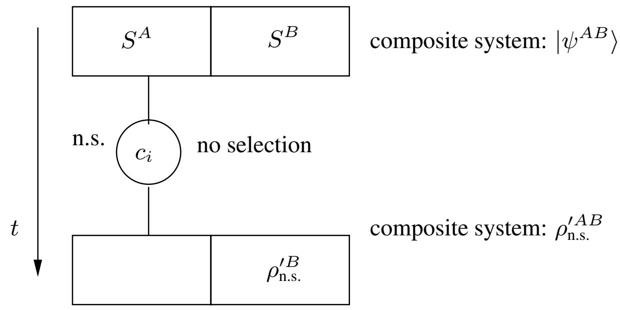

Figure 7.1: A non-selective measurement on the subsystem S^A.

The resulting states of the subsystems The state of the subsystem S^A after the non-selective measurement is given by the reduced density operator. With Eqs. (7.60) and (7.52), we obtain

$$\rho^{A} \xrightarrow{n.s.} \rho'^{A}_{n.s.} = \mathrm{tr}_B[\rho'^{AB}_{n.s.}] = \mathrm{tr}_B[\sum_n \hat{P}^A_n \rho^{AB} \hat{P}^A_n] \ . \tag{7.62}$$

Carrying out the trace with $\hat{P}^A_n = P^A_n \otimes \mathbb{1}^B$ leads to

$$\rho^{A} \xrightarrow{n.s.} \rho'^{A}_{n.s.} = \sum_n P^A_n \rho^A P^A_n \ . \tag{7.63}$$

As we saw in Sect. 7.3.2, the state of a subsystem is represented by the corresponding reduced density operator. Probability statements are obtained by following the rules for density operators in Chap. 4. The comparison of Eq. (7.58) with Eq. (4.19) and Eq. (7.63) with

[1] We shall see in Sect. 8.1 that the resulting state is not entangled.

(4.25) shows that: *for the transitions between the reduced density operators as produced by selective or non-selective local measurements on a subsystem, the rules for density operators from Chap. 4 can be applied.*

What can we say about the other subsystem S^B? All the measurements on Bob's subsystem S^B in the case of non-selective measurements by Alice on S^A can be described by the reduced density operator

$$\rho'^{B}_{n.s.} = \mathrm{tr}_A[\rho'^{AB}_{n.s.}] \ . \tag{7.64}$$

We reformulate it with the aid of Eqs. (7.60) and (7.52) and find by using $\sum_n \hat{P}_n^A = \mathbb{1}^{AB}$ the result:

$$\rho'^{B}_{n.s} = \mathrm{tr}_A[\sum_n \tilde{\rho}'^{AB}_n] = \mathrm{tr}_A[\sum_n \hat{P}_n^A \rho^{AB}] = \mathrm{tr}_A[(\sum_n \hat{P}_n^A)\rho^{AB}] = \mathrm{tr}_A \rho^{AB} = \rho^B \ . \tag{7.65}$$

The density operator $\rho'^{B}_{n.s}$ of the subsystem S^B after the non-selective measurement on S^A is the same as the density operator ρ^B before the measurement.

This is a remarkable result. Let us consider the situation in which the system S^A is at Alice's location and the system S^B at Bob's (spatially separated) location. In a preparation procedure, bipartite systems are often produced in the state ρ^{AB}. It can be entangled. It is then left open to Alice as to whether she carries our measurements of some observable C on her system or not. *Bob cannot determine in any manner by measurements on his subsystem S^B whether or not Alice has carried out measurements.* The analogous statement for unitary manipulations on the system S^A has already been derived in Sect. 7.4.1.

7.5 Separate Manipulations on both Subsystems

7.5.1 Pairs of Selective Measurements

First, Alice carries out a measurement and obtains the result c_n with the probability $p(c_n) = \langle \tilde{w}_n^B | \tilde{w}_n^B \rangle$. The system is transformed after the selection described above into the composite state $|c_n^A, w_n^B\rangle$ (compare Fig. 7.2). If Bob makes a measurement following this selection, he obtains the value d_i with the conditional probability

$$p(d_i|c_n) = \frac{|\alpha_{ni}|^2}{p(c_n)} \ . \tag{7.66}$$

This can be read off from Eqs. (7.44) and (7.55). The composite system is then transformed into the product state $|c_n^A, d_i^B\rangle$ after another selection for which Bob informs Alice of the result he has obtained. If, in reverse, first Bob and then – after selection according to the measured value d_i – Alice makes a measurement, we obtain analogously (see Fig. 7.2) after the second selection the same final state for the pair of measured values (c_n, d_i). For the probabilities, we then have

$$p(c_n|d_i) = \frac{|\alpha_{ni}|^2}{p(d_i)} \ . \tag{7.67}$$

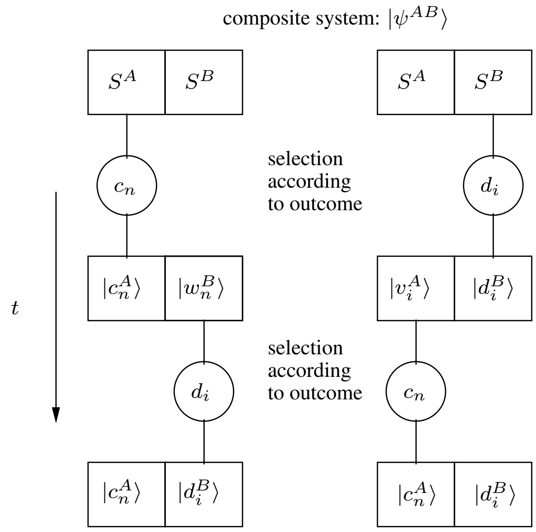

Figure 7.2: A selective measurement on the subsystems S^A and S^B. *Left*: the measurement is first carried out on S^A and then on S^B; *right* in the reverse order. A selection is carried out in each case according to the measured values d_i and c_n. The probability of obtaining the pair of measured values (c_n, d_i) and the corresponding final state $|c_n^A, d_i^B\rangle$ is the same in both cases.

The joint probability $p(c_n, d_i)$ with which the pair of measured values (c_n, d_i) is obtained from selective local measurements is independent of the order in which they are carried out. One finds

$$p(c_n, d_i) = p(c_n|d_i)p(d_i) = p(d_i|c_n)p(c_n) = |\alpha_{ni}|^2 = \langle\psi^{AB}|P_{ni}^{AB}|\psi^{AB}\rangle \qquad (7.68)$$

with the projection operator

$$P_{ni}^{AB} := |c_n^A, d_i^B\rangle\langle c_n^A, d_i^B| \,. \qquad (7.69)$$

The final state is $P_{ni}^{AB}|\psi^{AB}\rangle = |c_n^A, d_i^B\rangle$. Since the observables $\hat{C}^A$ and $\hat{D}^B$ of the local measurements commute, this could have been expected. We add that all of the statements made above for the pure initial state $|\psi^{AB}\rangle$ can be applied in the well-known manner when the initial state is a mixture with the density operator ρ^{AB}.

Instead of selecting after each local measurement, Alice and Bob can also find the state $|c_n, d_i\rangle$ after a large number of measurements by referring to the result (c_n, d_i). In this case, also, an exchange of information in both directions is necessary. It is a part of the preparation procedure for the state $|c_n^A, d_i^B\rangle$. In many cases, one is interested in the probabilities $p(c_n, d_i)$ with which the pairs of measured values (c_n, d_i) occur. To find them, Alice and Bob meet after carrying out measurements on many systems and determine the relative frequency of the combinations of measured values. These *correlations* of locally-obtained results are determined by the preparation procedure of the initial state (7.42).

Mean values The dyadic decomposition of the operators $C^A \otimes D^B$ has the form (compare Eq. (7.51))

$$C^A \otimes D^B = \sum_{n,i} c_n d_i |c_n^A, d_i^B\rangle\langle c_n^A, d_i^B| \, . \tag{7.70}$$

For its mean value in the state ρ^{AB}, we have

$$\sum_{n,i} \mathrm{tr}_{AB} \left[P_{ni}^{AB} \rho^{AB} \right] c_n d_i = \mathrm{tr}_{AB} \left[\left(C^A \otimes D^B \right) \rho^{AB} \right] \, . \tag{7.71}$$

The trace on the left side is the probability that in local measurements of $\hat{C}^A$ and $\hat{D}^B$ on the subsystems S^A and S^B, the pair of measured values (c_n, d_i) will be obtained. *The mean value of the products of correlated local measured values is the same as the mean value of the product operator.* We will make use of this fact, especially in Sects. 9.2.2 and 10.1, in connection with non-local measurements.

7.5.2 Non-Local Effects: "Spooky Action at a Distance"?

For an improved understanding, it is helpful to confront the results of the preceding sections with a popular catchword. We consider the following situation: we carry out local measurements of the same observable C on a system in the state

$$|\psi^{AB}\rangle = \frac{1}{\sqrt{2}} (|c_1^A, c_1^B\rangle + |c_2^A, c_2^B\rangle) \tag{7.72}$$

(conventions as in Sect. 7.4.2). The possible results are c_1 or c_2. The probabilities of occurrence of the pairs of measured values are

$$p(c_1, c_1) = p(c_2, c_2) = \frac{1}{2} \tag{7.73}$$

$$p(c_1, c_2) = p(c_2, c_1) = 0 \, . \tag{7.74}$$

The measurements on S^A yield, for example, the value c_1. Then one often says, in an abbreviated and sometimes misleading manner of speech, that the measurement has transformed the composite system S^{AB} into the state $|c_1^A, c_2^B\rangle$ and thereby the subsystem S^B into the state $|c_1^B\rangle$. This holds independently of the spatial separation between the system S^A at Alice's location and S^B at Bob's. In popular-scientific descriptions, this is often referred to as "spooky action at a distance"[2]. Is the situation of quantum physics correctly characterised by this term?

We have seen the the preparation of quantum objects in a state $|c_1^A, c_1^B\rangle$ requires a selection by Alice, a communication at most at the velocity of light between Alice and Bob, and a selection by Bob. This is most certainly not a case of instantaneous action at a distance.

[2] A. Einstein wrote concerning the quantum theory: "Ich kann aber deshalb nicht ernsthaft daran glauben, weil die Theorie mit dem Grundsatz unvereinbar ist, dass die Physik eine Wirklichkeit in Zeit und Raum darstellen soll, ohne spukhafte Fernwirkung". (A. Einstein in a letter to M. Born dated 3.3.1947 [EB 69]). Born's translation: "I cannot seriously believe in it because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance". We shall return to what Einstein understood to be "reality" in Chap. 10.

Perhaps the catchword is not meant to refer to states, and thereby to preparation procedures, but rather to measured values. If Alice obtains the value c_1, then Bob, according to Eq. (7.74), will with certainty find the value c_1. This is also the case when the two measurements are carried out simultaneously. For a system prepared in the state (7.72), measurements of the observable C on S^A and S^B yield completely correlated results. However, the occurrence of the two results is not causally related. We are familiar with a similar situation in the case of classical systems: as a preparation procedure, either a red or a blue ball is placed into each of two boxes. If the preparation procedure is known, then after opening one of the boxes, it can be predicted with certainty what the result of a simultaneous observation of the ball in the other box will be. It is not necessary that the colour of the one ball be somehow connected with the colour of the other via some interaction which propagates with more than the velocity of light. *Correlations are already determined by the preparation procedure*.

By the comparison to the two-ball experiment, we wished to emphasize that in this situation, the correlations are decisive. Not all statements about composite quantum systems can be simulated by classical systems such as e.g. coloured balls. We will discuss this in detail in Chap. 10. The section after the next contains a first demonstration of this.

How Alice prepares Bob's subsystem in a state of her choice For every given ONB $\{|s\rangle, |t\rangle\}$, the state $|\Phi_+^{AB}\rangle = \frac{1}{\sqrt{2}}(|0^A, 1^B\rangle - |1^A, 2^B\rangle)$ can always be written in the form

$$|\Phi_+^{AB}\rangle = \frac{1}{\sqrt{2}}(|s^A, t^B\rangle - |t^A, s^B\rangle) \ . \tag{7.75}$$

Alice wants to transform the subsystem S^B at Bob's location into the state $|s^B\rangle$. To do so, she makes a measurement on her system S^A in the ONB $\{|s^A\rangle|t^A\rangle\}$ and informs Bob if her measurement yields the result associated with $|t^A\rangle$. Then Bob can select accordingly among his subsystems. The result is a preparation procedure which leads Bob to quantum objects in the state $|s^B\rangle$. For this purpose, no quantum objects need be transmitted between Alice and Bob. The entangled state serves as a tool (similar procedures are described in Sect. 11.2 and in connection with quantum teleportation in Sect. 11.3).

7.6 The Unitary Dynamics of Composite Systems

We consider unitary transformations of the composite system. The von Neumann equation (4.9) or (4.10) can be applied to composite systems according to the postulates

$$i\hbar \frac{\mathrm{d}\rho^{AB}}{\mathrm{d}t} = [H^{AB}, \rho^{AB}(t)]_- \qquad\qquad i\frac{\mathrm{d}\rho^{AB}}{\mathrm{d}t} = \mathcal{L}^{AB}\rho^{AB}(t) \ . \tag{7.76}$$

with the Liouville operator $\mathcal{L}^{AB} \in \mathbb{L}^A \otimes \mathbb{L}^B$. We employ the Schrödinger representation. If an interaction described by the Hamiltonian $H_{\text{int}}^{AB} \neq 0$ is present between the subsystems S^A and S^B, then the individual subsystems are *open* quantum systems. The overall Hamiltonian then has the form

$$H^{AB} = H^A \otimes \mathbb{1}^B + \mathbb{1}^A \otimes H^B + H_{\text{int}}^{AB} \ . \tag{7.77}$$

The associated Liouville operator is found to be

$$\mathcal{L}^{AB} = \mathcal{L}^{A} + \mathcal{L}^{B} + \mathcal{L}^{AB}_{\text{int}} \tag{7.78}$$

and it follows for the von Neumann equation:

$$i\frac{\mathrm{d}\rho^{AB}}{\mathrm{d}t} = (\mathcal{L}^{A} + \mathcal{L}^{B} + \mathcal{L}^{AB}_{\text{int}})\rho^{AB}(t)\ . \tag{7.79}$$

This leads to a differential equation for the reduced density operator ρ^A

$$i\frac{\mathrm{d}\rho^{A}}{\mathrm{d}t} = \mathcal{L}^{A}\rho^{A}(t) + \mathrm{tr}_B[\mathcal{L}^{AB}_{\text{int}}\rho^{AB}(t)]\ . \tag{7.80}$$

To determine $\rho^A(t)$, the complete equation (7.79) must be integrated. There are various approximation methods to accomplish this. In Sect. 13.1 and Chap. 14, we will encounter an in-out approach to the dynamics of open systems, which is not based upon the differential time dependence of $\rho^A(t)$ described by Eq. (7.80). Instead, it relates the final state $\rho^A(t_{\text{out}})$ to the initial state $\rho^A(t_{\text{in}})$ by means of a superoperator.

7.7 A First Application of Entanglement: a Conjuring Trick

In the coming chapters, we will demonstrate repeatedly that entanglement is a central tool on which the effects of quantum information theory are based. Entanglement and the quantum correlations which arise from it can however also be a tool for the study of the fundamentals of quantum theory. We wish to demonstrate this in answering the following underlying question: can effects of quantum theory be explained by means of classical physics – possibly in the framework of theories which have yet to be formulated? This will give us directly an example of an application for the formalism introduced in the preceding sections. In a wider context, we will come back to this question in Chap. 10.

7.7.1 The Conjuring Trick

A magician amazes his audience with the following trick: the audience sees the magician give something to his two assistants Alice and Bob. Alice and Bob then each go into separate rooms which are perfectly insulated against any exchange of information. In each room is an audience. In each, a coin is tossed and, depending on the result of the toss, a question is asked of Alice or Bob. If the result is "heads", then the question concerns the favourite colour; it can be answered with either "red" or "green". If the tossed coin gives "tails", then the audience is to ask the question, "What is your favourite vegetable", and the answer can be either "carrots" or "peas". The question and answer are written down; one round is then finished. Alice, Bob and the magician meet again, enter the question-and-answer pair in a list with the audience as witnesses, and repeat the whole procedure again from the beginning. A large number of such rounds is completed. At the end, the combined audience analyses the question-and-answer pairs, looking for correlations.

There are four pairs of questions which can be divided into three cases: both are asked for a colour, one for a colour and the other for a vegetable, or both are asked for vegetables. In each performance, the following correlations are found:

	Alice	**Bob**
$1st$ Case	*colour?*	*colour?*
	green!	green!

To the pair of questions (colour?, colour?), the pair of answers (green!, green!) is given with a non-vanishing frequency.

$2nd$ Case	*colour?*	*vegetable?*
	green!	peas!
	vegetable?	*colour?*
	peas!	green!

When one answers this combination of questions with “green!”, then the other always gives the appropriate answer “peas”.

$3rd$ Case	*vegetable?*	*vegetable?*

To this combination of questions, one of the two assistants with certainty gives the answer “carrots!”.

This is what the audience records.

7.7.2 Classical Correlations can give no Explanation

The audience sees that pairs of answers are given with a certain regularity. How did the magician arrange this? What was his trick? The audience presumes that Alice and Bob were given slips of paper on which a colour and a vegetable were written. They then read off the answers to the questions correspondingly. The magician had prepared pairs of paper slips with different abundances, so that precisely the correlations between the answers as listed above would be observed.

Assuming this were the case, then to produce the first case in the table, the magician must have given out slips which both had the colour “green” written on them. In order that the pairs of questions (colour?, vegetable?) and (vegetable?, colour?) would always be answered correctly, the vegetable on both of these slips would have to be “peas” (second case). However, this would contradict the requirement that in the case of the third possible combination of questions (vegetable?, vegetable?) at least one of the assistants would answer with “carrots”. The method with slips of paper thus does not yield the results observed. We want to demonstrate that the magician nevertheless need not possess paranormal abilities in

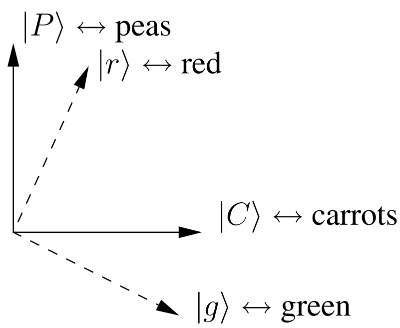

Figure 7.3: Bases in which the measurements are carried out for the conjuring trick. When pairs of photons are used, these are the analyser devices.

order to cause the observed correlations of the answers. It suffices for him to have some knowledge of entangled states.

7.7.3 The Trick

The magician's trick consists of the fact that he does not use correlated classical systems such as pairs of paper slips, but instead he makes use of entangled quantum systems. He gave to Alice and Bob each a subsystem of a bipartite systems, which was prepared to be in the state

$$|\chi^{AB}\rangle = N(|r^A,\, r^B\rangle - a^2|P^A,\, P^B\rangle) \tag{7.81}$$

with $a \in \mathbb{R}$ and $a \neq 0, a \neq 1$. N is a normalisation factor and $\{|r\rangle, |g\rangle\}$ and $\{|C\rangle, |P\rangle\}$ are orthonormal bases of $\mathcal{H}^2$, which are rotated relative to one another (see Fig. 7.3).

$$|r\rangle = a|P\rangle + b|C\rangle \tag{7.82}$$
$$|g\rangle = b|P\rangle - a|C\rangle \tag{7.83}$$

with $b \in \mathbb{R}$ and $a^2 + b^2 = 1$. Resolution leads to

$$|P\rangle = a|r\rangle + b|g\rangle \tag{7.84}$$
$$|C\rangle = b|r\rangle - a|g\rangle\ . \tag{7.85}$$

If Alice or Bob is asked for the colour, he or she carries our a measurement on the subsystem in the $\{|P\rangle, |C\rangle\}$ basis and interprets the result with respect to the state to which the measurement leads, according to the rule $|P\rangle \leftrightarrow$ "peas!" and $|C\rangle \leftrightarrow$ "carrots!". Correspondingly, when the question is for the vegetable, the measurement is carried out in the rotated $\{|r\rangle, |g\rangle\}$ basis and the answer is given according to the rule $|r\rangle \leftrightarrow$ "red!" and $|g\rangle \leftrightarrow$ "green!". We can read off from the state $|\chi^{AB}\rangle$ the probabilities with which particular pairs of answers will be given.

To find the probabilities, we insert $|P\rangle$ from Eq. (7.84) in various places into Eq. (7.81):

$$|\chi^{AB}\rangle = N(|r^A,\, r^B\rangle - a^2(a|r^A\rangle + b|g^A\rangle)(a|r^B\rangle + b|g^B\rangle)) \tag{7.86}$$
$$|\chi^{AB}\rangle = N(|r^A,\, r^B\rangle - a^2(a|r^A\rangle + b|g^A\rangle)|P^B\rangle) \tag{7.87}$$
$$|\chi^{AB}\rangle = N(|r^A,\, r^B\rangle - a^2|P^A\rangle(a|r^B\rangle + b|g^B\rangle))\ . \tag{7.88}$$

$|r\rangle$ from Eq. (7.82) inserted into $|\chi^{AB}\rangle$ leads to:

$$\begin{aligned}|\chi^{AB}\rangle &= N[(a|P^A\rangle + b|C^A\rangle)(a|P^B\rangle + b|C^B\rangle) - a^2|P^A,\, P^B\rangle] \\ &= N(b^2|C^A,\, C^B\rangle + ab(|C^A,\, P^B\rangle + |P^A,\, C^B\rangle))\ . \end{aligned} \tag{7.89}$$

From Eqs. (7.86)–(7.89), we find the probabilities for the possible pairs of measurement results and thus for the pairs of answers (compare Eq. (7.68)). From Eq. (7.86), for the pair of questions in the first case, the observed result is found:

$$p(g^A,\, g^B) = Na^4b^4 \neq 0\ . \tag{7.90}$$

Eqs. (7.87) and (7.88) lead to

$$p(g^A,\, C^B) = 0, \quad p(C^A,\, g^B) = 0\ . \tag{7.91}$$

Therefore, when the pair of questions (colour?, vegetable?) is asked, and Alice answers with "green!", then Bob always answers with "peas!" and *vice versa*. This reproduces the second case. Finally, we verify the third case with Eq. (7.89):

$$p(P^A,\, P^B) = 0\ . \tag{7.92}$$

Entanglement is the tool with which quantum magicians can carry out the tricks which classical magicians cannot master.

Consequences The experimental implementations of the basic idea of the conjuring trick are not so simple as we described in Sect. 7.7.1. The results however agree well with the quantum theoretical predictions (compare Sect. 10.8).

The conjuring trick can be carried out in principle, but not with the means and methods of classical physics. *Systems for which the results of measurements (Alice's and Bob's answers) are already predetermined before the measurement (Einstein's reality) on the corresponding subsystems (Einstein's locality), such as is the case for the slips of paper, cannot be the cause of the observed correlations.* This shows that local-realistic theories and the quantum theory can lead to differing predictions. In our example: "the conjuring trick cannot be carried out" or "the conjuring trick can be carried out"; but only the predictions of quantum theory can be experimentally verified. Thus, local-realistic *alternative theories* to the quantum theory are refuted. We will describe additional experiments in Chap. 10 and then give more precise definitions for the concepts of *Einstein reality* and *Einstein locality*.

7.8 Quantum Gates for Multiple Qubit Systems

7.8.1 Entanglement via a CNOT Gate

The processing of quantum information is often explained schematically without reference to an experimental implementation with the aid of *quantum circuits*. The essential devices which are needed are: *quantum wires*; these are special quantum channels through which quantum systems can propagate without being modified; and *quantum gates*, which effect

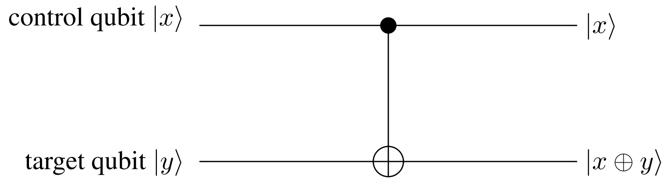

Figure 7.4: A CNOT gate.

unitary transformations of quantum systems. The systems are multi-qubits from the spaces $\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2 \ldots \otimes \mathcal{H}_2$. *Measurements* permit reading out of the information. Owing to the unitarity of their operations, quantum gates represent reversible processes. Measurements are, in contrast, irreversible. *Quantum computers* are a network of quantum gates. We have already encountered quantum gates for quantum systems in $\mathcal{H}_2$ in Sect. 3.4. We now move on to product spaces. In Chap. 12, we will assemble quantum circuits into quantum computers.

Entanglement via a CNOT gate A simple quantum gate which transforms a qubit product state into an entangled state is the *CNOT gate* or controlled NOT gate, also called an XOR gate. Its action on the computational basis of $\mathcal{H}_2^A \otimes \mathcal{H}_2^B$ is defined by

$$|x, y\rangle \rightarrow |x, y \oplus x\rangle \tag{7.93}$$

with $x, y, \ldots \in \{0, 1\}$. This determines the action on an arbitrary vector from $\mathcal{H}_2^A \otimes \mathcal{H}_2^B$. The symbol $\oplus$ denotes addition modulo 2, i. e. $1 \oplus 1 = 0$. In detail, this means that:

$$|0^A, 0^B\rangle \stackrel{\text{CNOT}}{\longrightarrow} |0^A, 0^B\rangle \tag{7.94}$$

$$|0^A, 1^B\rangle \stackrel{\text{CNOT}}{\longrightarrow} |0^A, 1^B\rangle \tag{7.95}$$

$$|1^A, 0^B\rangle \stackrel{\text{CNOT}}{\longrightarrow} |1^A, 1^B\rangle \tag{7.96}$$

$$|1^A, 1^B\rangle \stackrel{\text{CNOT}}{\longrightarrow} |1^A, 0^B\rangle \; . \tag{7.97}$$

From this, it follows that

$$(\text{CNOT}) \cdot (\text{CNOT}) = \mathbb{1} \; . \tag{7.98}$$

Applying the matrix representation in the computational basis,

$$\text{CNOT} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; , \tag{7.99}$$

one can readily verify the unitarity property:

$$(\text{CNOT})^\dagger = (\text{CNOT})^{-1} \; . \tag{7.100}$$

The qubits of the system A or B are called *control qubits* or *target qubits* (see Fig. 7.4).

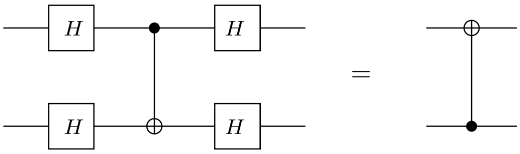

Figure 7.5: Two equivalent networks.

A simple example shows that *the CNOT gate transforms superpositions of control qubits into entanglements of control and target qubits:*

$$\left(\alpha|0^A\rangle \pm \beta|1^A\rangle\right)|0^B\rangle \quad \overset{\text{CNOT}}{\longrightarrow} \quad \alpha|0^A, 0^B\rangle \pm \beta|1^A, 1^B\rangle\,, \tag{7.101}$$

$$\left(\alpha|0^A\rangle \pm \beta|1^A\rangle\right)|1^B\rangle \quad \overset{\text{CNOT}}{\longrightarrow} \quad \alpha|0^A, 1^B\rangle \pm \beta|1^A, 0^B\rangle\,. \tag{7.102}$$

For $\alpha = \beta = \frac{1}{\sqrt{2}}$, in this manner four Bell states are formed. The reduced density operator of the target qubit is in this case $\rho^B = \frac{1}{2}\mathbb{1}^B$ (and correspondingly for the control qubit). Measurement in an arbitrary ONB of $\mathcal{H}_2^B$ yields the two measured values and states in perfect randomness with the probabilities $\frac{1}{2}$.

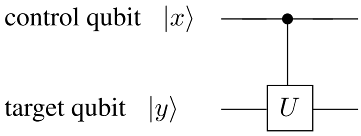

Figure 7.6: A controlled U gate.

A CNOT gate and four Hadamard gates can be combined to give the inverse of a CNOT gate (see Fig. 7.5). The CNOT gate is a special case of a *controlled U gate* (see Fig. 7.6). It leaves $|0,0\rangle$ and $|0,1\rangle$ unchanged. $|1,y\rangle$ with $y = 0,1$ goes over to $|1\rangle \otimes U|y\rangle$. CNOT corresponds to $U = \sigma_x$.

7.8.2 Toffoli, SWAP, and Deutsch Gates

The *Toffoli gate* in Fig. 7.7 is also called a *CCNOT gate* (controlled-controlled NOT) or doubly-controlled NOT gate. In this case, the NOT gate acts on the target qubit if and only if both control qubits are in the state $|1\rangle$. The action of CCNOT is

$$|x, y, z\rangle \to |x, y, z \oplus xy\rangle\;. \tag{7.103}$$

The *SWAP gate* exchanges qubit states

$$\text{SWAP}|x^A, y^B\rangle = |y^A, x^B\rangle\;. \tag{7.104}$$

Analogously, one can construct a doubly-controlled U gate (see Fig. 7.8). It can be implemented with three CNOT gates (cf. Fig. 7.9):

$$|x, y\rangle \to |x, x \oplus y\rangle \to |y, x \oplus y\rangle \to |y, x\rangle\,. \tag{7.105}$$

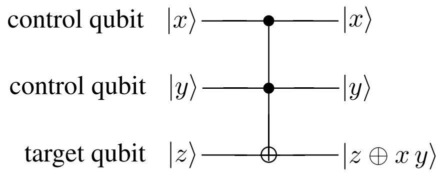

Figure 7.7: A Toffoli gate.

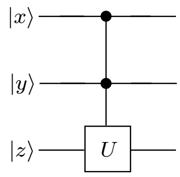

Figure 7.8: A doubly-controlled U gate.

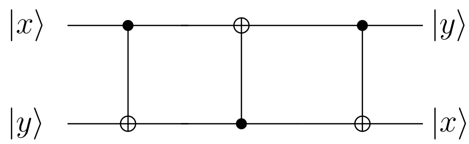

Figure 7.9: Exchange of two qubits (a SWAP gate).

Universal quantum gates are a series of quantum gates with which one can carry out every unitary transformation on $\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_2$. It can be shown that e. g. the *Deutsch gate* suffices for this purpose ([Deu 89]). In the case of this gate, the unitary transformation U in Fig. 7.8 has the form

$$U = -i \exp\left(i\frac{\theta}{2}\sigma_x\right) \quad . \tag{7.106}$$

There are other universal gates (compare Sect. 7.10). We shall return at length to this topic in Sect. 12.9.

7.9 Systems of Identical Particles*

In connection with systems whose subsystems contain elementary particles of the same type – we consider as an example two spin-$\frac{1}{2}$ particles – the following questions are frequently asked:

*The sections marked with an asterisk * can be skipped over in a first reading.

The particles are Fermions and their composite states must be antisymmetric with respect to exchange of the states of the individual particles. Therefore, they must take a form similar to the Bell vectors $|\Phi_-\rangle$ or $|\Psi_-\rangle$. Why can we not construct e.g. a teleportation procedure based on this always-present "natural" entanglement? And conversely, how can we implement teleportation with Fermions in a symmetrically-entangled state $|\Phi_+\rangle$?

Identical particles *Identical particles* have the same values of all their *intrinsic properties* such as mass, charge, spin etc. Electrons for example can be distinguished from positrons but not from each other. They cannot be marked and therefore have no individuality. An identification is not possible.

To describe systems of identical particles, one can begin with enumerated distinguishable particles and then remove their distinguishability. We restrict our considerations to 2-particle systems. The generalisation to more particles is straightforward. The state vectors of two distinguishable particles with the numbers (1) and (2) lie in the product space $\mathcal{H}^{(1)(2)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. According to the postulates, the states of identical particles are either completely symmetric in their particle numbers (Bosons, with integral spins), or completely antisymmetric (Fermions, with half-integral spins). Their state vectors lie correspondingly in subspaces of $\mathcal{H}^{(1)(2)}$, which we denote by $\mathcal{H}_+^{(1)(2)}$ and $\mathcal{H}_-^{(1)(2)}$. These subspaces are themselves not product spaces. If $\{|u^{(1)}\rangle\}$ and $\{|v^{(2)}\rangle\}$ are ONB of $\mathcal{H}^{(1)}$ or $\mathcal{H}^{(2)}$, respectively, then the bases of e.g. $\mathcal{H}_-^{(1)(2)}$ are given by $|n,i\rangle_- := \frac{1}{\sqrt{2}}(|n^{(1)}, i^{(2)}\rangle - |i^{(1)}, n^{(2)}\rangle)$ with $1 \le n \le \dim\mathcal{H}^{(1)}$ and $1 \le i \le \dim\mathcal{H}^{(2)}$. Without any interactions at all, the symmetry postulate leads to states which are formally entangled in terms of their non-observable particle numbers.

Even with the aid of observables, no identification of the particles is allowed. Observables must therefore be invariant under permutations of the particle numbers. The postulates for projection measurements apply. They are formulated with respect to the spaces $\mathcal{H}_+^{(1)(2)}$ or $\mathcal{H}_-^{(1)(2)}$. A consequence of this is that measurements or unitary dynamical evolutions cannot produce transitions between Bosons and Fermions.

Particles in two different regions of space We clarify the essential points using the example of two spin-$\frac{1}{2}$ particles with external degrees of freedom. $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are thus already assumed to be product spaces for the external degrees of freedom (vectors $|\alpha\rangle$ and $|\beta\rangle$) and for the spins (vectors $|0\rangle$ and $|1\rangle$). A possible state is then e.g.

$$|\Lambda^{(1)(2)}\rangle = \frac{1}{\sqrt{2}}\left(|\alpha,0\rangle^{(1)} \otimes |\beta,1\rangle^{(2)} - |\beta,1\rangle^{(1)} \otimes |\alpha,0\rangle^{(2)}\right) . \tag{7.107}$$

In order to make the connection to the preceding sections, we discuss a situation in which $\langle\alpha|\beta\rangle = 0$ holds. The states $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal. This is the case e.g. when the particles have differing directions of their momenta or when the wavefunctions $\langle\vec{r}|\alpha\rangle$ and $\langle\vec{r}|\beta\rangle$ are nonzero only in a restricted spatial region G_α or G_β, where G_α and G_β do not overlap ($G_\alpha \cap G_\beta = \emptyset$). Then, one can register a Fermion only in G_α or in G_β, but not outside them. However, statements about the particle numbers (1) or (2) are not possible. G_α or G_β can be e.g. different locations at which Alice A or Bob B have set up their measurement apparatus which can carry out measurements in spin space. If Alice's apparatus registers a signal, this at the same time represents a measurement in configuration space, i.e. G_α is registered. For the description of this situation, we can introduce an abbreviated form which reflects the fact that in the case $G_\alpha \cap G_\beta = \emptyset$, the state $|\alpha\rangle$ (i.e. the location G_α, Alice) is always correlated with $|0\rangle$ and the state $|\beta\rangle$ (i.e. the location G_β, Bob) is always correlated with $|1\rangle$:

$$|\Lambda^{(1)(2)}\rangle \leftrightarrow |\Lambda^{AB}\rangle = |0^A, 1^B\rangle \, . \tag{7.108}$$

With respect to all the measurements which can be carried out by Alice and Bob, the product state $|\Lambda^{AB}\rangle$ is equivalent to the state $|\Lambda^{(1)(2)}\rangle$. If Alice measures the observable σ_z, she always finds the spin state $|0\rangle$. This is the content of Eq. (7.107).

If Alice measures the observable σ_x, the result of the measurement can yield e.g. the eigenvalue $|1_x\rangle$ and the 2-Fermion system is transformed after selection into the state

$$|\Lambda^{AB}\rangle \rightarrow |\Lambda'^{AB}\rangle = |1_x^A, 1^B\rangle \, . \tag{7.109}$$

$|\Lambda'^{AB}\rangle$ can again be written in the complete form $|\Lambda'^{(1)(2)}\rangle$. To this end, we replace $|0\rangle$ on the right-hand side of Eq. (7.107) by $|1_x\rangle$. The probability of this result is $|\langle\Lambda^{(1)(2)}|\Lambda'^{(1)(2)}\rangle|^2 = |\langle\Lambda^{AB}|\Lambda'^{AB}\rangle|^2$. As a result of the orthonormalisation $\langle\alpha|\beta\rangle = 0$, the vector $|\Lambda^{AB}\rangle$ in Eq. (7.108) is a product vector in $\mathcal{H}^{AB}$.

Utilisable and non-utilisable entanglement The state introduced above, $|n, i\rangle_-$, is a superposition, from which measurable interference effects can result in particular physical situations. The fact that the particles cannot be distinguished has physical consequences. The energy spectrum of the helium atom is an example of this. In the following sections, however, we shall discuss other physical questions. The entanglement for example in the state $\frac{1}{\sqrt{2}}(|0^{(1)}, 1^{(2)}\rangle - |1^{(1)}, 0^{(2)}\rangle)$ is related to the indistinguishable particle numbers. They do not denote subsystems. Since this formal entanglement cannot be used, it cannot serve as a tool for quantum-mechanical information processing. It is not utilisable for this purpose.

Only the entanglement with the states $|\alpha\rangle$ and $|\beta\rangle$ with $\langle\alpha|\beta\rangle = 0$, as in the state $|\Lambda^{(1)(2)}\rangle$ of Eq. (7.107), opens up the possibility of intercession via $|\alpha\rangle$ and $|\beta\rangle$. As we have already seen, then $|\Lambda^{(1)(2)}\rangle$ becomes equivalent to a non-entangled product state $|\Lambda^{AB}\rangle$. If we now form e.g. by superposition with an additional state

$$|\Omega^{(1)(2)}\rangle = \frac{1}{\sqrt{2}}\Big(|\alpha, 1\rangle^{(1)} \otimes |\beta, 0\rangle^2 - |\beta, 0\rangle^{(1)} \otimes |\alpha, 1\rangle^{(2)}\Big) \leftrightarrow |\Omega^{AB}\rangle = |1^A, 0^B\rangle \tag{7.110}$$

the state vector

$$|\Psi_+^{(1)(2)}\rangle := \frac{1}{\sqrt{2}}\Big(|\Lambda^{(1)(2)}\rangle + |\Omega^{(1)(2)}\rangle\Big) \, , \tag{7.111}$$

then we can read off from the abbreviated notation

$$|\Psi_+^{(1)(2)}\rangle \leftrightarrow |\Psi_+^{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0^A, 1^B\rangle + |1^A, 0^B\rangle\right) \tag{7.112}$$

that the Bell state $|\Psi_+^{AB}\rangle$ has been formed. In spite of the addition in Eq. (7.112), it has the symmetry properties required for the description of two identical Fermions. At the same time, a utilisable entanglement has come about by the superposition described by Eq. (7.111).

If the condition $\langle\alpha|\beta\rangle = 0$ is not fulfilled in a physical situation, it can be expected that additional effects will occur in the course of the information processing, which are due to the indistinguishability of the particles. One can also switch on and off the coupling $\langle\alpha|\beta\rangle \neq 0$ in the form of a time-dependent *exchange coupling* and thereby produce an entanglement only at certain times. We mention also that the symmetry or antisymmetry property is automatically taken into account within the framework of the *second quantisation*.

7.10 Complementary Topics and Further Reading

- For "proper mixtures" and "improper mixtures": [d'Es 95], [d'Es 99].
- Local measurements and the requirements of the theory of relativity: [PT 04].
- The idea that the whole is more than the sum of its parts is referred to in philosophy as *holism*. There is a whole series of philosophical analyses in which the attempt is made to give this idea a precise meaning in many different fields from sociology to physics, and to investigate its consequences. For the natural-philosophical question as to whether there is a holism in physics, quite new aspects have resulted from the study of entangled states in composite systems (see [Pri 81, Sects. 3.7, 5.6, 6.3]). Two differing analyses of this question are introduced in [Esf 04] and [See 04] (cf. [Esf 06]). There, more detailed literature is also cited. See also [Hea 99].
- For the conjuring trick: [Har 93], [Har 98].
- Experiments on the conjuring trick: [Har 92], [TBM 95], [DMB 97], [BBD 97].
- An overview of quantum gates for qubits: [DiV 98], [Bra 02].
- In utilising coupled *quantum dots* or neutral atoms in *microtraps* as tools for quantum information processing, effects occur which are based upon the indistinguishability of particles. For details and further literature, see: [ESB 02].

7.11 Problems for Chapter 7

Prob. 7.1 [for 7.3.2]: Show that ρ^A and ρ^B in Eq. (7.34) have the properties required of a density operator.

Prob. 7.2 [for 7.4 and 7.5]: Confirm the results of Sects. 7.4 and 7.5 for the case that the initial state was not a pure state $|\psi^{AB}\rangle$, but rather a mixture, ρ^{AB}.

Prob. 7.3 [for 7.5.2]: Prove Eq. (7.75).

Prob. 7.4 [for 7.8]: Show in each case the equivalence of the networks asserted in Figs. 7.5 and 7.10.

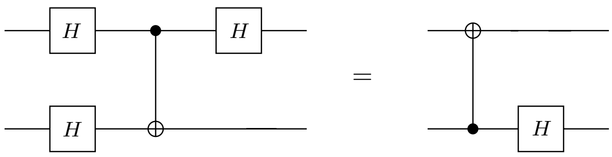

Figure 7.10: Two equivalent networks.

Prob. 7.5 [for 7.8:] Show that the network in Fig. 7.11 converts pairs of Bell states into pairs of Bell states.

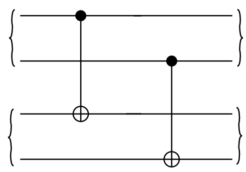

Figure 7.11: Mapping of Bell states onto Bell states.

Prob. 7.6 [for 7.8]: Show that one can construct a *quantum adder* from a Toffoli gate and a CNOT gate.

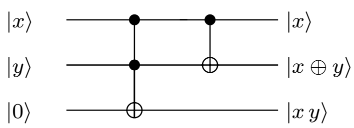

Figure 7.12: A quantum adder. The first two bits are added modulo 2. The circuitry is reversible.