1
Introduction

1.1
The Motivations

Coherent states were first studied by Schrödinger in 1926 [1] and were rediscovered by Klauder [2–4], Glauber [5–7], and Sudarshan [8] at the beginning of the 1960s. The term "coherent" itself originates in the terminology in use in quantum optics (e.g., coherent radiation, sources emitting coherently). Since then, coherent states and their various generalizations have disseminated throughout quantum physics and related mathematical methods, for example, nuclear, atomic, and condensed matter physics, quantum field theory, quantization and dequantization problems, path integrals approaches, and, more recently, quantum information through the questions of entanglement or quantum measurement.

The purpose of this book is to explain the notion of coherent states and of their various generalizations, since Schrödinger up to the most recent conceptual advances and applications in different domains of physics, with some incursions into signal analysis. This presentation, illustrated by various selected examples, does not have the pretension to be exhaustive, of course. Its main feature is a unifying method of construction of coherent states, of minimal complexity and of probabilistic nature. The procedure followed allows one to establish a simple and natural link between practically all families of coherent states proposed until now. It embodies the originality of the book in regard to well-established constructions derived essentially from group theory (e.g., coherent state family viewed as the orbit under the action of a group representation) or algebraic constraints (e.g., coherent states viewed as eigenvectors of some lowering operator), and comprehensively presented in previous treatises [10, 11], reviews [9, 12–14], an extensive collection of important papers [15], and proceedings [16].

As early as 1926, at the very beginning of quantum mechanics, Schrödinger [1] was interested in studying quantum states, which mimic their classical counterparts through the time evolution of the position operator:

$$Q(t) = e^{\frac{i}{\hbar}Ht}\, Q\, e^{-\frac{i}{\hbar}Ht}\,. \tag{1.1}$$

In this relation, $H = P^2/2m + V(Q)$ is the quantum Hamiltonian of the system. Schrödinger understood classical behavior to mean that the average or expected

Coherent States in Quantum Physics. Jean-Pierre Gazeau

ISBN: 978-3-527-40709-5

value of the position operator,

$$\overline{q}(t) = \langle \text{coherent state}|Q(t)|\text{coherent state}\rangle \,,$$

in the desired state, would obey the classical equation of motion:

$$m\ddot{\overline{q}}(t) + \overline{\frac{\partial V}{\partial q}} = 0 \,. \tag{1.2}$$

Schrödinger was originally concerned with the harmonic oscillator, $V(q) = \frac{1}{2}m^2\omega^2 q^2$. The states parameterized by the complex number $z = |z|e^{i\varphi}$, and denoted by $|z\rangle$, are defined in a way such that one recovers the familiar sinusoidal solution

$$\langle z|Q(t)|z\rangle = 2Q_o|z|\cos(\omega t - \varphi) \,, \tag{1.3}$$

where $Q_o = (\hbar/2m\omega)^{1/2}$ is a fundamental quantum length built from the universal constant $\hbar$ and the constants m and ω characterizing the quantum harmonic oscillator under consideration.

In this way, states $|z\rangle$ mediate a "smooth" transition from classical to quantum mechanics. But one should not be misled: coherent states are rigorously quantum states (witness the constant $\hbar$ appearing in the definition of Q_o), yet they allow for a classical "reading" in a host of quantum situations. This unique qualification results from a set of properties satisfied by these Schrödinger–Klauder–Glauber coherent states, also called *canonical coherent states* or *standard coherent states*.

The most important among them are the following:

(CS1) The states $|z\rangle$ saturate the Heisenberg inequality:

$$\langle \Delta Q\rangle_z \, \langle \Delta P\rangle_z = \tfrac{1}{2}\hbar \,, \tag{1.4}$$

where $\langle \Delta Q\rangle_z := [\langle z|Q^2|z\rangle - \langle z|Q|z\rangle^2]^{1/2}$.

(CS2) The states $|z\rangle$ are eigenvectors of the annihilation operator, with eigenvalue z:

$$a|z\rangle = z|z\rangle, \quad z \in \mathbb{C} \,, \tag{1.5}$$

where $a = (2m\hbar\omega)^{-1/2}\,(m\omega Q + iP)$.

(CS3) The states $|z\rangle$ are obtained from the ground state $|0\rangle$ of the harmonic oscillator by a unitary action of the Weyl–Heisenberg group. The latter is a key Lie group in quantum mechanics, whose Lie algebra is generated by $\{Q, P, I\}$, with $[Q, P] = i\hbar I_d$ (which implies $[a, a^\dagger] = I$):

$$|z\rangle = e^{(za^\dagger - \overline{z}a)}|0\rangle. \tag{1.6}$$

(CS4) The coherent states $\{|z\rangle\}$ constitute an overcomplete family of vectors in the Hilbert space of the states of the harmonic oscillator. This property is encoded in the following resolution of the identity or unity:

$$I_d = \frac{1}{\pi}\int_{\mathbb{C}} d\,\mathrm{Re}\,z \; d\,\mathrm{Im}\,z \; |z\rangle\langle z| \,. \tag{1.7}$$

These four properties are, to various extents, the basis of the many generalizations of the canonical notion of coherent states, illustrated by the family $\{|z\rangle\}$. Property (CS4) is in fact, both historically and conceptually, the one that survives. As far as physical applications are concerned, this property has gradually emerged as the one most fundamental for the analysis, or decomposition, of states in the Hilbert space of the problem, or of operators acting on this space. Thus, property (CS4) will be a sort of *motto* for the present volume, like it was in the previous, more mathematically oriented, book by Ali, Antoine, and the author [11]. We shall explain in much detail this point of view in the following pages, but we can say very schematically, that given a measure space (X, ν) and a Hilbert space $\mathcal{H}$, a family of coherent states $\{|x\rangle \mid x \in X\}$ must satisfy the operator identity

$$\int_X |x\rangle\langle x| \, \nu(dx) = I_d \,. \tag{1.8}$$

Here, the integration is carried out on projectors and has to be interpreted in a *weak sense*, that is, in terms of expectation values in arbitrary states $|\psi\rangle$. Hence, the equation in (1.8) is understood as

$$\langle\psi| \int_X x\rangle\langle x| \, \nu(dx) \, |\psi\rangle = \int_X |\langle x|\psi\rangle|^2 \, \nu(dx) = |\psi|^2 \,. \tag{1.9}$$

In the ultimate analysis, what is desired is to make the family $\{|x\rangle\}$ operational through the identity (1.8). This means being able to use it as a "frame", through which one reads the information contained in an arbitrary state in $\mathcal{H}$, or in an operator on $\mathcal{H}$, or in a setup involving both operators and states, such as an evolution equation on $\mathcal{H}$. At this point one can say that (1.8) realizes a "quantization" of the "classical" space (X, ν) and the measurable functions on it through the operator-valued maps:

$$x \mapsto |x\rangle\langle x| \,, \tag{1.10}$$

$$f \mapsto A_f \stackrel{\text{def}}{=} \int_X f(x) \, |x\rangle\langle x| \, \nu(dx) \,. \tag{1.11}$$

The second part of this volume contains a series of examples of this quantization procedure.

As already stressed in [11], the family $\{|x\rangle\}$ allows a "classical reading" of operators A acting on $\mathcal{H}$ through their expected values in coherent states, $\langle x|A|x\rangle$ ("lower symbols"). In this sense, a family of coherent states provides the opportunity to study quantum reality through a framework formally similar to classical reality. It was precisely this symbolic formulation that enabled Glauber and others to treat a quantized boson or fermion field like a classical field, particularly for computing correlation functions or other quantities of statistical physics, such as partition functions and derived quantities. In particular, one can follow the dynamical evolution of a system in a "classical" way, elegantly going back to the study of classical "trajectories" in the space X.

The formalisms of quantum mechanics and signal analysis are similar in many aspects, particularly if one considers the identities (1.8) and (1.9). In signal analysis, $\mathcal{H}$ is a Hilbert space of finite energy signals, (X, ν) a space of parameters, suitably chosen for emphasizing certain aspects of the signal that may interest us in particular situations, and (1.8) and (1.9) bear the name of "conservation of energy". Every signal contains "noise", but the nature and the amount of noise is different for different signals. In this context, choosing $(X, \nu, \{|x\rangle\})$ amounts to selecting a part of the signal that we wish to isolate and interpret, while eliminating or, at least, strongly damping a noise that has (once and for all) been regarded as unessential . Here too we have in effect chosen a frame. Perfect illustrations of the deep analogy between quantum mechanics and signal processing are *Gabor analysis* and *wavelet analysis*. These analyses yield a time–frequency ("Gaboret") or a time-scale (wavelet) representation of the signal. The built-in scaling operation makes it a very efficient tool for analyzing singularities in a signal, a function, an image, and so on – that is, the portion of the signal that contains the most significant information. Now, not surprisingly, Gaborets and wavelets can be viewed as coherent states from a group-theoretical viewpoint. The first ones are associated with the Weyl–Heisenberg group, whereas the latter are associated with the affine group of the appropriate dimension, consisting of translations, dilations, and also rotations if we deal with dimensions higher than one.

Let us now give an overview of the content and organization of the book.

Part One. Coherent States

The first part of the book is devoted to the construction and the description of different families of coherent states, with the chapters organized as follows.

Chapter 2. The Standard Coherent States: the Basics

In the second chapter, we present the basics of the Schrödinger–Glauber–Klauder–Sudarshan or "standard" coherent states $|z\rangle \equiv |q, p\rangle$ introduced as a specific superposition of all energy eigenstates of the one-dimensional harmonic oscillator. We do this through four representations of this system, namely, "position", "momentum", "Fock" or "number", and "analytical" or "Fock–Bargmann". We then describe the specific role coherent states play in quantum mechanics and in quantum optics, for which those objects are precisely the coherent states of a radiation quantum field.

Chapter 3. The Standard Coherent States: the (Elementary) Mathematics

In the third chapter, we focus on the main elementary mathematical features of the standard coherent states, particularly that essential property of being a continuous *frame*, resolving the unit operator in an "overcomplete" fashion in the space of quantum states, and also their relation to the Weyl–Heisenberg group. Appendix B is devoted to Lie algebra, Lie groups, and their representations on a very basic level to help the nonspecialist become familiar with such notions. Next, we state the probabilistic content of the coherent states and describe their links with three important quantum distributions, namely, the "P", "Q" distribution and the Wigner distribution. Appendix A is devoted to probabilities and will also help the reader grasp these essential aspects. Finally, we indicate the way in which coherent states naturally occur in the Feynman path integral formulation of quantum mechanics. In more mathematical language, we tentatively explain in intelligible terms the coherent state properties such as (CS1)–(CS4) and others characterizing on a mathematical level the standard coherent states.

Chapter 4. Coherent States in Quantum Information

Chapter 4 gives an account of a recent experimental evidence of a feedback-mediated quantum measurement aimed at discriminating between optical coherent states under photodetection. The description of the experiment and of its theoretical motivations is aimed at counterbalancing the abstract character of the mathematical formalism presented in the previous two chapters.

Chapter 5. Coherent States: a General Construction

In Chapter 5 we go back to the formalism by presenting a general method of construction of coherent states, starting from some observations on the structure of coherent states as superpositions of number states. Given a set X, equipped with a measure ν and the resulting Hilbert space $L^2(X,\nu)$ of square-integrable functions on X, we explain how the choice of an orthonormal system of functions in $L^2(X,\nu)$, precisely $\{\phi_j(x) \mid j \in \text{index set } \mathcal{J}\}$, $\int_X \overline{\phi_j(x)}\phi_{j'}(x)\,\nu(dx) = \delta_{jj'}$, carrying a probabilistic content, $\sum_{j\in\mathcal{J}} |\phi_j(x)|^2 = 1$, determines the family of coherent states $|x\rangle = \sum_j \overline{\phi_j(x)}|\phi_j\rangle$. The relation to the underlying existence of a reproducing kernel space will be clarified.

This coherent state construction is the main guideline ruling the content of the subsequent chapters concerning each family of coherent states examined (in a generalized sense). As an elementary illustration of the method, we present the coherent states for the quantum motion of a particle on the circle.

Chapter 6. Spin Coherent States

Chapter 6 is devoted to the second most known family of coherent states, namely, the so-called spin or Bloch or atomic coherent states. The way of obtaining them follows the previous construction. Once they have been made explicit, we describe their main properties: that is, we depict and comment on the sequence of properties like we did in the third chapter, the link with $SU(2)$ representations, their classical aspects, and so on.

Chapter 7. Selected Pieces of Applications of Standard and Bloch Coherent States

In Chapter 7 we proceed to a (small, but instructive) panorama of applications of the standard coherent states and spin coherent states in some problems encountered in physics, quantum physics, statistical physics, and so on. The selected papers that are presented as examples, despite their ancient publication, were chosen by virtue of their high pedagogical and illustrative content.

Application to the Driven Oscillator This is a simple and very pedagogical model for which the Weyl–Heisenberg displacement operator defining standard coherent states is identified with the S matrix connecting ingoing and outgoing states of a driven oscillator.

Application in Statistical Physics: Superradiance This is another nice example of application of the coherent state formalism. The object pertains to atomic physics: two-level atoms in resonant interaction with a radiation field (Dicke model and superradiance).

Application to Quantum Magnetism We explain how the spin coherent states can be used to solve exactly or approximately the Schrödinger equation for some systems, such as a spin interacting with a variable magnetic field.

Classical and Thermodynamical Limits Coherent states are useful in thermodynamics. For instance, we establish a representation of the partition function for systems of quantum spins in terms of coherent states. After introducing the so-called Berezin–Lieb inequalities, we show how that coherent state representation makes crossed studies of classical and thermodynamical limits easier.

Chapter 8. $SU(1,1)$, $SL(2,\mathbb{R})$, and $Sp(2,\mathbb{R})$ Coherent States
Chapter 8 is devoted to the third most known family of coherent states, namely, the $SU(1,1)$ Perelomov and Barut–Girardello coherent states. Again, the way of obtaining them follows the construction presented in Chapter 5. We then describe the main properties of these coherent states: probabilistic interpretation, link with $SU(1,1)$ representations, classical aspects, and so on. We also show the relationship between wavelet analysis and the coherent states that emerge from the unitary irreducible representations of the affine group of the real line viewed as a subgroup of $SL(2,\mathbb{R}) \sim SU(1,1)$.

Chapter 9. $SU(1,1)$ Coherent States and the Infinite Square Well
In Chapter 9 we describe a direct illustration of the $SU(1,1)$ Barut–Girardello coherent states, namely, the example of a particle trapped in an infinite square well and also in Pöschl–Teller potentials of the trigonometric type.

Chapter 10. $SU(1,1)$ Coherent States and Squeezed States in Quantum Optics
Chapter 10 is an introduction to the squeezed coherent states by insisting on their relations with the unitary irreducible representations of the symplectic groups $Sp(2,\mathbb{R}) \simeq SU(1,1)$ and their importance in quantum optics (reduction of the uncertainty on one of the two noncommuting observables present in the measurements of the electromagnetic field).

Chapter 11. Fermionic Coherent States

In Chapter 11 we present the so-called fermionic coherent states and their utilization in the study of many-fermion systems (e.g., the Hartree–Fock–Bogoliubov approach).

Part Two. Coherent State Quantization

This second part is devoted to what we call "coherent state quantization". This procedure of quantization of a measure space is quite straightforward and can be applied to many physical situations, such as motions in different geometries (line, circle, interval, torus, etc.) as well as to various geometries themselves (interval, circle, sphere, hyperboloid, etc.), to give a noncommutative or "fuzzy" version for them.

Chapter 12. Coherent State Quantization: The Klauder–Berezin Approach

We explain in Chapter 12 the way in which standard coherent states allow a natural quantization of a large class of functions and distributions, including tempered distributions, on the complex plane viewed as the phase space of the particle motion on the line. We show how they offer a classical-like representation of the evolution of quantum observables. They also help to set Heisenberg inequalities concerning the "phase operator" and the number operator for the oscillator Fock states. By restricting the formalism to the finite dimension, we present new quantum inequalities concerning the respective spectra of "position" and "momentum" matrices that result from such a coherent state quantization scheme for the motion on the line.

Chapter 13. Coherent State or Frame Quantization

In Chapter 13 we extend the procedure of standard coherent state quantization to any measure space labeling a total family of vectors solving the identity in some Hilbert space. We thus advocate the idea that, to a certain extent, quantization pertains to a larger discipline than just being restricted to specific domains of physics such as mechanics or field theory. We also develop the notion of lower and upper symbols resulting from such a quantization scheme, and we discuss the probabilistic content of the construction.

Chapter 14. Elementary Examples of Coherent State Quantization

The examples which are presented in Chapter 14 are, although elementary, rather unusual. In particular, we start with measure sets that are not necessarily phase spaces. Such sets are far from having any physical meaning in the common sense.

Finite Set We first consider a two-dimensional quantization of a N-element set that leads, for $N \geq 4$, to a Pauli algebra of observables.

Unit Interval We study two-dimensional (and higher-dimensional) quantizations of the unit segment.

Unit Circle We apply the same quantization procedure to the unit circle in the plane. As an interesting byproduct of this "fuzzy circle", we give an expression for the phase or angle operator, and we discuss its relevance in comparison with various phase operators proposed by many authors.

Chapter 15. Motions on Simple Geometries
Two examples of coherent state quantization of classical motions taking place in simple geometries are presented in Chapter 15.

Motion on the Circle Quantization of the motion of a particle on the circle (like the quantization of polar coordinates in a plane) is an old question with so far no really satisfactory answers. Many questions concerning this subject have been addressed, more specifically devoted to the problem of angular localization and related Heisenberg inequalities. We apply our scheme of coherent state quantization to this particular problem.

Motion on the Hyperboloid Viewed as a $1+1$ de Sitter Space-Time To a certain extent, the motion of a massive particle on a $1 + 1$ de Sitter background, which means a one-sheeted hyperboloid embedded in a 2+1 Minkowski space, has characteristics similar to those of the phase space for the motion on the circle. Hence, the same type of coherent state is used to perform the quantization.

Motion in an Interval We revisit the quantum motion in an infinite square well with our coherent state approach by exploiting the fact that the quantization problem is similar, to a certain extent, to the quantization of the motion on the circle S^1. However, the boundary conditions are different, and this leads us to introduce vector coherent states to carry out the quantization.

Motion on a Discrete Set of Points We end this series of examples by the consideration of a problem inspired by modern quantum geometry, where geometrical entities are treated as quantum observables, as they have to be in order for them to be promoted to the status of objects and not to be simply considered as a substantial arena in which physical objects "live".

Chapter 16. Motion on the Torus
Chapter 16 is devoted to the coherent states associated with the discrete Weyl–Heisenberg group and to their utilization for the quantization of the chaotic motion on the torus.

Chapter 17. Fuzzy Geometries: Sphere and Hyperboloid
In Chapter 17, we end this series of examples of coherent state quantization with the application of the procedure to familiar geometries, yielding a noncommutative or "fuzzy" structure for these objects.

Fuzzy Sphere This is an extension to the sphere S^2 of the quantization of the unit circle. It is a nice illustration of noncommutative geometry (approached in a rather pedestrian way). We show explicitly how the coherent state quantization of the ordinary sphere leads to its fuzzy geometry. The continuous limit at infinite spins restores commutativity.

Fuzzy Hyperboloid We then describe the construction of the two-dimensional fuzzy de Sitter hyperboloids by using a coherent state quantization.

Chapter 18. Conclusion and Outlook
In this last chapter we give some final remarks and suggestions for future developments of the formalism presented.