Contents

Preface  XIII

Part One  Coherent States  1

1  Introduction  3

1.1  The Motivations  3

2  The Standard Coherent States: the Basics  13

2.1  Schrödinger Definition  13

2.2  Four Representations of Quantum States  13

2.2.1  Position Representation  14

2.2.2  Momentum Representation  14

2.2.3  Number or Fock Representation  15

2.2.4  A Little (Lie) Algebraic Observation  16

2.2.5  Analytical or Fock–Bargmann Representation  16

2.2.6  Operators in Fock–Bargmann Representation  17

2.3  Schrödinger Coherent States  18

2.3.1  Bergman Kernel as a Coherent State  18

2.3.2  A First Fundamental Property  19

2.3.3  Schrödinger Coherent States in the Two Other Representations  19

2.4  Glauber–Klauder–Sudarshan or Standard Coherent States  20

2.5  Why the Adjective Coherent?  20

3  The Standard Coherent States: the (Elementary) Mathematics  25

3.1  Introduction  25

3.2  Properties in the Hilbertian Framework  26

3.2.1  A “Continuity” from the Classical Complex Plane to Quantum States  26

3.2.2  “Coherent” Resolution of the Unity  26

3.2.3  The Interplay Between the Circle (as a Set of Parameters) and the Plane (as a Euclidean Space)  27

3.2.4  Analytical Bridge  28

3.2.5  Overcompleteness and Reproducing Properties  29

3.3  Coherent States in the Quantum Mechanical Context  30

3.3.1  Symbols  30

3.3.2  Lower Symbols  30

Coherent States in Quantum Physics
Jean-Pierre Gazeau
Copyright © 2009 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim
ISBN: 978-3-527-40709-5
VI | Contents

3.3.3 Heisenberg Inequalities 31
3.3.4 Time Evolution and Phase Space 32
3.4 Properties in the Group-Theoretical Context 35
3.4.1 The Vacuum as a Transported Probe . . . 35
3.4.2 Under the Action of . . . 36
3.4.3 . . . the $D$-Function 37
3.4.4 Symplectic Phase and the Weyl–Heisenberg Group 37
3.4.5 Coherent States as Tools in Signal Analysis 38
3.5 Quantum Distributions and Coherent States 40
3.5.1 The Density Matrix and the Representation “$R$” 41
3.5.2 The Density Matrix and the Representation “$Q$” 41
3.5.3 The Density Matrix and the Representation “$P$” 42
3.5.4 The Density Matrix and the Wigner(–Weyl–Ville) Distribution 43
3.6 The Feynman Path Integral and Coherent States 44

4 Coherent States in Quantum Information: an Example of Experimental Manipulation 49
4.1 Quantum States for Information 49
4.2 Optical Coherent States in Quantum Information 50
4.3 Binary Coherent State Communication 51
4.3.1 Binary Logic with Two Coherent States 51
4.3.2 Uncertainties on POVMs 51
4.3.3 The Quantum Error Probability or Helstrom Bound 52
4.3.4 The Helstrom Bound in Binary Communication 53
4.3.5 Helstrom Bound for Coherent States 53
4.3.6 Helstrom Bound with Imperfect Detection 54
4.4 The Kennedy Receiver 54
4.4.1 The Principle 54
4.4.2 Kennedy Receiver Error 55
4.5 The Sasaki–Hirota Receiver 56
4.5.1 The Principle 56
4.5.2 Sasaki–Hirota Receiver Error 56
4.6 The Dolinar Receiver 57
4.6.1 The Principle 57
4.6.2 Photon Counting Distributions 58
4.6.3 Decision Criterion of the Dolinar Receiver 58
4.6.4 Optimal Control 59
4.6.5 Dolinar Hypothesis Testing Procedure 60
4.7 The Cook–Martin–Geremia Closed-Loop Experiment 61
4.7.1 A Theoretical Preliminary 61
4.7.2 Closed-Loop Experiment: the Apparatus 63
4.7.3 Closed-Loop Experiment: the Results 65
4.8 Conclusion 67

5 Coherent States: a General Construction 69
5.1 Introduction 69
## Contents

5.2 A Bayesian Probabilistic Duality in Standard Coherent States 70  
5.2.1 Poisson and Gamma Distributions 70  
5.2.2 Bayesian Duality 71  
5.2.3 The Fock–Bargmann Option 71  
5.2.4 A Scheme of Construction 72  
5.3 General Setting: “Quantum” Processing of a Measure Space 72  
5.4 Coherent States for the Motion of a Particle on the Circle 76  
5.5 More Coherent States for the Motion of a Particle on the Circle 78

6 **The Spin Coherent States** 79  
6.1 Introduction 79  
6.2 Preliminary Material 79  
6.3 The Construction of Spin Coherent States 80  
6.4 The Binomial Probabilistic Content of Spin Coherent States 82  
6.5 Spin Coherent States: Group-Theoretical Context 82  
6.6 Spin Coherent States: Fock–Bargmann Aspects 86  
6.7 Spin Coherent States: Spherical Harmonics Aspects 86  
6.8 Other Spin Coherent States from Spin Spherical Harmonics 87  
6.8.1 Matrix Elements of the $SU(2)$ Unitary Irreducible Representations 87  
6.8.2 Orthogonality Relations 89  
6.8.3 Spin Spherical Harmonics 89  
6.8.4 Spin Spherical Harmonics as an Orthonormal Basis 91  
6.8.5 The Important Case: $\sigma = j$ 91  
6.8.6 Transformation Laws 92  
6.8.7 Infinitesimal Transformation Laws 92  
6.8.8 “Sigma-Spin” Coherent States 93  
6.8.9 Covariance Properties of Sigma-Spin Coherent States 95

7 **Selected Pieces of Applications of Standard and Spin Coherent States** 97  
7.1 Introduction 97  
7.2 Coherent States and the Driven Oscillator 98  
7.3 An Application of Standard or Spin Coherent States in Statistical Physics: Superradiance 103  
7.3.1 The Dicke Model 103  
7.3.2 The Partition Function 105  
7.3.3 The Critical Temperature 106  
7.3.4 Average Number of Photons per Atom 108  
7.3.5 Comments 109  
7.4 Application of Spin Coherent States to Quantum Magnetism 109  
7.5 Application of Spin Coherent States to Classical and Thermodynamical Limits 111  
7.5.1 Symbols and Traces 112  
7.5.2 Berezin–Lieb Inequalities for the Partition Function 114  
7.5.3 Application to the Heisenberg Model 116
## Contents

### 8 SU(1,1) or SL(2,R) Coherent States

8.1 Introduction 117  
8.2 The Unit Disk as an Observation Set 117  
8.3 Coherent States 119  
8.4 Probabilistic Interpretation 120  
8.5 Poincaré Half-Plane for Time-Scale Analysis 121  
8.6 Symmetries of the Disk and the Half-Plane 122  
8.7 Group-Theoretical Content of the Coherent States 123  
8.7.1 Cartan Factorization 123  
8.7.2 Discrete Series of SU(1,1) 124  
8.7.3 Lie Algebra Aspects 126  
8.7.4 Coherent States as a Transported Vacuum 127  
8.8 A Few Words on Continuous Wavelet Analysis 129

### 9 Another Family of SU(1,1) Coherent States for Quantum Systems

9.1 Introduction 135  
9.2 Classical Motion in the Infinite-Well and Pöschl–Teller Potentials 135  
9.2.1 Motion in the Infinite Well 136  
9.2.2 Pöschl–Teller Potentials 138  
9.3 Quantum Motion in the Infinite-Well and Pöschl–Teller Potentials 141  
9.3.1 In the Infinite Well 141  
9.3.2 In Pöschl–Teller Potentials 142  
9.4 The Dynamical Algebra su(1,1) 143  
9.5 Sequences of Numbers and Coherent States on the Complex Plane 146  
9.6 Coherent States for Infinite-Well and Pöschl–Teller Potentials 150  
9.6.1 For the Infinite Well 150  
9.6.2 For the Pöschl–Teller Potentials 152  
9.7 Physical Aspects of the Coherent States 153  
9.7.1 Quantum Revivals 153  
9.7.2 Mandel Statistical Characterization 155  
9.7.3 Temporal Evolution of Symbols 158  
9.7.4 Discussion 162

### 10 Squeezed States and Their SU(1,1) Content

10.1 Introduction 165  
10.2 Squeezed States in Quantum Optics 166  
10.2.1 The Construction within a Physical Context 166  
10.2.2 Algebraic (su(1,1)) Content of Squeezed States 171  
10.2.3 Using Squeezed States in Molecular Dynamics 175

### 11 Fermionic Coherent States

11.1 Introduction 179  
11.2 Coherent States for One Fermionic Mode 179  
11.3 Coherent States for Systems of Identical Fermions 180  
11.3.1 Fermionic Symmetry SU(r) 180  
11.3.2 Fermionic Symmetry SO(2r) 185
Contents
IX

11.3.3 Fermionic Symmetry $SO(2r+1)$ 187
11.3.4 Graphic Summary 188
11.4 Application to the Hartree–Fock–Bogoliubov Theory 189

Part Two Coherent State Quantization 191
12 Standard Coherent State Quantization: the Klauder–Berezin Approach 193
12.1 Introduction 193
12.2 The Berezin–Klauder Quantization of the Motion of a Particle on the Line 193
12.3 Canonical Quantization Rules 196
12.3.1 Van Hove Canonical Quantization Rules [161] 196
12.4 More Upper and Lower Symbols: the Angle Operator 197
12.5 Quantization of Distributions: Dirac and Others 199
12.6 Finite-Dimensional Canonical Case 202

13 Coherent State or Frame Quantization 207
13.1 Introduction 207
13.2 Some Ideas on Quantization 207
13.3 One more Coherent State Construction 209
13.4 Coherent State Quantization 211
13.5 A Quantization of the Circle by $2 \times 2$ Real Matrices 214
13.5.1 Quantization and Symbol Calculus 214
13.5.2 Probabilistic Aspects 216
13.6 Quantization with k-Fermionic Coherent States 218
13.7 Final Comments 220

14 Coherent State Quantization of Finite Set, Unit Interval, and Circle 223
14.1 Introduction 223
14.2 Coherent State Quantization of a Finite Set with Complex $2 \times 2$ Matrices 223
14.3 Coherent State Quantization of the Unit Interval 227
14.3.1 Quantization with Finite Subfamilies of Haar Wavelets 227
14.3.2 A Two-Dimensional Noncommutative Quantization of the Unit Interval 228
14.4 Coherent State Quantization of the Unit Circle and the Quantum Phase Operator 229
14.4.1 A Retrospective of Various Approaches 229
14.4.2 Pegg–Barnett Phase Operator and Coherent State Quantization 234
14.4.3 A Phase Operator from Two Finite-Dimensional Vector Spaces 235
14.4.4 A Phase Operator from the Interplay Between Finite and Infinite Dimensions 237

15 Coherent State Quantization of Motions on the Circle, in an Interval, and Others 241
15.1 Introduction 241
15.2 Motion on the Circle 241
15.2.1 The Cylinder as an Observation Set 241
15.2.2 Quantization of Classical Observables 242
15.2.3 Did You Say Canonical? 243
15.3 From the Motion of the Circle to the Motion on 1 + 1 de Sitter Space-Time 244
15.4 Coherent State Quantization of the Motion in an Infinite-Well Potential 245
15.4.1 Introduction 245
15.4.2 The Standard Quantum Context 246
15.4.3 Two-Component Coherent States 247
15.4.4 Quantization of Classical Observables 249
15.4.5 Quantum Behavior through Lower Symbols 253
15.4.6 Discussion 254
15.5 Motion on a Discrete Set of Points 256
16 Quantizations of the Motion on the Torus 259
16.1 Introduction 259
16.2 The Torus as a Phase Space 259
16.3 Quantum States on the Torus 261
16.4 Coherent States for the Torus 265
16.5 Coherent States and Weyl Quantizations of the Torus 267
16.5.1 Coherent States (or Anti-Wick) Quantization of the Torus 267
16.5.2 Weyl Quantization of the Torus 267
16.6 Quantization of Motions on the Torus 269
16.6.1 Quantization of Irrational and Skew Translations 269
16.6.2 Quantization of the Hyperbolic Automorphisms of the Torus 270
16.6.3 Main Results 271
17 Fuzzy Geometries: Sphere and Hyperboloid 273
17.1 Introduction 273
17.2 Quantizations of the 2-Sphere 273
17.2.1 The 2-Sphere 274
17.2.2 The Hilbert Space and the Coherent States 274
17.2.3 Operators 275
17.2.4 Quantization of Observables 275
17.2.5 Spin Coherent State Quantization of Spin Spherical Harmonics 276
17.2.6 The Usual Spherical Harmonics as Classical Observables 276
17.2.7 Quantization in the Simplest Case: $j = 1$ 276
17.2.8 Quantization of Functions 277
17.2.9 The Spin Angular Momentum Operators 277
17.3 Link with the Madore Fuzzy Sphere 278
17.3.1 The Construction of the Fuzzy Sphere à la Madore 278
17.3.2 Operators 280
17.4 Summary 282
17.5 The Fuzzy Hyperboloid 283
Contents

18 Conclusion and Outlook 287

Appendix A The Basic Formalism of Probability Theory 289
A.1 Sigma-Algebra 289
A.1.1 Examples 289
A.2 Measure 290
A.3 Measurable Function 290
A.4 Probability Space 291
A.5 Probability Axioms 291
A.6 Lemmas in Probability 292
A.7 Bayes’s Theorem 292
A.8 Random Variable 293
A.9 Probability Distribution 293
A.10 Expected Value 294
A.11 Conditional Probability Densities 294
A.12 Bayesian Statistical Inference 295
A.13 Some Important Distributions 296
A.13.1 Degenerate Distribution 296
A.13.2 Uniform Distribution 296

Appendix B The Basics of Lie Algebra, Lie Groups, and Their Representations 303
B.1 Group Transformations and Representations 303
B.2 Lie Algebras 304
B.3 Lie Groups 306
B.3.1 Extensions of Lie algebras and Lie groups 310

Appendix C SU(2) Material 313
C.1 SU(2) Parameterization 313
C.2 Matrix Elements of SU(2) Unitary Irreducible Representation 313
C.3 Orthogonality Relations and 3 j Symbols 314
C.4 Spin Spherical Harmonics 315
C.5 Transformation Laws 317
C.6 Infinitesimal Transformation Laws 318
C.7 Integrals and 3 j Symbols 319
C.8 Important Particular Case: j = 1 320
C.9 Another Important Case: σ = j 321

Appendix D Wigner–Eckart Theorem for Coherent State Quantized Spin Harmonics 323

Appendix E Symmetrization of the Commutator 325

References 329

Index 339