1
Vector Wave Equations

Introduction

Starting from the general Maxwell equations, we shall establish the inhomogeneous vector wave equations for the case of dielectric media, which normally constitute waveguides of light. For these materials, that admit neither charge nor current, we shall deduce the homogeneous vector wave equations. Furthermore, if the propagation medium is translation invariant, the solutions to these equations describe the fields of the modes of light that are propagated along the waveguides.

In the case of step-index waveguides, the mode fields can be expressed analytically in terms of Bessel and modified Bessel functions. In the case of one-dimensional waveguides, the solutions are expressed in terms of circular and exponential functions. However, only numerical solutions are generally admitted in the case of gradient-index profiles.

1.1
Maxwell Equations for Dielectric Media

In general, the electric field **E** and the magnetic field **H** of an electromagnetic and monochromatic wave are written

$$\begin{cases} \mathbf{E}(\mathbf{r}, t) = \mathbf{E}(x, y, z)\exp(-i\omega t), \\ \mathbf{H}(\mathbf{r}, t) = \mathbf{H}(x, y, z)\exp(-i\omega t), \end{cases} \tag{1.1}$$

in Cartesian coordinates, or

$$\begin{cases} \mathbf{E}(\mathbf{r}, t) = \mathbf{E}(r, \phi, z)\exp(-i\omega t), \\ \mathbf{H}(\mathbf{r}, t) = \mathbf{H}(r, \phi, z)\exp(-i\omega t), \end{cases} \tag{1.2}$$

in cylindrical coordinates.

Dielectric media are characterized by a dielectric permittivity $\varepsilon(\mathbf{r}) = n^2(\mathbf{r})\ \varepsilon_0$ and a magnetic permeability μ. In practice, the magnetism is so weak that the permeability is considered to be equal to that of the vacuum, thus $\mu = \mu_0$. The *Maxwell equations* that link the space derivatives of one field to the time

Guided Optics: Optical Fibers and All-fiber Components. Jacques Bures

ISBN: 978-3-527-40796-5

derivatives of the other are [1, 2]:

$$\begin{cases} \nabla \wedge \mathbf{E} = -\mu_0 \dfrac{\partial \mathbf{H}}{\partial t} = i\omega\mu_0 \mathbf{H} = i\sqrt{\dfrac{\mu_0}{\varepsilon_0}} k\mathbf{H}, \\ \nabla \wedge \mathbf{H} = \mathbf{J} + \varepsilon_0 n^2 \dfrac{\partial \mathbf{E}}{\partial t} = \mathbf{J} - i\omega\varepsilon_0 n^2 \mathbf{E} = \mathbf{J} - i\sqrt{\dfrac{\varepsilon_0}{\mu_0}} kn^2 \mathbf{E}, \end{cases} \tag{1.3}$$

and the divergences are

$$\begin{cases} \nabla \cdot (\varepsilon_0 n^2 \mathbf{E}) = \sigma, \\ \nabla \cdot (\mu_0 \mathbf{H}) = 0, \end{cases} \tag{1.4}$$

where $\mathbf{J}$ is the current density, σ the charge density, and $\omega = 2\pi\, c/\lambda = k/\sqrt{\varepsilon_0\mu_0}$.

In the MKS system of units, $\varepsilon_0 = 10^7/4\pi\, c^2$ F m^{-1} and $\mu_0 = 4\pi\, 10^{-7}$ H m^{-1}. The factor $\sqrt{\mu_0/\varepsilon_0}$ has the units of an impedance and corresponds to 377 Ω, the impedance of a vacuum.

1.2 Inhomogeneous Vector Wave Equations [3]

The vector wave equations can be expressed solely in terms of $\mathbf{E}$ or $\mathbf{H}$ by eliminating either of these fields in the two equations of (1.3). These equations are considered *inhomogeneous* since they still contain the current density vector $\mathbf{J}$. In order to establish them, we shall make use of the following vector identities

$$\begin{cases} \nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A}, \\ \nabla \cdot (\nabla \wedge \mathbf{A}) = 0, \\ \nabla \wedge (\Psi \mathbf{A}) = \Psi(\nabla \wedge \mathbf{A}) + \nabla\Psi \wedge \mathbf{A}, \end{cases} \tag{1.5}$$

where $\mathbf{A}$ is a vector, Ψ a scalar, ∇ the gradient operator, and $\boldsymbol{\nabla}^2$ the *vector Laplacian operator*, which must not be confused with the *scalar Laplacian operator* ∇^2.

For the electric field, we apply the curl to the first equation of (1.3) and, by substituting the second, we get

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = i\sqrt{\frac{\mu_0}{\varepsilon_0}} k\, \nabla \wedge \mathbf{H} = i\sqrt{\frac{\mu_0}{\varepsilon_0}} \left\{ k\,\mathbf{J} - i\sqrt{\frac{\varepsilon_0}{\mu_0}}\, k^2\, n^2 \mathbf{E} \right\}.$$

With the help of the first identity (1.5), it follows that

$$\nabla(\nabla \cdot \mathbf{E}) - \boldsymbol{\nabla}^2 \mathbf{E} = i\sqrt{\frac{\mu_0}{\varepsilon_0}} k\,\mathbf{J} + k^2 n^2 \mathbf{E}, \text{ therefore}$$

$$(\boldsymbol{\nabla}^2 + k^2 n^2)\; \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - i\sqrt{\frac{\mu_0}{\varepsilon_0}} k\,\mathbf{J}. \tag{1.6}$$

By applying the second identity to $\nabla \wedge \mathbf{H}$, we can now elaborate the $\nabla \cdot \mathbf{E}$ term

$$\begin{aligned} \nabla \cdot (\nabla \wedge \mathbf{H}) &= \nabla \cdot \mathbf{J} - i\sqrt{\frac{\varepsilon_0}{\mu_0}}\, k\, \nabla \cdot\, (n^2 \mathbf{E}) \\ &= \nabla \cdot \mathbf{J} - i\sqrt{\frac{\varepsilon_0}{\mu_0}}\, k\, n^2 \nabla \cdot \mathbf{E} - i\sqrt{\frac{\varepsilon_0}{\mu_0}}\, k\, \mathbf{E} \cdot \nabla n^2 = 0, \end{aligned}$$

therefore

$$\nabla \cdot \mathbf{E} = -\frac{i}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\nabla \cdot \mathbf{J}}{n^2} - \mathbf{E} \cdot \frac{\nabla n^2}{n^2} = -\frac{i}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\nabla \cdot \mathbf{J}}{n^2} - \mathbf{E} \cdot \nabla \ln n^2$$

and finally

$$\nabla(\nabla \cdot \mathbf{E}) = -\frac{i}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\nabla\left(\frac{\nabla \cdot \mathbf{J}}{n^2}\right) - \nabla(\mathbf{E} \cdot \nabla \ln n^2).$$

By substituting this result into (1.6), we obtain the following expression for **E**:

$$\boxed{(\nabla^2 + k^2n^2)\,\mathbf{E} = -\nabla(\mathbf{E} \cdot \nabla \ln n^2) - i\sqrt{\frac{\mu_0}{\varepsilon_0}}\left\{k\mathbf{J} + \frac{1}{k}\nabla\left(\frac{\nabla \cdot \mathbf{J}}{n^2}\right)\right\}.} \quad (1.7)$$

For the magnetic field, we apply the curl to the second equation of (1.3) and, by substituting the first, we get

$$\nabla \wedge (\nabla \wedge \mathbf{H}) = \nabla \wedge \mathbf{J} - i\sqrt{\frac{\varepsilon_0}{\mu_0}}k\,\nabla \wedge n^2\mathbf{E}.$$

From the second and third identities (1.5) it follows that,

$$\nabla(\nabla \cdot \mathbf{H}) - \nabla^2\mathbf{H} = \nabla \wedge \mathbf{J} - i\sqrt{\frac{\varepsilon_0}{\mu_0}}k\,\{n^2\nabla \wedge \mathbf{E} + \nabla n^2 \wedge \mathbf{E}\},$$

and by substituting the expression for $\nabla \wedge \mathbf{E}$ given by the first equation of (1.3)

$$\nabla(\nabla \cdot \mathbf{H}) - \nabla^2\mathbf{H} = \nabla \wedge \mathbf{J} + k^2n^2\mathbf{H} - i\sqrt{\frac{\varepsilon_0}{\mu_0}}k\,\nabla n^2 \wedge \mathbf{E}.$$

Therefore, with $\nabla \cdot \mathbf{H} = 0$,

$$(\nabla^2 + k^2n^2)\mathbf{H} = -\nabla \wedge \mathbf{J} + i\sqrt{\frac{\varepsilon_0}{\mu_0}}k\,\nabla n^2 \wedge \mathbf{E}. \quad (1.8)$$

Isolating **E** from the second equation of (1.3) and substituting it into (1.8)

$$\begin{aligned}(\nabla^2 + k^2n^2)\mathbf{H} &= -\nabla \wedge \mathbf{J} + \frac{\nabla n^2}{n^2} \wedge (\mathbf{J} - \nabla \wedge \mathbf{H}) \\ &= -\nabla \wedge \mathbf{J} + \nabla \ln n^2 \wedge (\mathbf{J} - \nabla \wedge \mathbf{H}).\end{aligned}$$

Then, by reversing the order of the last vector product, we finally obtain the following for **H**

$$\boxed{(\nabla^2 + k^2n^2)\mathbf{H} = (\nabla \wedge \mathbf{H}) \wedge \nabla \ln n^2 - \nabla \wedge \mathbf{J} - \mathbf{J} \wedge \nabla \ln n^2.} \quad (1.9)$$

1.3
Homogeneous Vector Wave Equations

In the absence of current, the current density **J** is zero and the two inhomogeneous equations (1.7) and (1.9) are reduced to two *homogeneous* vector wave equations. These new equations only have terms which contain the refractive index n^2 and **E** or **H**, thus

$$\begin{cases} (\boldsymbol{\nabla}^2 + k^2 n^2)\mathbf{E} = -\boldsymbol{\nabla}(\mathbf{E} \cdot \boldsymbol{\nabla} \ln n^2), \\ (\boldsymbol{\nabla}^2 + k^2 n^2)\mathbf{H} = (\boldsymbol{\nabla} \wedge \mathbf{H}) \wedge \boldsymbol{\nabla} \ln n^2, \end{cases} \tag{1.10}$$

and these fields must satisfy the following boundary conditions, where $\hat{\mathbf{n}}$ is a unit vector normal to the boundary,

$$\boxed{\begin{cases} \text{continuity of the normal components } \hat{\mathbf{n}} \cdot (n^2\mathbf{E}) \text{ and } \hat{\mathbf{n}} \cdot (\mathbf{H}), \\ \text{continuity of the tangential components } \hat{\mathbf{n}} \wedge \mathbf{E} \text{ and } \hat{\mathbf{n}} \wedge \mathbf{H}. \end{cases}} \tag{1.11}$$

In the case of a homogeneous medium, the right-hand sides of (1.10) are null because the index n is everywhere constant. With the identity $\boldsymbol{\nabla}^2\mathbf{A} = \nabla^2\mathbf{A}$ in Cartesian components, these equations reduce to the Helmholtz scalar wave equation [4]

$$(\nabla^2 + k^2 n^2)\ \Psi(x, y, z) = 0, \tag{1.12}$$

where $\Psi(x, y, z)$ is the amplitude of **E** or **H**, since both are proportional to each other.

1.4
Translation-invariant Waveguides and Propagation Modes

For a waveguide that is invariant from $-\infty < z < \infty$, the refractive index profile n is z-independent. The electric and magnetic fields can thus be expressed with a superposition of fields written in a separable form; in Cartesian components the fields are written

$$\begin{cases} \mathbf{E}(x, y, z) = \mathbf{e}(x, y) \exp(i\beta z), \\ \mathbf{e}(x, y) = \mathbf{e}_t + \hat{\mathbf{z}} e_z = \hat{\mathbf{x}} e_x + \hat{\mathbf{y}} e_y + \hat{\mathbf{z}} e_z, \\ \mathbf{H}(x, y, z) = \mathbf{h}(x, y) \exp(i\beta z), \\ \mathbf{h}(x, y) = \mathbf{h}_t + \hat{\mathbf{z}} h_z = \hat{\mathbf{x}} h_x + \hat{\mathbf{y}} h_y + \hat{\mathbf{z}} h_z, \end{cases} \tag{1.13}$$

and in cylindrical polar components

$$\begin{cases} \mathbf{E}(r, \phi, z) = \mathbf{e}(r, \phi) \exp(i\beta z), \\ \mathbf{e}(r, \phi) = \mathbf{e}_t + \hat{\mathbf{z}} e_z = \hat{\mathbf{r}} e_r + \hat{\boldsymbol{\varphi}} e_\phi + \hat{\mathbf{z}} e_z, \\ \mathbf{H}(r, \phi, z) = \mathbf{h}(r, \phi) \exp(i\beta z), \\ \mathbf{h}(r, \phi) = \mathbf{h}_t + \hat{\mathbf{z}} h_z = \hat{\mathbf{r}} h_r + \hat{\boldsymbol{\varphi}} h_\phi + \hat{\mathbf{z}} h_z. \end{cases} \tag{1.14}$$

E and **H**, given by (1.13) and (1.14), define the electric and magnetic fields of a propagation *mode* characterized by the *propagation constant* β along the z-axis and the amplitudes $\mathbf{e}$ and $\mathbf{h}$ which are invariant in z. These are inhomogeneous plane waves in the sense that surfaces of the same phase are planar.

Since the refractive index n and the fields $\mathbf{e}$ and $\mathbf{h}$ are not z-dependent:

- The gradient operator $\boldsymbol{\nabla}$ can be written

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_{\mathrm{t}} + \hat{\mathbf{z}}\,\frac{\partial}{\partial z} = \boldsymbol{\nabla}_{\mathrm{t}} + i\beta\hat{\mathbf{z}},$$

  where $\boldsymbol{\nabla}_{\mathrm{t}}$ is the transverse gradient operator.
- The vector Laplacian operator $\boldsymbol{\nabla}^2$, when applied to $\mathbf{E}$ (or $\mathbf{H}$), gives

$$\boldsymbol{\nabla}^2\mathbf{E} = \boldsymbol{\nabla}_{\mathrm{t}}^2\mathbf{E} + \frac{\partial^2\mathbf{E}}{\partial z^2} = \boldsymbol{\nabla}_{\mathrm{t}}^2\mathbf{E} - \beta^2\mathbf{E},$$

  where $\boldsymbol{\nabla}_{\mathrm{t}}^2$ is the transverse vector Laplacian.
- $\boldsymbol{\nabla}\,\ln n^2$ reduces to $\boldsymbol{\nabla}_{\mathrm{t}}\,\ln n^2$ and $\mathbf{E}\cdot\boldsymbol{\nabla}\,\ln n^2$ to $\mathbf{E}_{\mathrm{t}}\cdot\boldsymbol{\nabla}_{\mathrm{t}}\,\ln n^2$.

With these simplifications, the homogeneous equations (1.10) become the *modal* vector wave equations [3]

$$\boxed{\begin{cases} (\boldsymbol{\nabla}_{\mathrm{t}}^2 + k^2n^2 - \beta^2)\mathbf{e} = -(\boldsymbol{\nabla}_{\mathrm{t}} + i\beta\hat{\mathbf{z}})(\mathbf{e}_t\cdot\boldsymbol{\nabla}_{\mathrm{t}}\ln n^2), \\ (\boldsymbol{\nabla}_{\mathrm{t}}^2 + k^2n^2 - \beta^2)\mathbf{h} = \{(\boldsymbol{\nabla}_{\mathrm{t}} + i\beta\hat{\mathbf{z}})\wedge\mathbf{h}\}\wedge\boldsymbol{\nabla}_{\mathrm{t}}\ln n^2, \end{cases}} \tag{1.15}$$

the solutions of which give β, and the expressions for $\mathbf{e}$ and $\mathbf{h}$ fields of the propagation modes.

We must now separately consider the cases of cylindrical components $(\hat{\mathbf{r}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}})$ and Cartesian components $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$.

1.4.1 Cylindrical Polar Components

The system of cylindrical polar coordinates is particularly well adapted for the case of standard optical fibers that have circular symmetry. This symmetry implies that the refractive index profile *only depends on* r, thus $n(r)$. The field components are

$$\begin{cases} \mathbf{e}(r,\phi) = \mathbf{e}_{\mathrm{t}} + \hat{\mathbf{z}}e_z = \hat{\mathbf{r}}e_r(r,\phi) + \hat{\boldsymbol{\varphi}}e_\phi(r,\phi) + \hat{\mathbf{z}}e_z(r,\phi), \\ \mathbf{h}(r,\phi) = \mathbf{h}_{\mathrm{t}} + \hat{\mathbf{z}}h_z = \hat{\mathbf{r}}h_r(r,\phi) + \hat{\boldsymbol{\varphi}}h_\phi(r,\phi) + \hat{\mathbf{z}}h_z(r,\phi). \end{cases}$$

Coupled differential equations

The transverse vector Laplacian operator $\boldsymbol{\nabla}_{\mathrm{t}}^2$ yields

$$\begin{cases} \boldsymbol{\nabla}_{\mathrm{t}}^2\mathbf{e} = \hat{\mathbf{r}}\left\{\nabla_{\mathrm{t}}^2 e_r - \dfrac{2}{r^2}\dfrac{\partial e_\phi}{\partial\phi} - \dfrac{e_r}{r^2}\right\} + \hat{\boldsymbol{\varphi}}\left\{\nabla_{\mathrm{t}}^2 e_\phi + \dfrac{2}{r^2}\dfrac{\partial e_r}{\partial\phi} - \dfrac{e_\phi}{r^2}\right\} + \hat{\mathbf{z}}\nabla_{\mathrm{t}}^2 e_z, \\ \boldsymbol{\nabla}_{\mathrm{t}}^2\mathbf{h} = \hat{\mathbf{r}}\left\{\nabla_{\mathrm{t}}^2\, h_r - \dfrac{2}{r^2}\dfrac{\partial h_\phi}{\partial\phi} - \dfrac{h_r}{r^2}\right\} + \hat{\boldsymbol{\varphi}}\left\{\nabla_{\mathrm{t}}^2 h_\phi + \dfrac{2}{r^2}\dfrac{\partial h_r}{\partial\phi} - \dfrac{h_\phi}{r^2}\right\} + \hat{\mathbf{z}}\nabla_{\mathrm{t}}^2 h_z, \end{cases}$$

where ∇_t^2 is the transverse scalar Laplacian operator

$$\nabla_t^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}.$$

Since n is only a function of r, the transverse gradient operator $\boldsymbol{\nabla}_t$ gives us

$$\boldsymbol{\nabla}_t = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\varphi}}}{r}\frac{\partial}{\partial \phi},\ \boldsymbol{\nabla}_t \ln n^2 = \hat{\mathbf{r}}\frac{\mathrm{d}\ln n^2}{\mathrm{d}r} \text{ and } \mathbf{e}_t \cdot \boldsymbol{\nabla}_t \ln n^2 = e_r\frac{\mathrm{d}\ln n^2}{\mathrm{d}r},$$

and it follows that, with $\partial/\partial z = i\beta$, we obtain

$$(\boldsymbol{\nabla}_t + i\beta\hat{\mathbf{z}})\ (\mathbf{e}_t \cdot \boldsymbol{\nabla}_t \ln n^2) = \hat{\mathbf{r}}\frac{\partial}{\partial r}\left(e_r\frac{\mathrm{d}\ln n^2}{\mathrm{d}r}\right) + \frac{\hat{\boldsymbol{\varphi}}}{r}\left(\frac{\partial e_r}{\partial \phi}\frac{\mathrm{d}\ln n^2}{\mathrm{d}r}\right) + i\,\hat{\mathbf{z}}\,\beta\, e_r\frac{\mathrm{d}\ln n^2}{\mathrm{d}r}.$$

Next, with

$$(\boldsymbol{\nabla}_t + i\beta\hat{\mathbf{z}}) \wedge \mathbf{h} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial h_z}{\partial \phi} - i\,\beta\, h_\phi\right) + \hat{\boldsymbol{\varphi}}\left(i\,\beta\, h_r - \frac{\partial h_z}{\partial r}\right) + \frac{\hat{\mathbf{z}}}{r}\left(\frac{\partial(rh_\phi)}{\partial r} - \frac{\partial h_r}{\partial \phi}\right),$$

we obtain

$$\{(\boldsymbol{\nabla}_t + i\beta\hat{\mathbf{z}}) \wedge \mathbf{h}\} \wedge \boldsymbol{\nabla}_t \ln n^2 = \hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\mathrm{d}\ln n^2}{\mathrm{d}r}\left(\frac{\partial(rh_\phi)}{\partial r} - \frac{\partial h_r}{\partial \phi}\right) + \hat{\mathbf{z}}\frac{\mathrm{d}\ln n^2}{\mathrm{d}r}\left(\frac{\partial h_z}{\partial r} - i\,\beta\, h_r\right).$$

Upon expanding the terms in the equations (1.15), and collecting the terms in $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\varphi}}$ and $\hat{\mathbf{z}}$, there results *two systems of coupled differential equations* [3] which, respectively, link the three components of $\mathbf{e}$ and $\mathbf{h}$. Thus for the electric field we have

$$\begin{cases} \nabla_t^2 e_r - \dfrac{2}{r^2}\dfrac{\partial e_\phi}{\partial \phi} - \dfrac{e_r}{r^2} + \{n^2k^2 - \beta^2\}e_r + \dfrac{\partial}{\partial r}\left\{e_r\dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r}\right\} = 0, \\ \nabla_t^2 e_\phi + \dfrac{2}{r^2}\dfrac{\partial e_r}{\partial \phi} - \dfrac{e_\phi}{r^2} + \{n^2k^2 - \beta^2\}e_\phi + \dfrac{1}{r}\dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r}\dfrac{\partial e_r}{\partial \phi} = 0, \\ \nabla_t^2 e_z + \{n^2k^2 - \beta^2\}e_z + i\beta e_r\dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r} = 0, \end{cases} \tag{1.16}$$

and for the *magnetic field*

$$\begin{cases} \nabla_t^2 h_r - \dfrac{2}{r^2}\dfrac{\partial h_\phi}{\partial \phi} - \dfrac{h_r}{r^2} + \{n^2k^2 - \beta^2\}h_r = 0, \\ \nabla_t^2 h_\phi + \dfrac{2}{r^2}\dfrac{\partial h_r}{\partial \phi} - \dfrac{h_\phi}{r^2} + \{n^2k^2 - \beta^2\}h_\phi + \dfrac{1}{r}\dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r}\left(\dfrac{\partial h_r}{\partial \phi} - \dfrac{\partial(rh_\phi)}{\partial r}\right) = 0, \\ \nabla_t^2 h_z + \{n^2k^2 - \beta^2\}h_z + \dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r}\left(i\beta h_r - \dfrac{\partial h_z}{\partial r}\right) = 0. \end{cases} \tag{1.17}$$

Relations between the components of e and h

We isolate the field **E** from the second Maxwell equation

$$\mathbf{E} = i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{kn^2}\nabla \wedge \mathbf{H}.$$

By explicitly writing all the components of **E** and **H**, and with $\partial/\partial z = i\beta$, we get

$$\hat{\mathbf{r}}e_r + \hat{\boldsymbol{\varphi}}\, e_\phi + \hat{\mathbf{z}}e_z = i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{kn^2}\frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ \partial/\partial r & \partial/\partial\phi & i\beta \\ h_r & rh_\phi & h_z \end{vmatrix},$$

thus

$$\hat{\mathbf{r}}e_r + \hat{\boldsymbol{\varphi}}\, e_\phi + \hat{\mathbf{z}}e_z = i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{kn^2}\left\{\hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial h_z}{\partial\phi} - i\beta\, h_\phi\right) + \hat{\boldsymbol{\varphi}}\left(i\beta\, h_r - \frac{\partial h_z}{\partial r}\right)\right.$$
$$\left. + \,\hat{\mathbf{z}}\,\frac{1}{r}\left(\frac{\partial(rh_\phi)}{\partial r} - \frac{\partial h_r}{\partial\phi}\right)\right\}.$$

We repeat the same procedure using **H** from the first Maxwell equation

$$\mathbf{H} = -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{1}{k}\nabla \wedge \mathbf{E},$$

and by ordering the terms we obtained the desired equations that link the components of e and h

$$\begin{cases} e_r = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\left(\dfrac{1}{r}\dfrac{\partial ih_z}{\partial\phi} + \beta h_\phi\right), \\ e_\phi = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\left(\beta h_r + \dfrac{\partial ih_z}{\partial r}\right), \\ ie_z = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2 r}\left(\dfrac{\partial(rh_\phi)}{\partial r} - \dfrac{\partial h_r}{\partial\phi}\right). \end{cases} \quad \begin{cases} h_r = -\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\left(\dfrac{1}{r}\dfrac{\partial ie_z}{\partial\phi} + \beta e_\phi\right), \\ h_\phi = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\left(\beta e_r + \dfrac{\partial ie_z}{\partial r}\right), \\ ih_z = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{kr}\left(\dfrac{\partial(re_\phi)}{\partial r} - \dfrac{\partial e_r}{\partial\phi}\right). \end{cases} \quad (1.18)$$

Constructing the transverse components from the longitudinal components e_z and h_z [5]

In the equations (1.18) we successively replace

- h_ϕ in e_r and isolate e_r,
- h_r in e_ϕ and isolate e_ϕ,
- e_ϕ in h_r and isolate h_r,
- and e_r in h_ϕ from which we isolate h_ϕ.

We are thus able to express the four transverse components only in terms of the *derivatives of the longitudinal components* ie_z and ih_z

$$\boxed{\begin{aligned} e_r &= \frac{1}{n^2k^2-\beta^2}\left\{\beta\frac{\partial ie_z}{\partial r}+\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{k}{r}\frac{\partial ih_z}{\partial\phi}\right\},\\ e_\phi &= \frac{1}{n^2k^2-\beta^2}\left\{\frac{\beta}{r}\frac{\partial ie_z}{\partial\phi}-\sqrt{\frac{\mu_0}{\varepsilon_0}}k\frac{\partial ih_z}{\partial r}\right\},\\ h_r &= \frac{1}{n^2k^2-\beta^2}\left\{\beta\frac{\partial ih_z}{\partial r}-\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{kn^2}{r}\frac{\partial ie_z}{\partial\phi}\right\},\\ h_\phi &= \frac{1}{n^2k^2-\beta^2}\left\{\frac{\beta}{r}\frac{\partial ih_z}{\partial\phi}+\sqrt{\frac{\varepsilon_0}{\mu_0}}kn^2\frac{\partial ie_z}{\partial r}\right\}. \end{aligned}} \tag{1.19}$$

These equations allow for the reconstruction of the transverse field components if we know the derivatives of the longitudinal components with respect to r and ϕ – this is the basis for *all numerical vector calculations for fibers.*

Coupled differential equations between the longitudinal components e_z and h_z
Substituting the expressions of (1.19) for e_r and h_r into the third equations of (1.16) and (1.17), we find the coupled equations between e_z and h_z.

$$\boxed{\begin{aligned} &\nabla_t^2 e_z+\{n^2k^2-\beta^2\}e_z-\frac{\mathrm{d}\ln n^2}{\mathrm{d}r}\frac{\beta}{(n^2k^2-\beta^2)}\left\{\beta\frac{\partial e_z}{\partial r}+\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{k}{r}\frac{\partial h_z}{\partial\phi}\right\}=0,\\ &\nabla_t^2 h_z+\{n^2k^2-\beta^2\}h_z-\frac{\mathrm{d}\ln n^2}{\mathrm{d}r}\frac{n^2k^2}{(n^2k^2-\beta^2)}\left\{\frac{\partial h_z}{\partial r}-\frac{\beta}{kr}\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\partial e_z}{\partial\phi}\right\}=0. \end{aligned}} \tag{1.20}$$

This system of coupled differential equations can be solved numerically if the index profile $n^2(r)$ is known, along with the values e_z and h_z (as well as their derivatives) at $r = 0$. Afterwards, the transverse components (1.19) can be calculated from the values of the longitudinal components.

1.4.2
Cartesian Components

The field components now depend on x and y

$$\begin{cases} \mathbf{e}(x,y)=\mathbf{e}_t+e_z\hat{\mathbf{z}}=e_x(x,y)\hat{\mathbf{x}}+e_y(x,y)\hat{\mathbf{y}}+e_z(x,y)\hat{\mathbf{z}},\\ \mathbf{h}(x,y)=\mathbf{h}_t+h_z\hat{\mathbf{z}}=h_x(x,y)\hat{\mathbf{x}}+h_y(x,y)\hat{\mathbf{y}}+h_z(x,y)\hat{\mathbf{z}}, \end{cases}$$

but here the index profile $n(x, y)$ does not necessarily have rotation symmetry; the results of the following section will thus be more general than those of the previous section.

In this system of components, the vector Laplacian operator $\boldsymbol{\nabla}^2$ can be replaced by the scalar Laplacian operator ∇^2 in the homogeneous equations (1.15) thanks to the identity

$$\boldsymbol{\nabla}^2\mathbf{A} = \nabla^2\mathbf{A} = \hat{\mathbf{x}}\,(\nabla^2 A_x) + \hat{\mathbf{y}}\,(\nabla^2 A_y) + \hat{\mathbf{z}}\,(\nabla^2 A_z),$$

with

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \boldsymbol{\nabla}_{\mathrm{t}}^2 + \frac{\partial^2}{\partial z^2} = \boldsymbol{\nabla}_{\mathrm{t}}^2 - \beta^2.$$

Thus the homogeneous equations (1.15) become

$$\begin{cases} (\nabla_{\mathrm{t}}^2 + n^2k^2 - \beta^2)\mathbf{e} = -(\boldsymbol{\nabla}_{\mathrm{t}} + i\beta\hat{\mathbf{z}})(\mathbf{e}_{\mathrm{t}}\cdot\boldsymbol{\nabla}_{\mathrm{t}}\ln n^2), \\ (\nabla_{\mathrm{t}}^2 + n^2k^2 - \beta^2)\mathbf{h} = \{(\boldsymbol{\nabla}_{\mathrm{t}} + i\beta\hat{\mathbf{z}})\wedge\mathbf{h}\}\wedge\boldsymbol{\nabla}_{\mathrm{t}}\ln n^2, \end{cases} \tag{1.21}$$

where the vector operators ∇_t^2 of the left hand sides of (1.15) are replaced with the scalar operators ∇_{t}^2, and the transverse gradient operator is now written $\boldsymbol{\nabla}_{\mathrm{t}} = \hat{\mathbf{x}}\,\dfrac{\partial}{\partial x} + \hat{\mathbf{y}}\,\dfrac{\partial}{\partial y}$.

Coupled differential equations

With

$$\boldsymbol{\nabla}_{\mathrm{t}}\ \ln n^2 = \hat{\mathbf{x}}\frac{\partial\ \ln n^2}{\partial x} + \hat{\mathbf{y}}\frac{\partial\ \ln n^2}{\partial y}$$

and

$$\mathbf{e}_{\mathrm{t}}\cdot\boldsymbol{\nabla}_{\mathrm{t}}\ \ln n^2 = e_x\frac{\partial\ \ln n^2}{\partial x} + e_y\frac{\partial\ \ln n^2}{\partial y},$$

we obtain

$$(\boldsymbol{\nabla}_{\mathrm{t}} + i\beta\hat{\mathbf{z}})(\mathbf{e}_{\mathrm{t}}\cdot\boldsymbol{\nabla}_{\mathrm{t}}\ \ln n^2) = \left\{\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + i\beta\hat{\mathbf{z}}\right\}\left\{e_x\frac{\partial\ \ln n^2}{\partial x} + e_y\frac{\partial\ \ln n^2}{\partial y}\right\}.$$

Afterwards, with

$$(\boldsymbol{\nabla}_{\mathrm{t}} + i\beta\hat{\mathbf{z}})\wedge\mathbf{h} = \hat{\mathbf{x}}\left(\frac{\partial h_z}{\partial y} - i\beta\,h_y\right) + \hat{\mathbf{y}}\left(i\beta\,h_x - \frac{\partial h_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right),$$

we obtain

$$\begin{aligned} &\{(\boldsymbol{\nabla}_{\mathrm{t}} + i\beta\hat{\mathbf{z}})\wedge\mathbf{h}\}\wedge\boldsymbol{\nabla}_{\mathrm{t}}\ln n^2 \\ &\quad = -\hat{\mathbf{x}}\left\{\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right)\frac{\partial\ln n^2}{\partial y}\right\} + \hat{\mathbf{y}}\left\{\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right)\frac{\partial\ln n^2}{\partial x}\right\} \\ &\qquad + \hat{\mathbf{z}}\left\{\left(\frac{\partial h_z}{\partial y} - i\beta h_y\right)\frac{\partial\ln n^2}{\partial y} - \left(i\beta h_x - \frac{\partial h_z}{\partial x}\right)\frac{\partial\ln n^2}{\partial x}\right\}. \end{aligned}$$

Each component of the homogeneous equations is expanded and the terms in $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are grouped together. There results two systems of coupled differential equations, respectively, relating the three components of **e** and **h**.

For the electric field we have

$$\begin{cases} \nabla_{\mathrm{t}}^2 e_x + (n^2k^2 - \beta^2)e_x + \dfrac{\partial}{\partial x}\left\{e_x \dfrac{\partial \ln n^2}{\partial x} + e_y \dfrac{\partial \ln n^2}{\partial y}\right\} = 0, \\ \nabla_{\mathrm{t}}^2 e_y + (n^2k^2 - \beta^2)e_y + \dfrac{\partial}{\partial y}\left\{e_x \dfrac{\partial \ln n^2}{\partial x} + e_y \dfrac{\partial \ln n^2}{\partial y}\right\} = 0, \\ \nabla_{\mathrm{t}}^2 e_z + (n^2k^2 - \beta^2)e_z + i\beta\left\{e_x \dfrac{\partial \ln n^2}{\partial x} + e_y \dfrac{\partial \ln n^2}{\partial y}\right\} = 0, \end{cases} \tag{1.22}$$

and for the magnetic field

$$\begin{cases} \nabla_{\mathrm{t}}^2\, h_x + (n^2k^2 - \beta^2)h_x + \left(\dfrac{\partial h_y}{\partial x} - \dfrac{\partial h_x}{\partial y}\right)\dfrac{\partial \ln n^2}{\partial y} = 0, \\ \nabla_{\mathrm{t}}^2 h_y + (n^2k^2 - \beta^2)h_y - \left(\dfrac{\partial h_y}{\partial x} - \dfrac{\partial h_x}{\partial y}\right)\dfrac{\partial \ln n^2}{\partial x} = 0, \\ \nabla_{\mathrm{t}}^2\, h_z + (n^2k^2 - \beta^2)h_z - \left(\dfrac{\partial h_z}{\partial y} - i\beta h_y\right)\dfrac{\partial \ln n^2}{\partial y} \\ \quad + \left(i\beta h_x - \dfrac{\partial h_z}{\partial x}\right)\dfrac{\partial \ln n^2}{\partial x} = 0. \end{cases} \tag{1.23}$$

Relations between the components of e and h

We isolate the field **E** from the second Maxwell equation

$$\mathbf{E} = i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{kn^2}\boldsymbol{\nabla} \wedge \mathbf{H}.$$

By explicitly writing all the components, we get

$$\hat{\mathbf{x}}\, e_x + \hat{\mathbf{y}}\, e_y + \hat{\mathbf{z}} e_z = i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{kn^2}\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + i\,\beta\hat{\mathbf{z}}\right) \wedge (\hat{\mathbf{x}}\, h_x + \hat{\mathbf{y}}\, h_y + \hat{\mathbf{z}} h_z),$$

$$\hat{\mathbf{x}}\, e_x + \hat{\mathbf{y}}\, e_y + \hat{\mathbf{z}} e_z = i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{kn^2}\left\{\hat{\mathbf{x}}\left(\frac{\partial h_z}{\partial x} - i\,\beta h_y\right) + \hat{\mathbf{y}}\left(i\,\beta h_x - \frac{\partial h_z}{\partial x}\right)\right.$$
$$\left. + \,\hat{\mathbf{z}}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right)\right\}.$$

By repeating the same procedure with **H** from the first Maxwell equation

$$\mathbf{H} = -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{1}{k}\nabla \wedge \mathbf{E},$$

and by grouping the like terms, we obtain the desired relations between the components of **e** and **h**

$$\begin{cases} e_x = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\left(\dfrac{\partial ih_z}{\partial y} + \beta h_y\right), \\ e_y = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\left(\beta h_x + \dfrac{\partial ih_z}{\partial x}\right), \\ ie_z = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\left(\dfrac{\partial h_y}{\partial x} - \dfrac{\partial h_x}{\partial y}\right). \end{cases} \quad \begin{cases} h_x = -\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\left(\dfrac{\partial ie_z}{\partial y} + \beta e_y\right), \\ h_y = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\left(\beta e_x + \dfrac{\partial ie_z}{\partial x}\right), \\ ih_z = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\left(\dfrac{\partial e_y}{\partial x} - \dfrac{\partial e_x}{\partial y}\right). \end{cases} \tag{1.24}$$

Constructing the transverse components from the longitudinal components e_z and h_z [5]

In (1.24) we successively replace

- h_y in e_x and isolate e_x,
- h_x in e_y and isolate e_y,
- e_y in h_x and isolate h_x,
- and e_x into h_y from which we isolate h_y.

We thus obtain the transverse components only in terms of the *derivatives of the longitudinal components ie_z* and ih_z

$$\boxed{\begin{aligned} e_x &= \frac{1}{n^2k^2-\beta^2}\left\{\beta\frac{\partial ie_z}{\partial x} + \sqrt{\frac{\mu_0}{\varepsilon_0}}\,k\frac{\partial ih_z}{\partial y}\right\}, \\ e_y &= \frac{1}{n^2k^2-\beta^2}\left\{\beta\frac{\partial ie_z}{\partial y} - \sqrt{\frac{\mu_0}{\varepsilon_0}}\,k\frac{\partial ih_z}{\partial x}\right\}, \\ h_x &= \frac{1}{n^2k^2-\beta^2}\left\{\beta\frac{\partial ih_z}{\partial x} - \sqrt{\frac{\varepsilon_0}{\mu_0}}\,kn^2\frac{\partial ie_z}{\partial y}\right\}, \\ h_y &= \frac{1}{n^2k^2-\beta^2}\left\{\beta\frac{\partial ih_z}{\partial y} + \sqrt{\frac{\varepsilon_0}{\mu_0}}\,kn^2\frac{\partial ie_z}{\partial x}\right\}. \end{aligned}} \tag{1.25}$$

Coupled differential equations between the longitudinal components e_z and h_z

By substituting the expressions of (1.25) for e_x, h_x, e_y, and h_y into the third equations of (1.22) and (1.23), we find the coupled equations for e_z

and h_z

$$\begin{aligned}
&\nabla_t^2 e_z + pe_z - \frac{\beta}{p}\left\{\frac{\partial \ln n^2}{\partial x}\left(\beta\frac{\partial e_z}{\partial x} + \sqrt{\frac{\mu_0}{\varepsilon_0}}k\frac{\partial h_z}{\partial y}\right)\right.\\
&\qquad\left. + \frac{\partial \ln n^2}{\partial y}\left(\beta\frac{\partial e_z}{\partial y} - \sqrt{\frac{\mu_0}{\varepsilon_0}}k\frac{\partial h_z}{\partial x}\right)\right\} = 0,\\
&\nabla_t^2\, h_z + ph_z - \frac{n^2k^2}{p}\left\{\frac{\partial \ln n^2}{\partial y}\left(\frac{\partial h_z}{\partial y} + \frac{\beta}{k}\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\partial e_z}{\partial x}\right)\right.\\
&\qquad\left. + \frac{\partial \ln n^2}{\partial x}\left(\frac{\partial h_z}{\partial x} - \frac{\beta}{k}\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\partial e_z}{\partial y}\right)\right\} = 0,\\
&\qquad\qquad \text{with} \quad p = n^2k^2 - \beta^2.
\end{aligned} \tag{1.26}$$

1.5
TE and TM modes

In general, the vector modes will have six non-vanishing components $(e_r, e_\phi e_z, h_r, h_\phi h_z)$ or $(e_x, e_y, e_z, h_x, h_y, h_z)$. However, there exist two families of modes for which one of the two longitudinal components is null. Thus the transverse electric modes (TE modes) have $e_z = 0$ and the transverse magnetic modes (TM modes) have $h_z = 0$. Their properties will depend on the symmetry and the geometry of the guides.

It should be noted that this nomenclature holds a different meaning from that of the two *eigen-states* of polarization, also referred to as TE and TM waves, that are encountered in the case of the reflection and refraction of a plane-polarized wave on a diopter [6]. In this latter case, the electric field **E** of the TE wave is perpendicular to the plane of incidence yz, hence $E_y = E_z = 0$. Furthermore, the TM wave has its magnetic field **H** perpendicular to the plane of incidence, hence $H_y = H_z = 0$. Sometimes the TE wave is referred to as the ***s** wave*, and the TM wave as the ***p** wave*. Note, however, that these TE and TM waves are not guided modes; therefore they do not have the properties of guided modes.

1.5.1
The case of y and z Invariant Planar Waveguides

For these waveguides, x is the only variable that intervenes – **e**, **h** and n are only functions of x and everything is invariant in y and z. The previous equations (1.22) and (1.23) simplify considerably [7]:

$$\begin{cases} \dfrac{d^2 e_x}{dx^2} + (n^2k^2 - \beta^2)e_x + \dfrac{d}{dx}\left\{e_x \dfrac{d\ln n^2}{dx}\right\} = 0, \\ \dfrac{d^2 e_y}{dx^2} + (n^2k^2 - \beta^2)e_y = 0, \\ \dfrac{d^2 e_z}{dx^2} + (n^2k^2 - \beta^2)e_z + i\beta e_x \dfrac{d\ln n^2}{dx} = 0, \end{cases} \tag{1.27}$$

$$\begin{cases} \dfrac{d^2 h_x}{dx^2} + (n^2k^2 - \beta^2)h_x = 0, \\ \dfrac{d^2 h_y}{dx^2} + (n^2k^2 - \beta^2)h_y - \dfrac{d\ln n^2}{dx}\dfrac{dh_y}{dx} = 0, \\ \dfrac{d^2 h_z}{dx^2} + (n^2k^2 - \beta^2)h_z + \dfrac{d\ln n^2}{dx}\left(i\beta h_x - \dfrac{dh_z}{dx}\right) = 0. \end{cases} \tag{1.28}$$

and the equations (1.24), grouped into triplets of components (e_y, h_x, ih_z) and (e_x, h_y, ie_z), become

$$\begin{cases} e_y = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\left(\beta h_x + \dfrac{dih_z}{dx}\right), \\ h_x = -\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{\beta}{k}e_y, \\ ih_z = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\dfrac{de_y}{dx}. \end{cases} \qquad \begin{cases} e_x = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{\beta}{kn^2}h_y, \\ h_y = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\left(\beta e_x + \dfrac{die_z}{dx}\right), \\ ie_z = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\dfrac{dh_y}{dx}. \end{cases} \tag{1.29}$$

Note that these triplets of equations are *independent* since there is *no coupling* between the two groups. Moreover, the differential equation for e_y in (1.27) and that for h_y in (1.28) are also not coupled and independent of each other. These equations,

$$\begin{cases} \dfrac{d^2 e_y}{dx^2} + (n^2k^2 - \beta^2)e_y = 0, \\ \dfrac{d^2 h_y}{dx^2} + (n^2k^2 - \beta^2)h_y - \dfrac{d\ln n^2}{dx}\dfrac{dh_y}{dx} = 0, \end{cases}$$

are inconsistent because the substitution of the triplets (1.29) into the corresponding differential equations yields two *different* solutions for β. In order to obtain *consistent* solutions to the Maxwell equations, it is necessary for one of the two triplets of components to be zero. Thus we have the two following cases:

- either $e_y, h_x, h_z = 0$ and $e_x, h_y, e_z \neq 0$ – these are the transverse magnetic TM modes ($h_z = 0$),
- or $e_x, h_y, e_z = 0$ and $e_y, h_x, h_z \neq 0$ – these are the transverse electric TE modes ($e_z = 0$).

These cases further simplify (1.27) and (1.28) by reducing the number of necessary equations. Thus, for each family of modes TE and TM, we respectively obtain [8]:

TE modes

$$\begin{cases} h_x = -\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{\beta}{k}e_y, \\ ih_z = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\dfrac{\mathrm{d}e_y}{\mathrm{d}x}, \\ e_x = e_z = h_y = 0. \end{cases} \quad \begin{cases} \dfrac{\mathrm{d}^2 e_y}{\mathrm{d}x^2} + (n^2k^2 - \beta^2)e_y = 0, \\ \dfrac{\mathrm{d}^2 h_x}{\mathrm{d}x^2} + (n^2k^2 - \beta^2)h_x = 0, \\ \dfrac{\mathrm{d}^2 h_z}{\mathrm{d}x^2} + (n^2k^2 - \beta^2)h_z \\ \quad - \dfrac{n^2k^2}{(n^2k^2 - \beta^2)}\dfrac{\mathrm{d}\ln n^2}{\mathrm{d}x}\dfrac{\mathrm{d}h_z}{\mathrm{d}x} = 0. \end{cases} \tag{1.30}$$

TM modes

$$\begin{cases} e_x = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{\beta}{kn^2}h_y, \\ ie_z = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\dfrac{\mathrm{d}h_y}{\mathrm{d}x}, \\ e_y = h_x = h_z = 0. \end{cases} \quad \begin{cases} \dfrac{\mathrm{d}^2 e_x}{\mathrm{d}x^2} + (n^2k^2 - \beta^2)e_x + \dfrac{\mathrm{d}}{\mathrm{d}x}\left\{e_x\dfrac{\mathrm{d}\ln n^2}{\mathrm{d}x}\right\} = 0, \\ \dfrac{\mathrm{d}^2 h_y}{\mathrm{d}x^2} + (n^2k^2 - \beta^2)h_y - \dfrac{\mathrm{d}\ln n^2}{\mathrm{d}x}\dfrac{\mathrm{d}h_y}{\mathrm{d}x} = 0, \\ \dfrac{\mathrm{d}^2 e_z}{\mathrm{d}x^2} + (n^2k^2 - \beta^2)e_z \\ \quad - \dfrac{\beta^2}{(n^2k^2 - \beta^2)}\dfrac{\mathrm{d}\ln n^2}{\mathrm{d}x}\dfrac{\mathrm{d}e_z}{\mathrm{d}x} = 0. \end{cases} \tag{1.31}$$

At first sight, the differential equations for e_x and h_y in (1.31) appear to be different. However, it should be noted that these equations can be deduced from each other. It is relatively straightforward to show that the first differential equation can be obtained from the second with the following function change, $h_y = n^2\, e_x$. Therefore these two equations are one and the same.

In the case of an arbitrary index profile $n(x, y)$, the planar waveguide loses its invariance in the y dimension and the equations (1.24) no longer reduce to two triplets of independent components like those of (1.29). Moreover, the longitudinal components e_z and h_z are no longer null, therefore these waveguides only support hybrid modes and not TE or TM modes.

1.5.2 The case of a Circularly Symmetric Refractive Index Profile $n(r)$

Let us return to the cylindrical polar components and the equations (1.18). In a similar way to the previous case with the Cartesian coordinates (x, y), in the case where the components h_z, h_r, e_z and e_r are independent of ϕ, we find two groups of coordinates that are mutually independent

$$\begin{cases} e_r = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{\beta}{kn^2}h_\phi, \\ h_\phi = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{kn^2}\left(\beta e_r + \dfrac{\mathrm{d}ie_z}{\mathrm{d}r}\right), \\ ie_z = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2 r}\dfrac{\mathrm{d}(rh_\phi)}{\mathrm{d}r}. \end{cases} \qquad \begin{cases} e_\phi = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\left(\beta h_r + \dfrac{\mathrm{d}ih_z}{\mathrm{d}r}\right), \\ h_r = -\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{\beta}{k}e_\phi, \\ ih_z = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{kr}\dfrac{\mathrm{d}(re_\phi)}{\mathrm{d}r}. \end{cases} \quad (1.32)$$

Again, one of these two triplets must necessarily consist of null components, therefore we have the two following cases:

- either $e_\phi, h_r, h_z = 0$ and $e_r, h_\phi, e_z \neq 0$ – these are the transverse magnetic TM modes ($h_z = 0$),
- or $e_r, h_\phi, e_z = 0$ and $e_\phi, h_r, h_z \neq 0$ – these are the transverse electric TE modes ($e_z = 0$).

By rewriting (1.16), (1.17) and (1.20) for each of these cases, and by eliminating the derivatives with respect to ϕ, we, respectively, obtain the following for each family of modes TE and TM.

TE modes

$$\begin{cases} e_\phi = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2}\left(\beta h_r + \dfrac{\mathrm{d}ih_z}{\mathrm{d}r}\right), \\ h_r = -\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{\beta}{k}e_\phi, \\ ih_z = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{kr}\dfrac{\mathrm{d}(re_\phi)}{\mathrm{d}r}, \\ e_z = e_r = h_\phi = 0. \end{cases} \qquad \begin{cases} \nabla_t^2 e_\phi - \dfrac{e_\phi}{r^2} + (n^2k^2 - \beta^2)e_\phi = 0, \\ \nabla_t^2 h_r - \dfrac{h_r}{r^2} + (n^2k^2 - \beta^2)h_r = 0, \\ \nabla_t^2 h_z + (n^2k^2 - \beta^2)h_z \\ \quad - \dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r}\dfrac{n^2k^2}{(n^2k^2 - \beta^2)}\dfrac{\mathrm{d}h_z}{\mathrm{d}r} = 0. \end{cases} \quad (1.33)$$

The second equation of the left-hand column indicates that the h_r and e_ϕ components are proportional to each other, thus e_ϕ is also independent of ϕ. Finally we remark that for TE modes the electric field is reduced to this single azimuthal component, therefore the lines of polarization of the electric field form circles in the cross-section that are perpendicular to the purely radial transverse components h_r.

TM modes

$$\begin{cases} e_r = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{\beta}{kn^2}h_\phi, \\ h_\phi = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{k}\left(\beta e_r + \dfrac{\mathrm{d}ie_z}{\mathrm{d}r}\right), \\ ie_z = -\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{kn^2 r}\dfrac{\mathrm{d}(rh_\phi)}{\mathrm{d}r}, \\ h_z = h_r = e_\phi = 0. \end{cases} \quad \begin{cases} \nabla_t^2 e_r - \dfrac{e_r}{r^2} + (n^2k^2 - \beta^2)e_r \\ \quad + \dfrac{\mathrm{d}}{\mathrm{d}r}\left\{e_r \dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r}\right\} = 0, \\ \nabla_t^2 h_\phi - \dfrac{h_\phi}{r^2} + (n^2k^2 - \beta^2)h_\phi \\ \quad - \dfrac{1}{r}\dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r}\dfrac{\mathrm{d}(rh_\phi)}{\mathrm{d}r} = 0, \\ \nabla_t^2 e_z + (n^2k^2 - \beta^2)e_z \\ \quad - \dfrac{\mathrm{d}\ln n^2}{\mathrm{d}r}\dfrac{\beta^2}{(n^2k^2 - \beta^2)}\dfrac{\mathrm{d}e_z}{\mathrm{d}r} = 0. \end{cases} \tag{1.34}$$

The first equations of the left-hand column indicate that the e_r and h_ϕ components are proportional to each other, thus h_ϕ is also independent of ϕ. Finally we remark that, for TM modes, the magnetic field is reduced to this single azimuthal component, therefore the lines of polarization of the magnetic field form circles in the cross-section that are perpendicular to the purely radial transverse components e_r.

1.5.3
Concluding Remarks on TE and TM Modes

This type of mode can only exist in a very specific class of waveguides: y- and z-invariant planar waveguides and fibers with circular symmetry. These are the only two cases where the field components can be grouped into two independent families in which one family consists of null components. Other than these two cases e_z and h_z have no reason to be null and generally it is hybrid modes that are guided.

Finally, it should be noted that it is impossible to guide vectorial TEM modes. Indeed, if $e_z = h_z = 0$, then it is clear from (1.25) or (1.19) that all the transverse components would become identically null.

1.6
Nature of the Solutions to Vector Wave Equations

The solutions to the vector wave equations (1.15) are the propagation modes of light, which are characterized by their propagation constant β. Except for very specific cases, it is very difficult to solve these vector wave equations analytically and one must proceed numerically. The solutions will yield the values of β as well as expressions for the six components of the fields $\mathbf{e}$ and $\mathbf{h}$.

The equations (1.19) and (1.25) indicate that the transverse components, $(\hat{\mathbf{r}},\ \hat{\boldsymbol{\varphi}})$ or $(\hat{\mathbf{x}},\ \hat{\mathbf{y}})$, and the longitudinal components are related by the imaginary number i. We are thus faced with a choice, and by convention we choose *real transverse components* and *imaginary longitudinal components*.

According to the values of β, the solutions to the equations (1.15) can be classed into two large families of modes:

1. The *guided* modes correspond the *real* and *discrete* values of β. These are equivalent to the guided light rays in geometrical optics. These modes propagate through the waveguide without loss. In the cross-section plane, the fields far from the waveguide are evanescent and tend towards zero at infinity.
2. The other modes correspond to *radiation* modes which can be decomposed into three parts:
   a) Those corresponding to *complex* and *discrete* values of β. These are the *leaky modes*, or guided pseudo-modes [9, 10], which are equivalent to the tunneling rays of geometrical optics [11]. These tunneling rays, as illustrated in Fig. 1.1, are evanescent only for the turning-point caustic or the core-cladding interface, all the way to the radiation caustic. They propagate like guided modes. However, they are attenuated more or less slowly along z because of the imaginary components β^{i} of the propagation constants β that result in an $\exp(-\beta^{\mathrm{i}}z)$ decrease of the amplitudes.
   b) Those corresponding to a *continuum* of *real* values for β [12]. These are the equivalents to the refracted rays, which can be considered as tunneling rays in the limit where the radiation caustic tends to the turning-point caustic (the core-cladding interface). Contrary to the leaky modes, these modes attenuate very rapidly.
   c) Those corresponding to a *continuum* of *purely imaginary* values for β [12]. These are the *evanescent* modes along z, which do not

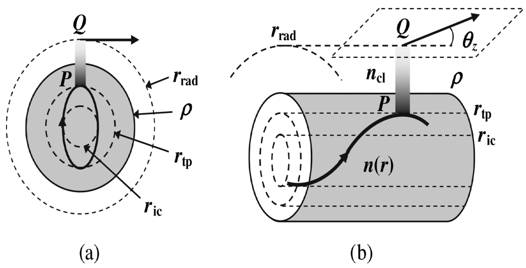

Fig. 1.1 Illustration of a tunneling ray which is characterized by three caustics: the inner (r_{ic}), turning-point (r_{tp}), and radiation (r_{rad}) caustics. a) Projection onto the cross-section plane; and b) Perspective view. The evanescent wave is represented by the gradient of gray between the points P and Q. Upon exiting the point Q, the tunneling ray is tangent to the radiation caustic but makes a certain angle θ_z with respect to the axis of the fiber (adapted from Snyder and Love [11]).

propagate within the waveguide because β does not have a real part. These modes describe the energy stored in the immediate vicinity of waveguide discontinuities; like for example at the extremities of a fiber, in the vicinity of sources, or in the plane of a splice between two fibers.

Contrary to the guided modes, the radiation modes are not evanescent in the regions that are far away from the waveguide in the cross-section plane. Therefore the fields do not vanish at infinity, which can result in certain normalization problems. These modes can be *neglected*, however, if we are *sufficiently far* from the regions where they can be excited – like sources, splices, the waveguide extremities, or anything that results in a break of the z invariance. Figure 1.2 illustrates the entire set of solutions for the two larges families of modes in the complex plane of β values.

Taken as a whole, all these solutions form a *complete* basis for the decomposition of the fields. It is worth noting that a *reduced* basis, considering only guided modes, for instance, can lead to considerable errors, as we shall see in Chapter 8.

In the case of guided modes, the transverse resonance in the cross-section of the waveguide is analogous to the vibration modes of a membrane. This state, determined by the boundary conditions (1.11), will propagate invariantly in the longitudinal direction z with a propagation constant β.

In the case of fiber optics, the six field components will generally exist and form hybrid modes, named EH or HE. If one of the two longitudinal components vanishes we get either the transverse electric modes TE ($e_z = 0$) or the transverse magnetic modes TM ($h_z = 0$). In the case of one-dimensional planar waveguides, only the transverse modes TE and TM exist. However, the two longitudinal components can never vanish simultaneously. In the case of a *free wave*, the field vectors **E** and **H** are orthogonal to the direction of propagation $\hat{\mathbf{z}}$. In the absence of longitudinal field components, the wave

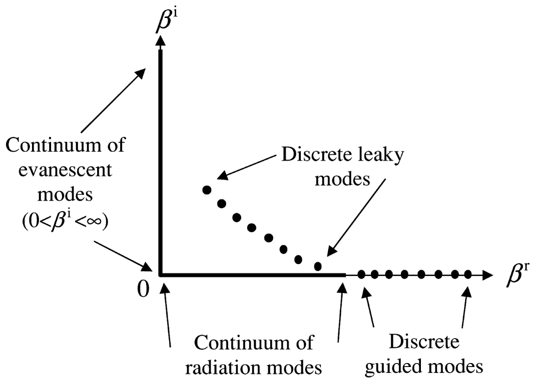

Fig. 1.2 Illustration of the modal solutions in the complex plane β^i, β^r.

is rigorously TEM ($e_z = h_z = 0$). However, in contrast to the free waves, guided waves cannot be rigorously TEM. Nevertheless, we shall see that in the case of weak guidance the guided wave becomes quasi-transverse, i.e., quasi-TEM.

For *telecommunications* it is evident that only the guided modes of the first family are of any practical interest, because the other modes are not guided or present substantial loss factors.

1.7 Conclusion

The translation invariance along the z-axis allows the electric and magnetic fields to be written in a separable form. Thus, the z dependence becomes a phase factor $\exp(i\beta_j z)$, where β_j is the propagation constant, which also corresponds to the eigenvalue of the j-th mode. The consequence of this factorization is that the amplitudes of the **e** and **h** fields of a mode are independent of z. All the differential equations and the relations between the six field components of a mode can subsequently be deduced.

As we shall see in Chapters 6, 7, 8, and 9, the translation invariance is no longer respected in fiber tapers, distributed Bragg gratings, fiber splices, and fused couplers. In the case of these devices, we shall continue to talk in terms of guided modes, but we will consider 'amplitudes and propagation constants that are functions of z'. This is somewhat of a misnomer. While the concept of 'variable' mode lacks rigor, it is nonetheless a very good approximation that allows us to understand and correctly model all of these devices.

References

1. A.W. Snyder and J.D. Love: *Optical Waveguide Theory*, Chapman and Hall, London New York, Chapter 30, p. 590 (1983).
2. J.A. Straton: *Electromagnetic Theory*, McGraw-Hill, New York (1941).
3. A.W. Snyder and J.D. Love: *Optical Waveguide Theory*, Chapman and Hall, London New York, Chapter 30, p. 594 (1983).
4. M. Born and E. Wolf: *Principles of Optics*, Pergamon Press, Sixth Edition, p. 375 (1980).
5. A.W. Snyder and J.D. Love: *Optical Waveguide Theory*, Chapman and Hall, London New York, Chapter 30, p. 593 (1983).
6. M. Born and E. Wolf: *Principles of Optics*, Pergamon Press, Sixth Edition, pp. 52–54 (1980).
7. A.W. Snyder and J.D. Love: *Optical Waveguide Theory*, Chapman and Hall, London New York, Chapter 30, p. 596 (1983).
8. A.W. Snyder and J.D. Love: *Optical Waveguide Theory*, Chapman and Hall, London New York, Chapter 30, p. 599 (1983).
9. A.W. Snyder and J.D. Love: *Optical Waveguide Theory*, Chapman and Hall, London New York, Chapter 24, pp. 488–91 (1983).
10. C. Vassalo: *Optical Waveguide Concepts*, Elsevier, Amsterdam,

Oxford, New York, Tokyo, page 157 (1991).

11. A.W. Snyder and J.D. Love: *Optical Waveguide Theory*, Chapman and Hall, London New York, Chapter 7, p. 140–8 (1983).

12. A.W. Snyder and J.D. Love: *Optical Waveguide Theory*, Chapman and Hall, London New York, Chapter 25, p. 515–16 (1983).