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Fundamentals of Mathematical Modeling of One-Dimensional Flows of Fluid and Gas in Pipelines

1.1
Mathematical Models and Mathematical Modeling

Examination of phenomena is carried out with the help of *models*. Each model represents a definite schematization of the phenomenon taking into account not all the characteristic factors but some of them governing the phenomena and characterizing it from some area of interest to the researcher.

For example, to examine the motion of a body the *material point model* is often used. In such a model the dimensions of the body are assumed to be equal to zero and the whole mass to be concentrated at a point. In other words we ignore a lot of factors associated with body size and shape, the material from which the body is made and so on. The question is: to what extent would such a schematization be efficient in examining the phenomenon? As we all know such a body does not exist in nature. Nevertheless, when examining the motion of planets around the sun or satellites around the earth, and in many other cases, the material point model gives brilliant results in the calculation of the trajectories of a body under consideration.

In the examination of oscillations of a small load on an elastic spring we meet with greater schematization of the phenomenon. First the load is taken as a point mass m, that is we use the material point model, ignoring body size and shape and the physical and chemical properties of the body material. Secondly, the elastic string is also schematized by replacing it by the so-called *restoring force* $F = -k \cdot x$, where $x(t)$ is the deviation of the material point modeling the load under consideration from the equilibrium position and k is the factor characterizing the elasticity of the string. Here we do not take into account the physical-chemical properties of the string, its construction and material properties and so on. Further schematization could be done by taking into account the drag arising from the air flow around the moving load and the rubbing of the load during its motion along the guide.

The use of the differential equation

$$m\frac{d^2x}{dt^2} = -k \cdot x, \tag{1.1}$$

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expressing the second Newtonian law is also a schematization of the phenomenon, since the motion is described in the framework of Euclidian geometry which is the model of our space without taking into account the relativistic effects of the relativity theory.

The fact that the load motion can begin from an arbitrary position with an arbitrary initial velocity may be taken into account in the schematization by specifying *initial conditions* at

$$t = 0: \; x = x_0; v = \left(\frac{dx}{dt}\right)_0 = v_0. \tag{1.2}$$

Equation (1.1) represents the closed *mathematical model* of the considered phenomenon and when the initial conditions are included (1.2) this is the *concrete mathematical model* in the framework of this model. In the given case we have the so-called *initial value (Cauchy) problem* allowing an exact solution. This solution permits us to predict the load motion at instants of time $t > 0$ and by so doing to discover regularities of its motion that were not previously evident. The latest circumstance contains the whole meaning and purpose of mathematical models.

It is also possible of course to produce another more general schematization of the same phenomenon which takes into account a great number of characteristic factors inherent to this phenomenon, that is, it is possible, in principle, to have another more general model of the considered phenomenon.

This raises the question, how can one tell about the correctness or incorrectness of the phenomenon schematization when, from the logical point of view, both schematizations (models) are consistent? The answer is: only from results obtained in the framework of these models. For example, the above-outlined model of load oscillation around an equilibrium position allows one to calculate the motion of the load as

$$x(t) = x_0 \cdot \cos\left(\sqrt{\frac{k}{m}} \cdot t\right) + \sqrt{\frac{m}{k}} \cdot v_0 \cdot \sin\left(\sqrt{\frac{k}{m}} \cdot t\right)$$

having undamped periodic oscillations. How can one evaluate the obtained result? On the one hand there exists a time interval in the course of which the derived result accords well with the experimental data. Hence the model is undoubtedly *correct and efficient*. On the other hand the same experiment shows that oscillations of the load are gradually damping in time and come to a stop. This means that the model (1.1) and the problem (1.2) do not take into account some factors which could be of interest for us, and the accepted schematization is inadequate.

Including in the number of forces acting on the load additional forces, namely the forces of dry $-f_0 \cdot \text{sign}(\dot{x})$ and viscous $-f_1 \cdot \dot{x}$ friction (where the symbol $\text{sign}(\dot{x})$ denotes the function $\dot{x}-$ sign equal to 1, at $\dot{x} > 0$; equal to -1, at $\dot{x} < 0$ and equal to 0, at $\dot{x} = 0$), that is using the equation

$$m\frac{d^2x}{dt^2} = -k \cdot x - f_0 \cdot \text{sign}(\dot{x}) - f_1 \cdot \dot{x} \tag{1.3}$$

instead of Eq. (1.1), one makes the schematization (model) more complete. Therefore it adequately describes the phenomenon.

But even the new model describes only *approximately* the model under consideration. In the case when the size and shape of the load strongly affect its motion, the motion itself is not one-dimensional, the forces acting on the body have a more complex nature and so on. Thus it is necessary to use more complex schematizations or in another words to exploit more complex models. Correct schematization frequently represents a challenging task, requiring from the researcher great experience, intuition and deep insight into the phenomenon to be studied (Sedov, 1965).

Of special note is the *continuum model,* which occupies a highly important place in the following chapters. It is known that all media, including liquids and gases, comprise a great collection of different atoms and molecules in permanent heat motion and with complex interactions. By molecular interactions we mean such properties of real media as compressibility, viscosity, heat conductivity, elasticity and others. The complexity of these processes is very high and the governing forces are not always known. Therefore such seemingly natural investigation of medium motion through a study of discrete molecules is absolutely unacceptable.

One of the general schematization methods for fluid, gas and other deformable media motion is based on the continuum model. Because each macroscopic volume of the medium under consideration contains a great number of molecules the medium could be approximately considered as if it fills the space continuously. Oil, oil products, gas, water or metals may be considered as a medium continuously filling one or another region of the space. That is why *a system of material points continuously filling a part of space is called a continuum.*

Replacement of a real medium consisting of separate molecules by a continuum represents of course a schematization. But such a schematization has proved to be very convenient in the use of the mathematical apparatus of continuous functions and, as was shown in practice, it is quite sufficient for studying the overwhelming majority of observed phenomena.

1.1.1 Governing Factors

In the examination of different phenomena the researcher is always restricted by a finite number of parameters called *governing factors (parameters)* within the limits of which the investigation is being studied. This brings up the question: How to reveal the system of governing parameters?

It could be done for example by formulating the problem mathematically or, in other words, by building a mathematical model of the considered phenomenon as was demonstrated in the above-mentioned example. In this problem the governing parameters are:

$$x,\ t,\ m,\ k, f_0, f_1,\ x_0,\ v_0.$$

But, in order to determine the system of governing parameters, there is no need for mathematical schematization of the process. It is enough to be guided, as has already been noted, by experience, intuition and understanding of the mechanism of the phenomenon.

Let us investigate the decrease in a parachutist's speed v in the air when his motion can be taken as steady. Being governed only by intuition it is an easy matter to assume the speed to be dependent on the mass of the parachutist m, acceleration due to g, the diameter of the parachute canopy D, the length L of its shroud and the air density ρ. The viscosity of the air flowing around the parachute during its descent can be taken into account or ignored since the force of viscous friction is small compared to parachute drag. Both cases represent only different schematizations of the phenomenon.

So the function sought could be assumed to have the following general form $v = f(m, g, D, L, \rho)$. Then the governing parameters are:

$$m, g, D, L, \rho.$$

The use of dimensional theory permits us to rewrite the formulated dependence in invariant form, that is, independent of the system of measurement units (see Chapters 6 and 7)

$$\frac{v}{\sqrt{gD}} = \widetilde{f}\left(\frac{m}{\rho D^3}, \frac{L}{D}\right), \Rightarrow v = \sqrt{gD} \cdot \widetilde{f}\left(\frac{m}{\rho D^3}, \frac{L}{D}\right).$$

Thus, among five governing parameters there are only two independent dimensionless combinations, $m/\rho D^3$ and L/D, defining the sought-for dependence.

1.1.2
Schematization of One-Dimensional Flows of Fluids and Gases in Pipelines

In problems of oil and gas transportation most often schematization of the flow process under the following conditions is used:

- oil, oil product and gas are considered as a continuum continuously filling the whole cross-section of the pipeline or its part;
- the flow is taken as one-dimensional, that is all governing parameters depend only on one space coordinate x measured along the pipeline axis and, in the general case, on time t;
- the governing parameters of the flow represent values of the corresponding physical parameters averaged over the pipeline cross-section;

- the profile of the pipeline is given by the dependence of the height of the pipeline axis above sea level on the linear coordinate $z(x)$;
- the area S of the pipeline cross-section depends, in the general case, on x and t. If the pipeline is assumed to be undeformable, then $S = S(x)$. If the pipeline has a constant diameter, then $S(x) = S_0 = \text{const.}$;
- the most important parameters are:
  $\rho(x, t)$ – density of medium to be transported, kg m^{-3};
  $v(x, t)$ – velocity of the medium, m s^{-1};
  $p(x, t)$ – pressure at the pipeline axis, Pa = N m^{-2};
  $T(x, t)$ – temperature of the medium to be transported, degrees;
  $\tau(x, t)$ – shear stress (friction force per unit area of the pipeline internal surface), Pa = N m^{-2};
  $Q(x, t) = vS$ – volume flow rate of the medium, m^3 s^{-1};
  $\dot{M}(x, t) = \rho v S$ – mass flow rate of the medium, kg s^{-1} and other.

Mathematical models of fluid and gas flows in the pipeline are based on the fundamental laws of physics (mechanics and thermodynamics) of a continuum, modeling a real fluid and a real gas.

1.2 Integral Characteristics of Fluid Volume

In what follows one needs the notion of *movable fluid volume* of the continuum in the pipeline. Let, at some instant of time, an arbitrary volume of the medium be transported between cross-sections x_1 and x_2 of the pipeline (Figure 1.1).

If the continuum located between these two cross-sections is identified with a system of material points and track is kept of its displacement in time, the boundaries x_1 and x_2 become dependent on time and, together with the pipeline surface, contain one and the same material points of the continuum. This volume of the transported medium is called the *movable fluid volume* or *individual volume*. Its special feature is that it always *consists of the same particles of the continuum* under consideration. If, for example, the transported medium is incompressible and the pipeline is non-deformable, then $S = S_0 = \text{const.}$ and the difference between the demarcation boundaries $(x_2 - x_1)$ defining the length of the fluid volume remains constant.

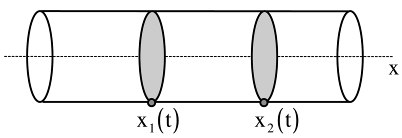

Figure 1.1 Movable fluid volume of the continuum.

Exploiting the notion of fluid or individual volume of the transported medium in the pipeline one can introduce the following integral quantities:

$$M = \int_{x_1(t)}^{x_2(t)} \rho(x,t) \cdot S(x,t)\,\mathrm{d}x - \text{mass of fluid volume (kg);}$$

$$I = \int_{x_1(t)}^{x_2(t)} \rho(x,t) \cdot v(x,t) \cdot S(x,t)\,\mathrm{d}x - \text{momentum of fluid volume (kg m s}^{-1}\text{);}$$

$$E_{\mathrm{kin}} = \int_{x_1(t)}^{x_2(t)} \alpha_{\mathrm{k}} \frac{\rho v^2}{2} S(x,t)\,\mathrm{d}x - \text{kinetic energy of the fluid volume } (J),$$

where α_{k} is the factor;

$$E_{\mathrm{in}} = \int_{x_1(t)}^{x_2(t)} \rho(x,t) \cdot e_{\mathrm{in}}(x,t) \cdot S(x,t)\,\mathrm{d}x - \text{internal energy of the fluid}$$

volume, where e_{in} is the density of the internal energy (J kg^{-1}), that is the internal energy per unit mass.

These quantities model the mass, momentum and energy of a material point system.

Since the main laws of physics are often formulated as connections between physical quantities and the rate of their change in time, we ought to adduce the rule of integral quantity differentiation with respect to time. The symbol of differentiation $\mathrm{d}()/\mathrm{d}t$ denotes the total derivative with respect to time, associated with *individual* particles of a continuum whereas the symbol $\partial()/\partial t$ denotes the local derivative with respect to time, that is the derivative of a flow parameter with respect to time at a *given space point*, e.g. $x = \text{const}$. The local derivative with respect to time gives the rate of flow parameter change at a given cross-section of the flow while, at two consecutive instances of time, different particles of the continuum are located in this cross-section.

The total derivative with respect to time is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1(t)}^{x_2(t)} A(x,t) \cdot S(x,t)\,\mathrm{d}x.$$

From mathematical analysis it is known how an integral containing a parameter, in the considered case it is t, is differentiated with respect to this parameter, when the integrand and limits of integration depend on this parameter. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1(t)}^{x_2(t)} A(x,t) \cdot S(x,t)\,\mathrm{d}x = \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial t}[A(x,t) \cdot S(x,t)]\,\mathrm{d}x + A(x,t) \cdot S(x,t)|_{x_2(t)} \cdot \frac{\mathrm{d}x_2}{\mathrm{d}t} - A(x,t) \cdot S(x,t)|_{x_1(t)} \cdot \frac{\mathrm{d}x_1}{\mathrm{d}t}.$$

First, at frozen upper and lower integration limits, the integrand is differentiated (the derivative being local) and then the integrand calculated at the upper and lower integration limits is multiplied by the rates of change of these limits $\mathrm{d}x_2/\mathrm{d}t$ and $\mathrm{d}x_1/\mathrm{d}t$, the first term having been taken with a plus sign and the second with a minus sign (see Appendix B).

For the case of the fluid volume of the medium the quantities $\mathrm{d}x_2/\mathrm{d}t$ and $\mathrm{d}x_1/\mathrm{d}t$ are the corresponding velocities $v_2(t)$ and $v_1(t)$ of the medium in the left and right cross-sections bounding the considered volume. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1(t)}^{x_2(t)} A(x,t)\cdot S(x,t)\,\mathrm{d}x = \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial t}[A(x,t)\cdot S(x,t)]\,\mathrm{d}x$$
$$+ A(x,t)\cdot v(x,t)\cdot S(x,t)|_{x_2(t)} - A(x,t)\cdot v(x,t)\cdot S(x,t)|_{x_1(t)}.$$

If, in addition, we take into account the well-known Newton–Leibniz formula, according to which

$$A(x,t)\cdot v(x,t)\cdot S(x,t)|_{x_2(t)} - A(x,t)\cdot v(x,t)\cdot S(x,t)|_{x_1(t)}$$
$$= \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial x}[A(x,t)\cdot v(x,t)\cdot S(x,t)]\,\mathrm{d}x,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1(t)}^{x_2(t)} A(x,t)\cdot S(x,t)\,\mathrm{d}x = \int_{x_1(t)}^{x_2(t)} \left(\frac{\partial AS}{\partial t} + \frac{\partial ASv}{\partial x}\right)\mathrm{d}x. \tag{1.4}$$

1.3 The Law of Conservation of Transported Medium Mass. The Continuity Equation

The density $\rho(x,t)$, the velocity of the transported medium $v(x,t)$ and the area of the pipeline cross-section $S(x,t)$ cannot be chosen arbitrarily since their values define the enhancement or reduction of the medium mass in one or another place of the pipeline. Therefore the first equation would be obtained when the transported medium is governed by the mass conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1(t)}^{x_2(t)} \rho(x,t)\cdot S(x,t)\,\mathrm{d}x = 0, \tag{1.5}$$

This equation should be obeyed for any fluid particle of the transported medium, that is for any values $x_1(t)$ and $x_2(t)$.

Applying to Eq. (1.4) the rule (1.5) of differentiation of integral quantity with regard to fluid volume, we obtain

$$\int_{x_1(t)}^{x_2(t)} \left(\frac{\partial \rho S}{\partial t} + \frac{\partial \rho v S}{\partial x}\right)\mathrm{d}x = 0.$$

Since the last relation holds for arbitrary integration limits we get the following differential equation

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho v S}{\partial x} = 0, \tag{1.6}$$

which is called *continuity equation* of the transported medium in the pipeline.

If the flow is stationary, that is the local derivative with respect to time is zero ($\partial ()/\partial t = 0$), the last equation is simplified to

$$\frac{\mathrm{d}\rho v S}{\mathrm{d}x} = 0 \Rightarrow \dot{M} = \rho v S = \text{const.} \tag{1.7}$$

This means that in stationary flow the mass flow rate $\dot{M}$ is constant along the pipeline.

If we ignore the pipeline deformation and take $S(x) \cong S_0 = \text{const.}$, from Eq. (1.7) it follows that $\rho v = \text{const.}$ From this follow two important consequences:

1. In the case of a homogeneous incompressible fluid (sometimes oil and oil product can be considered as such fluids) $\rho \cong \rho_0 = \text{const.}$ and the flow velocity $v(x) = \text{const.}$ Hence the *flow velocity of a homogeneous incompressible fluid in a pipeline of constant cross-section does not change along the length of the pipeline.*

Example. The volume flow rate of the oil transported by a pipeline with diameter $D = 820$ mm and wall thickness $\delta = 8$ mm is 2500 $\text{m}^3\ \text{h}^{-1}$. It is required to find the velocity v of the flow.

Solution. The internal diameter d of the oil pipeline is equal to

$$d = D - 2\delta = 0.82 - 2 \cdot 0.008 = 0.804 \text{ m};$$

$$v = 4Q/\pi d^2 = \text{const.}$$

$$v = 4 \cdot 2500/(3600 \cdot 3.14 \cdot 0.804^2) \cong 1.37 \text{ m s}^{-1}.$$

2. In the case of a compressible medium, e.g. a gas, the density $\rho(x)$ changes along the length of pipeline section under consideration. Since the density is as a rule connected with pressure, this change represents a monotonic function decreasing from the beginning of the section to its end. Then from the condition $\rho v = \text{const.}$ it follows that the velocity $v(x)$ of the flow also increases monotonically from the beginning of the section to its end. Hence the velocity of the *gas flow in a pipeline with constant diameter increases from the beginning of the section between compressor stations to its end.*

Example. The mass flow rate of gas transported along the pipeline ($D = 1020$ mm, $\delta = 10$ mm) is 180 kg s^{-1}. Find the velocity of the gas flow

v_1 at the beginning and v_2 at the end of the gas-pipeline section, if the density of the gas at the beginning of the section is 45 kg m^{-3} and at the end is 25 kg m^{-3}.

Solution. $v_1 = \dot{M}/(\rho_1\, S) = 4 \cdot 180/(45 \cdot 3.14 \cdot 1^2) \cong 5.1$ m s^{-1}; $v_2 = \dot{M}/(\rho_2\, S) = 4 \cdot 180/(25 \cdot 3.14 \cdot 1^2) \cong 9.2$ m s^{-1}, that is the gas flow velocity is enhanced by a factor 1.8 towards the end as compared with the velocity at the beginning.

1.4 The Law of Change in Momentum. The Equation of Fluid Motion

The continuity equation (1.6) contains several unknown functions, hence the use of only this equation is insufficient to find each of them. To get additional equations we can use, among others, the equation of the change in momentum of the system of material points comprising the transported medium. This law expresses properly the second Newton law applied to an arbitrary fluid volume of transported medium

$$\frac{dI}{dt} = \frac{d}{dt}\int_{x_1(t)}^{x_2(t)} v \cdot \rho S\, dx = (p_1\, S_1 - p_2\, S_2) + \int_{x_1(t)}^{x_2(t)} p\frac{\partial S}{\partial x}\, dx$$
$$- \int_{x_1(t)}^{x_2(t)} \pi d \cdot \tau_w\, dx - \int_{x_1(t)}^{x_2(t)} \rho g \sin\alpha(x) \cdot S\, dx. \qquad (1.8)$$

On the left is the total derivative of the fluid volume momentum of the transported medium with respect to time and on the right the sum of all external forces acting on the considered volume.

The first term on the right-hand side of the equation gives the difference in pressure forces acting at the ends of the single continuum volume. The second term represents the axial projection of the reaction force from the lateral surface of the pipe (this force differs from zero when $S \neq$ const.). The third term defines the friction force at the lateral surface of the pipe (τ_w is the shear stress at the pipe walls, that is the friction force per unit area of the pipeline internal surface, Pa). The fourth term gives the *sliding component* of the gravity force ($\alpha(x)$ is the slope of the pipeline axis to the horizontal, $\alpha > 0$ for ascending sections of the pipeline; $\alpha < 0$ for descending sections of the pipeline; g is the acceleration due to gravity).

Representing the pressure difference in the form of an integral over the length of the considered volume

$$p_1\, S_1 - p_2\, S_2 = -\int_{x_1(t)}^{x_2(t)} \frac{\partial pS}{\partial x}\, dx$$

and noting that

$$-\int_{x_1(t)}^{x_2(t)} \frac{\partial pS}{\partial x}\,\mathrm{d}x + \int_{x_1(t)}^{x_2(t)} p\frac{\partial S}{\partial x}\,\mathrm{d}x = -\int_{x_1(t)}^{x_2(t)} S\frac{\partial p}{\partial x}\,\mathrm{d}x,$$

we obtain the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1(t)}^{x_2(t)} \rho v S\,\mathrm{d}x = \int_{x_1(t)}^{x_2(t)} \left(-S\frac{\partial p}{\partial x} - S\cdot\frac{4}{d}\tau_w - S\rho g\sin\alpha(x)\right)\mathrm{d}x.$$

Now applying to the left-hand side of this equation the differentiation rule of fluid volume

$$\int_{x_1(t)}^{x_2(t)} \left(\frac{\partial \rho v S}{\partial t} + \frac{\partial \rho v^2 S}{\partial x}\right)\mathrm{d}x = \int_{x_1(t)}^{x_2(t)} \left(-S\frac{\partial p}{\partial x} - S\cdot\frac{4}{d}\tau_w - S\rho g\sin\alpha(x)\right)\mathrm{d}x.$$

As far as the limits of integration in the last relation are arbitrary one can discard the integral sign and get the differential equation

$$\frac{\partial \rho v S}{\partial t} + \frac{\partial \rho v^2 S}{\partial x} = S\cdot\left(-\frac{\partial p}{\partial x} - \frac{4}{d}\tau_w - \rho g\sin\alpha(x)\right). \tag{1.9}$$

If we represent the left-hand side of this equation in the form

$$v\left(\frac{\partial \rho S}{\partial t} + \frac{\partial \rho v S}{\partial x}\right) + \rho S\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right)$$

and take into account that in accordance with the continuity equation (1.6) the expression in the first brackets is equal to zero, the resulting equation may be written in a more simple form

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial x} - \frac{4}{d}\tau_w - \rho g\sin\alpha(x). \tag{1.10}$$

The expression in brackets on the left-hand side of Eq. (1.10) represents the total derivative with respect to time, that is the particle acceleration

$$w = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}. \tag{1.11}$$

Now the meaning of Eq. (1.10) becomes clearer: the product of unit volume mass of transported medium and its acceleration is equal to the sum of all forces acting on the medium, namely pressure, friction and gravity forces. So Eq. (1.10) expresses the Newton's Second Law and can therefore also be called the *flow motion equation*.

Remark. *about the connection between total and partial derivatives with respect to time.* The acceleration $w = \mathrm{d}v/\mathrm{d}t$ is a total derivative with respect to time (symbol $\mathrm{d}()/\mathrm{d}t$), since we are dealing with the velocity differentiation of one and the same fixed particle of the transported medium moving from one cross-section of the pipeline to another one, whereas the partial derivative with respect to time (symbol $\partial()/\partial t$) has the meaning of velocity differentiation at a given place in space, that is at a constant value of x. Thus such a derivative gives the change in velocity of different particles of the transported medium entering a given cross-section of the pipeline.

Let a particle of the medium at the instant of time t be in the cross-section x of the pipeline and so have velocity $v(x, t)$. In the next instant of time $t + \Delta t$ this particle will transfer to the cross-section $x + \Delta x$ and will have velocity $v(x + \Delta x, t + \Delta t)$. The acceleration w of this particle is defined as the limit

$$w = \frac{dv}{dt} = \lim_{\Delta t \Rightarrow 0} \frac{v(x + \Delta x, t + \Delta t) - v(x, t)}{\Delta t} = \left.\frac{\partial v}{\partial t}\right|_x + \left.\frac{\partial v}{\partial x}\right|_t \cdot \frac{dx}{dt}.$$

Since $dx/dt = v(x, t)$ is the velocity of the considered particle, from the last equality it follows that

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v \cdot \frac{\partial v}{\partial x}. \tag{1.12}$$

A similar relation between the total derivative (d/dt), or as it is also called the individual or Lagrangian derivative, and the partial derivative $(\partial/\partial t)$, or as it is also called the local or Eulerian derivative, has the form (1.12) no matter whether the case in point is velocity or any other parameter $A(x, t)$

$$\frac{dA(x, t)}{dt} = \frac{\partial A(x, t)}{\partial t} + v \cdot \frac{\partial A(x, t)}{\partial x}.$$

1.5
The Equation of Mechanical Energy Balance

Consider now what leads to the use of the mechanical energy change law as applied to the system of material points representing a fluid particle of the transported medium. This law is written as:

$$\frac{dE_{kin}}{dt} = \frac{dA^{ex}}{dt} + \frac{dA^{in}}{dt} \tag{1.13}$$

that is the change in kinetic energy of a system of material points dE_{kin} is equal to the sum of the work of the external dA^{ex} and internal dA^{in} forces acting on the points of this system.

We can calculate separately the terms of this equation but first we should define more exactly what meant by the kinetic energy E_{kin}. If the transported medium moves in the pipeline as a piston with equal velocity $v(x, t)$ over the cross-section then the kinetic energy would be expressed as the integral

$$E_{kin} = \int_{x_1(t)}^{x_2(t)} \frac{\rho v^2}{2} S \, dx.$$

But, in practice, such a schematization is too rough because, as experiments show, the velocity of the separate layers of the transported medium (fluid or gas) varies over the pipe cross-section. At the center of the pipe it reaches the greatest value, whereas as the internal surface of the pipe is approached the velocity decreases and at the wall itself it is equal to zero. Furthermore, if at a small velocity of the fluid the flow regime is *laminar,* with an increase in velocity the laminar flow changes into a *turbulent* one (pulsating and mixing flow) and the velocities of the separate particles differ significantly from the average velocity v of the flow. That is why models of the flow are, as a rule, constructed with regard to the difference in flow velocity from the average velocity over the cross-section.

The true velocity u of a particle of the transported medium is given as the sum $u = v + \Delta u$ of the average velocity over the cross-section $v(x, t)$ and the additive one (deviation) Δu representing the difference between the true velocity and the average one. The average value of this additive $\overline{\Delta u}$ is equal to zero, but the root-mean-square (rms) value of the additive $\overline{(\Delta u)^2}$ is non-vanishing. The deviation characterizes the kinetic energy of the relative motion of the continuum particle in the pipeline cross-section. Then the kinetic energy of the transported medium unit mass e_{kin} may be presented as the sum of two terms

$$e_{\text{kin}} = \frac{v^2}{2} + \frac{\overline{(\Delta u)^2}}{2}$$

namely the kinetic energy of the center of mass of the considered point system and the kinetic energy of the motion of these points relative to the center of mass. If the average velocity $v \neq 0$, then

$$\frac{\rho v^2}{2} + \frac{\rho\overline{(\Delta u)^2}}{2} = \frac{\rho v^2}{2} \cdot \left(1 + \frac{\overline{(\Delta u)^2}}{v^2}\right) = \alpha_{\text{k}} \cdot \frac{\rho v^2}{2}$$

where $\alpha_k = 1 + \overline{(\Delta u)^2}/v^2 > 1$. For laminar flow $\alpha_{\text{k}} = 4/3$, while for turbulent flow the value of α_{k} lies in the range 1.02–1.05.

Remark. *It should be noted that in one-dimensional theory, as a rule, the cases* $v = 0$ *and* $\overline{(\Delta u)^2} \neq 0$ *are not considered.*

With regard to the introduced factor the kinetic energy of any movable volume of transported medium may be represented as

$$E_{\text{kin}} = \int_{x_1(t)}^{x_2(t)} \alpha_{\text{k}} \cdot \frac{\rho v^2}{2} \cdot S\,\mathrm{d}x.$$

Let us turn now to the calculation of the terms in the mechanical energy equation (1.13). Let us calculate first the change in kinetic energy

$$\frac{\mathrm{d}E_{\text{kin}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{x_1(t)}^{x_2(t)} \alpha_{\text{k}} \cdot \frac{\rho v^2}{2} S\,\mathrm{d}x\right).$$

Employing the rule of integral quantity integration with reference to the fluid volume, that is an integral with variable integration limits, we get

$$\frac{\mathrm{d}E_{\mathrm{kin}}}{\mathrm{d}t} = \int_{x_1(t)}^{x_2(t)} \left[\frac{\partial}{\partial t}\left(\alpha_{\mathrm{k}} \cdot \frac{\rho v^2}{2} S \right) + \frac{\partial}{\partial x}\left(\alpha_{\mathrm{k}} \cdot \frac{\rho v^2}{2} S \cdot v \right) \right] \mathrm{d}x.$$

The work of the external forces (in this case they are the forces of pressure and gravity), including also the work of external mechanical devices, e.g. pumps if such are used, is equal to

$$\begin{aligned}\frac{\mathrm{d}A^{\mathrm{ex}}}{\mathrm{d}t} &= (p_1 S v_1 - p_2 S v_2) - \int_{x_1(t)}^{x_2(t)} \rho g \sin\alpha \cdot v \cdot S \,\mathrm{d}x + N_{\mathrm{mech}} \\ &= -\int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial x}(pSv)\,\mathrm{d}x - \int_{x_1(t)}^{x_2(t)} \rho g \sin\alpha \cdot v \cdot S \,\mathrm{d}x + N_{\mathrm{mech}}.\end{aligned}$$

The first term on the right-hand side of the last expression gives the work performed in unit time or, more precisely, the power of the pressure force applied to the initial and end cross-sections of the detached volume. The second term gives the power of the gravity force and the third term N_{mech} the power of the external mechanical devices acting on the transported medium volume under consideration.

The work of the internal forces (pressure and internal friction) executed in unit time is given by

$$\frac{\mathrm{d}A^{\mathrm{in}}}{\mathrm{d}t} = \int_{x_1(t)}^{x_2(t)} p\frac{\partial(Sv)}{\partial x}\,\mathrm{d}x + \int_{x_1(t)}^{x_2(t)} n^{in} \cdot \rho S\,\mathrm{d}x.$$

The first term on the right-hand side gives the work of the pressure force in unit time, that is the power, for compression of the particles of the medium, the factor $\partial(Sv)/\partial x \cdot \mathrm{d}x$ giving the rate of elementary volume change. The second term represents the power of the internal friction forces, that is the forces of mutual friction between the internal layers of the medium, n^{in} denoting specific power, that is per unit mass of the transported medium. In what follows it will be shown that this quantity characterizes the amount of mechanical energy converting into heat per unit time caused by mutual internal friction of the transported particles of the medium.

Gathering together all the terms of the mechanical energy equation we get

$$\begin{aligned}&\int_{x_1(t)}^{x_2(t)} \left[\frac{\partial}{\partial t}\left(\alpha_{\mathrm{k}} \cdot \frac{\rho v^2}{2} S \right) + \frac{\partial}{\partial x}\left(\alpha_{\mathrm{k}} \cdot \frac{v^2}{2} \rho v S \right) \right] \mathrm{d}x \\ &\quad = -\int_{x_1(t)}^{x_2(t)} \rho S v \left[\left(\frac{1}{\rho}\frac{\partial p}{\partial x} \right) + g\sin\alpha \right] \mathrm{d}x + \int_{x_1(t)}^{x_2(t)} n^{\mathrm{in}} \cdot \rho S\,\mathrm{d}x + N_{\mathrm{mech}}.\end{aligned}$$

If the transported medium is barotropic, that is the pressure in it depends only on the density $p = p(\rho)$, one can introduce a function $P(\rho)$ of the pressure such that $dP = dp/\rho$, $P(\rho) = \int dp/\rho$ and $\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial P(\rho)}{\partial x}$. If, moreover, we take into account the equality $\sin\alpha(x) = \partial z/\partial x$, where the function $z(x)$ is referred to as the pipeline profile, the last equation could be rewritten in the simple form

$$\int_{x_1(t)}^{x_2(t)} \left[\rho S \frac{\partial}{\partial t}\left(\frac{\alpha_k v^2}{2}\right) + \rho v S \frac{\partial}{\partial x}\left(\frac{\alpha_k v^2}{2} + P(\rho) + gz\right)\right] dx$$
$$= \int_{x_1(t)}^{x_2(t)} n^{in} \cdot \rho S \, dx + N_{mech}. \tag{1.14}$$

If we assume that in the region $[x_1(t), x_2(t)]$ external sources of mechanical energy are absent. Then $N_{mech} = 0$ and we can go from the integral equality (1.14) to a differential equation using, as before, the condition of arbitrariness of integration limits $x_1(t)$ and $x_2(t)$ in Eq. (1.14). Then the sign of the integral can be omitted and the corresponding differential equation is

$$\rho S \frac{\partial}{\partial t}\left(\frac{\alpha_k v^2}{2}\right) + \rho v S \frac{\partial}{\partial x}\left(\frac{\alpha_k v^2}{2} + P(\rho) + gz\right) = \rho S \cdot n^{in} \tag{1.15}$$

or

$$\frac{\partial}{\partial t}\left(\frac{\alpha_k v^2}{2}\right) + v \cdot \frac{\partial}{\partial x}\left(\frac{\alpha_k v^2}{2} + \int \frac{dp}{\rho} + gz\right) = n^{in}. \tag{1.16}$$

This is the sought differential equation expressing the law of mechanical energy change. It should be emphasized that this equation is not a consequence of the motion equation (1.10). It represents an independent equation for modeling one-dimensional flows of a transported medium in the pipeline.

If we divide both parts of Eq. (1.16) by g we get

$$\frac{\partial}{\partial t}\left(\frac{\alpha_k v^2}{2g}\right) + v \cdot \frac{\partial}{\partial x}\left(\frac{\alpha_k v^2}{2g} + \int \frac{dp}{\rho g} + z\right) = \frac{n^{in}}{g}.$$

The expression

$$H = \frac{\alpha_k v^2}{2g} + \int \frac{dp}{\rho g} + z \tag{1.17}$$

in the derivative on the left-hand side of the last equation has the dimension of length and is called the *total head*. The total head at the pipeline cross-section x consists of the *kinetic head (dynamic pressure)* $\alpha_k v^2/2g$, the *piezometric head* $\int dp/\rho g$ and the *geometric head* z. The concept of head is very important in the calculation of processes occurring in pipelines.

1.5.1 Bernoulli Equation

In the case of stationary flow of a *barotropic fluid or gas* in the pipeline the derivative $\partial()/\partial t = 0$, hence the following ordinary differential equations apply

$$v\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\alpha_k v^2}{2g} + \int \frac{\mathrm{d}p}{\rho g} + z\right) = \frac{n^{\mathrm{in}}}{g}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\alpha_k v^2}{2g} + \int \frac{\mathrm{d}p}{\rho g} + z\right) = \frac{n^{\mathrm{in}}}{gv} = i, \tag{1.18}$$

where i denotes the dimensional quantity n^{in}/gv called the *hydraulic gradient*

$$i = \frac{\mathrm{d}H}{\mathrm{d}x} = \frac{n^{\mathrm{in}}}{gv}.$$

Thus the hydraulic gradient, defined as the pressure loss per unit length of the pipeline, is proportional to the dissipation of mechanical energy into heat through internal friction between the transported medium layers ($i < 0$).

In integral form, that is as applied to transported medium located between two fixed cross-sections x_1 and x_2, Eq. (1.18) takes the following form

$$\left(\frac{\alpha_k v^2}{2g} + \int \frac{\mathrm{d}p}{\rho g} + z\right)_1 - \left(\frac{\alpha_k v^2}{2g} + \int \frac{\mathrm{d}p}{\rho g} + z\right)_2 = -\int_{x_1}^{x_2} i\,\mathrm{d}x. \tag{1.19}$$

This equation is called the *Bernoulli equation*. It is one of the fundamental equations used to describe the stationary flow of a barotropic medium in a pipeline.

For an *incompressible homogeneous fluid*, which under some conditions can be water, oil and oil product, $\rho = \text{const.}$, $\int \mathrm{d}p/\rho g = p/\rho g + \text{const.}$ Therefore the Bernoulli equation becomes

$$\left(\frac{\alpha_k v^2}{2g} + \frac{p}{\rho g} + z\right)_1 - \left(\frac{\alpha_k v^2}{2g} + \frac{p}{\rho g} + z\right)_2 = -\int_{x_1}^{x_2} i\,\mathrm{d}x.$$

If in addition we take $i = -i_0 = \text{const.}$ ($i_0 > 0$), then

$$\left(\frac{\alpha_\kappa v^2}{2g} + \frac{p}{\rho g} + z\right)_1 - \left(\frac{\alpha_\kappa v^2}{2g} + \frac{p}{\rho g} + z\right)_2 = i_0 \cdot l_{1-2} \tag{1.20}$$

where l_{1-2} is the length of the pipeline between cross-sections 1 and 2.

This last equation has a simple geometric interpretation (see Figure 1.2). This figure illustrates a pipeline profile (heavy broken line); the line $H(x)$ denoting the dependence of the total head H on the coordinate x directed along the axis of the pipeline (straight line) with constant slope β to the horizontal ($i = \mathrm{d}H/\mathrm{d}x = tg\beta = \text{const.}$) and three components of the total head at an

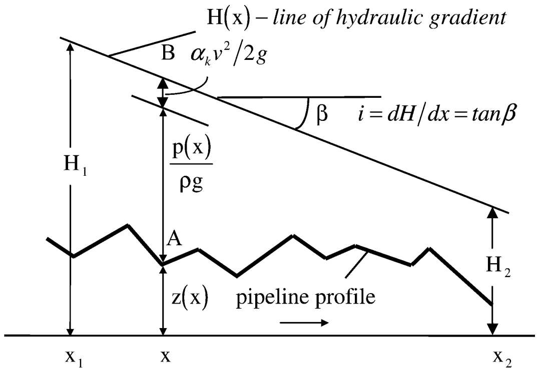

Figure 1.2 Geometric interpretation of the Bernoulli equation.

arbitrary cross-section of the pipeline: geometric head $z(x)$, piezometric head $p(x)/\rho g$ and kinetic head $\alpha_k v^2(x)/2g$.

The line $H(x)$ representing the dependence of the total head H on the coordinate x along the pipeline axis is called the *line of hydraulic gradient.*

It should be noted that if we neglect the dynamic pressure (in oil and oil product pipelines the value of the dynamic pressure does not exceed the pipeline diameter, e.g. at $v \approx 2\ \text{m s}^{-1}$, $\alpha_k \approx 1.05$ then $v^2/2g \cong 0.25$ m), and the length of the section between the pipeline profile and the line of hydraulic gradient multiplied by ρg gives the value of the pressure in the pipeline cross-section x. For example, when the length of the section AA (see Figure 1.2) is 500 m and diesel fuel with density $\rho = 840\ \text{kg m}^{-3}$ is transported along the pipeline, then

$$\frac{p}{840 \cdot 9.81} = 500 \Rightarrow p = 500 \cdot 840 \cdot 9.81 = 4\,120\,200\ \text{(Pa)}$$

or 4.12 MPa (≈42 atm).

1.5.2
Input of External Energy

In fluid flow in the pipeline the mechanical energy is dissipated into heat and the pressure decreases gradually. Devices providing pressure restoration or generation are called *compressors.*

Compressors installed separately or combined in a group form the pumping plant destined to set the fluid moving from the cross-section with lesser pressure to the cross-section with greater pressure. To do this it is required to expend, or deliver from outside to the fluid, energy whose power is denoted by N_{mech}.

Let index 1 in the Bernoulli equation refer to parameters at the cross-section x_1 of the pump entrance (suction line) and index 2 at the cross-section x_2 of the pump exit (discharge line). Since $\rho v S = \text{const.}$, the Bernoulli equation (1.14) may be written as:

$$\int_{x_1}^{x_2} \frac{d}{dx}\left[\rho v S \cdot \left(\frac{\alpha_k v^2}{2} + \frac{p}{\rho} + gz\right)\right] dx = \int_{x_1}^{x_2} n^{\text{in}} \cdot \rho S \, dx + N_{\text{mech}}.$$

Ignoring the difference between the kinetic and geometric heads we get

$$\rho v S \cdot \frac{p_2 - p_1}{\rho} - \int_{x_1}^{x_2} n^{\text{in}} \cdot \rho S \, dx = N_{\text{mech}}.$$

Denoting by $\Delta H = (p_2 - p_1)/\rho g$ the *differential head* produced by the pump or pumping plant and taking into account that $\rho v S = \rho Q = \text{const.}$ and $n^{\text{in}} = gv \cdot i$, we obtain

$$N_{\text{mech}} = \rho g Q \cdot \Delta H - \int_{x_1}^{x_2} \rho g Q \cdot i \, dx = \rho g Q \cdot \Delta H \cdot \left(1 - \int_{x_1}^{x_2} \frac{i}{\Delta H} \, dx\right).$$

The expression in parentheses characterizes the loss of mechanical energy within the pump. Usually this factor is taken into account by insertion of the pump efficiency η

$$\eta = \left(1 - \int_{x_1}^{x_2} i/\Delta H \; dx\right)^{-1} < 1$$

so that

$$N_{\text{mech}} = \frac{\rho g Q \cdot \Delta H}{\eta(Q)}. \tag{1.21}$$

The relation (1.21) is the main formula used to calculate the power of the pump generating head ΔH in fluid pumping with flow rate Q.

1.6
Equation of Change in Internal Motion Kinetic Energy

At the beginning of the previous section it was noted that the total kinetic energy of the transported medium consisted of two terms – the kinetic energy of the center of mass of the particle and the kinetic energy of the internal motion of the center of mass, so that the total energy of a particle is equal to $\alpha_k \rho v^2/2$, where $\alpha_k > 1$. Now we can derive an equation for the second component of the kinetic energy, namely the kinetic energy of the internal or relative motion in the flow of the transported medium.

Multiplication of motion equation (1.10) by the product vS yields

$$\rho S \frac{d}{dt}\left(\frac{v^2}{2}\right) = -\frac{\partial p}{\partial x} \cdot vS - \frac{4}{d}\tau_w \cdot vS - \rho g v S \cdot \sin\alpha(x).$$

Subtracting this equation term-by-term from the Bernoulli equation (1.15), one obtains

$$\rho S \frac{\mathrm{d}}{\mathrm{d}t}\left[(\alpha_{\mathrm{k}}-1)\frac{v^2}{2}\right] = \frac{4}{d}\tau_{\mathrm{w}} \cdot vS + \rho S \cdot n^{\mathrm{in}}.$$

Introduction of $n^{\mathrm{in}} = -gv \cdot i_0$ gives

$$\rho S \frac{\mathrm{d}}{\mathrm{d}t}\left[(\alpha_{\kappa}-1)\frac{v^2}{2}\right] = \left(\frac{4}{d}\tau_{\mathrm{w}} \cdot v\right) S - \rho g v S \cdot i_0. \tag{1.22}$$

This is the sought equation of change in *kinetic energy of internal motion* of one-dimensional flow of the transported medium. Its sense is obvious: *the power of the external friction forces* $(4\tau_{\mathrm{w}} \cdot vS/d)$ *in one-dimensional flow minus the power* $\rho g S(v \cdot i_0)$ *of internal friction forces between the particles causing transition of mechanical energy into heat is equal to the rate of change of internal motion kinetic energy in the flow of the transported medium.*

For stationary flow $(\mathrm{d}/\mathrm{d}t = 0 + v \cdot \partial/\partial x)$ of the transported medium Eq. (1.22) gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(\alpha_{\mathrm{k}}-1)\frac{v^2}{2}\right] = \frac{4}{d}\frac{\tau_{\mathrm{w}}}{\rho} - g \cdot i_0. \tag{1.23}$$

If $v \cong$ const., which for the flow of an incompressible medium in a pipeline with constant diameter is the exact condition, the left-hand part of the equation vanishes. This means that the tangential friction tension τ_{w} at the pipeline wall and the hydraulic gradient i_0 are connected by

$$\tau_{\mathrm{w}} = \frac{\rho g d}{4} \cdot i_0. \tag{1.24}$$

It must be emphasized that in the general case, including non-stationary flow, such a connection between τ_w and i_0 is absent (see Section 4.1).

1.6.1
Hydraulic Losses (of Mechanical Energy)

The quantity n^{in} entering into Eq. (1.16) denotes the specific power of the internal friction force, that is per unit mass of transported medium. This quantity is very important since it characterizes the loss of mechanical energy converted into heat owing to internal friction between layers of the medium. In order to derive this quantity theoretically one should know how the layers of transported medium move at each cross-section of the pipeline but this is not always possible. In the next chapter it will be shown that in several cases, in particular for laminar, flow such motion can be calculated and the quantity n^{in} can be found. In other cases, such as for turbulent flows of the transported medium, it is not possible to calculate the motion of the layers and other methods of determining n^{in} are needed.

The quantity of specific mechanical energy dissipation n^{in} has the following dimension (from now onwards dimension will be denoted by the symbol [ ])

$$[n^{in}] = \frac{W}{kg} = \frac{J}{s\ kg} = \frac{N\ m}{s\ kg} = \frac{kg\ m\ s^{-2}\ m}{s\ kg} = \frac{m^2}{s^3} = \left[\frac{v^3}{d}\right].$$

So the dimension of n^{in} is the same as the dimension of the quantity v^3/d, hence, without disturbance of generality, one can seek n^{in} in the form

$$n^{in} = -\frac{\lambda}{2} \cdot \frac{v^3}{d} \tag{1.25}$$

where λ is a dimensional factor ($\lambda > 0$), the minus sign shows that $n^{in} < 0$, that is the mechanical energy decreases thanks to the forces of internal friction. The factor 1/2 is introduced for the sake of convenience.

The presented formula does not disturb the generality of the consideration because the unknown dependence of n^{in} on the governing parameters of the flow is accounted for by the factor λ. This dependence is valid for any medium be it fluid, gas or other medium with complex specific properties, e.g. waxy crude oil, suspension or even pulp, that is a mixture of water with large rigid particles.

For stationary fluid or gas flow one can suppose the factor λ to be dependent on four main parameters: the flow velocity v (m s^{-1}), the kinematic viscosity of the flow ν (m^2 s^{-1}), the internal diameter of the pipeline d (m) and the mean height of the roughness of its internal surface Δ (mm or m), so that $\lambda = f(v, \nu, d, \Delta)$. The density of the fluid ρ and the acceleration due to gravity g are not included here because intuition suggests that the friction between fluid or gas layers will be dependent on neither their density nor the force of gravity.

Note that the quantity λ is dimensionless, that is its numerical value is independent of the system of measurement units, while the parameters v, ν, d, Δ are dimensional quantities and their numerical values depend on such a choice. The apparent contradiction is resolved by the well-known Buckingham I-theorem, in accordance with which any dimensionless quantity can depend only on dimensionless combinations of parameters governing this quantity (Lurie, 2001). In our case there are two such parameters

$$\frac{v \cdot d}{\nu} = Re \quad \text{and} \quad \frac{\Delta}{d} = \varepsilon,$$

the first is called the *Reynolds number* and the second the *relative roughness* of the pipeline internal surface. Thus

$$\lambda = \lambda(Re, \varepsilon).$$

The formula (1.25) acquires the form

$$n^{\text{in}} = -\lambda(Re, \varepsilon) \cdot \frac{1}{d} \cdot \frac{v^3}{2}. \tag{1.26}$$

The factor λ in this formula is called the *hydraulic resistance factor,* one of the most important parameters of hydraulics and pipeline transportation. Characteristic values of λ lie in the range 0.01–0.03. More detailed information about this factor and its dependence on the governing parameters will be presented below.

Turning to the hydraulic gradient i_0, one can write

$$i_0 = -\frac{n^{\text{in}}}{gv} = \lambda \cdot \frac{1}{d} \cdot \frac{v^2}{2g}. \tag{1.27}$$

Characteristic values of the hydraulic slope are 0.00005–0.005.

If we substitute Eq. (1.27) into the Bernoulli equation (1.20), we obtain

$$\left(\frac{\alpha_k v^2}{2g} + \frac{p}{\rho g} + z\right)_1 - \left(\frac{\alpha_k v^2}{2g} + \frac{p}{\rho g} + z\right)_2 = \lambda(Re, \varepsilon) \cdot \frac{l_{1-2}}{d}\frac{v^2}{2g}. \tag{1.28}$$

The expression $h_\tau = \lambda \cdot l_{1-2}/d \cdot v^2/2g$ on the right-hand side of this equation is called *the loss of head in Darcy-Veisbach form.*

Using Eq. (1.27) in the case of stationary flow of the transported medium permits us to get an expression for the tangential friction stress τ_w at the pipeline wall. Substitution of Eq. (1.27) into Eq. (1.24), yields

$$\tau_w = \frac{\rho g d}{4} \cdot i_0 = \frac{\rho g d}{4} \cdot \left(\lambda \frac{1}{d}\frac{v^2}{2g}\right) = \frac{\lambda}{4} \cdot \frac{\rho v^2}{2} = C_f \cdot \frac{\rho v^2}{2}, \tag{1.29}$$

$$C_f(Re, \varepsilon) = \frac{\lambda(Re, \varepsilon)}{4}$$

where the dimensional factor C_f is called the *friction factor* of the fluid on the internal surface of the pipeline or the *Funning factor* (Leibenson *et al.*, 1934).

1.6.2
Formulas for Calculation of the Factor $\lambda(Re, \varepsilon)$

Details of methods to find and calculate the factor of hydraulic resistance λ in Eqs. (1.26)–(1.29) and one of the primary factors in hydraulics and pipeline transportation will be given in Chapter 3. Here are shown several formulas exploiting the practice.

If the flow of fluid or gas in the pipeline is laminar, that is jetwise or layerwise (the Reynolds number Re should be less than 2300), then to determine λ the *Stokes formula* (see Section 3.1) is used

$$\lambda = \frac{64}{Re}. \tag{1.30}$$

As the Reynolds number increases ($Re > 2300$) the flow in the pipeline gradually loses hydrodynamic stability and becomes turbulent, that is vortex flow with mixing layers. The best known formula to calculate the factor λ in this case is the *Altshuler formula*:

$$\lambda = 0.11 \cdot \left(\varepsilon + \frac{68}{Re} \right)^{1/4} \tag{1.31}$$

valid over a wide range of Reynolds number from 10^4 up to 10^6 and higher.

If $10^4 < Re < 27/\varepsilon^{1.143}$ and $Re < 10^5$, the Altshuler formula becomes the *Blasius formula*:

$$\lambda = \frac{0.3164}{\sqrt[4]{Re}} \tag{1.32}$$

having the same peculiarity as the Stokes formula for laminar flow, which does not consider the relative roughness of the pipeline internal surface ε. This means that for the considered range of Reynolds numbers the pipeline behaves as a pipeline with a smooth surface. Therefore the fluid flow in this range is *flow in a hydraulic smooth pipe*. In this case the friction tension τ_w at the pipe wall is expressed by formula

$$\tau_w = -\frac{\lambda}{4} \cdot \frac{\rho v^2}{2} = -\frac{0.0791}{\sqrt[4]{vd/\nu}} \cdot \frac{\rho v^2}{2} \approx v^{1.75}$$

signifying that friction resistance is proportional to fluid mean velocity to the power of 1.75.

If $Re > 500/\varepsilon$, the second term in parentheses in the Altshuler formula can be neglected compared to the first one. Whence it follows that at great fluid velocities the fluid friction is caused chiefly by the smoothness of the pipeline internal surface, that is by the parameter ε. In such a case one can use the simpler *Shiphrinson formula* $\lambda = 0.11 \cdot \varepsilon^{0.25}$. Then

$$\tau_w = -\frac{\lambda}{4} \cdot \frac{\rho v^2}{2} = -\frac{0.11 \cdot \varepsilon^{1/4}}{4} \cdot \frac{\rho v^2}{2} \approx v^2.$$

From this it transpires that the friction resistance is proportional to the square of the fluid mean velocity and hence this type of flow is called *square flow*.

Finally, in the region of flow transition from laminar to turbulent, that is in the range of Reynolds number from 2320 up to 10^4 one can use the approximation formula

$$\lambda = \frac{64}{Re} \cdot (1 - \gamma_*) + \frac{0.3164}{\sqrt[4]{Re}} \cdot \gamma_*, \tag{1.33}$$

where $\gamma_* = 1 - e^{-0.002 \cdot (Re - 2320)}$ is the intermittency factor (Ginsburg, 1957). It is obvious that the form of the last formula assures continuous transfer from the Stokes formula for laminar flow to the Blasius formula for turbulent flow in the zone of hydraulic smooth pipes.

To calculate the hydraulic resistance factor λ of the gas flow in a gas main, where the Reynolds number *Re* is very large and this factor depends only on the condition of the pipeline internal surface, Eq. (1.34) is often used.

$$\lambda = 0.067 \cdot \left(\frac{2\Delta}{d}\right)^{0.2} \tag{1.34}$$

in which the absolute roughness Δ is equal to 0.03–0.05 mm.

Exercise 1. The oil ($\rho = 870\ \text{kg m}^{-3}$, $\nu = 15$ s St) flows along the pipeline ($D = 156$ mm; $\delta = 5$ mm; $\Delta = 0.1$ mm) with mean velocity $v = 0.2\ \text{m s}^{-1}$. Determine through the Reynolds criterion the flow regime; calculate factors λ and C_f.

Answer. Laminar; 0.033; 0.0083.

Exercise 2. Benzene ($\rho = 750\ \text{kg m}^{-3}$, $\nu = 0.7$ s St) flows along the pipeline ($D = 377$ mm; $\delta = 7$ mm; $\Delta = 0.15$ mm) with mean velocity $v = 1.4\ \text{m s}^{-1}$. Determine through the Reynolds criterion the flow regime; calculate factors λ and C_f.

Answer. Turbulent; 0.017; 0.0041.

Exercise 3. Diesel fuel ($\rho = 840\ \text{kg m}^{-3}$, $\nu = 6$ s St) flows along the pipeline ($D = 530$ mm; $\delta = 8$ mm; $\Delta = 0.25$ mm) with mean velocity $v = 0.8\ \text{m s}^{-1}$. Determine the flow regime; calculate factors λ and C_f.

Answer. Turbulent; 0.022; 0.0054.

1.7 Total Energy Balance Equation

Besides the law (1.13) of mechanical energy change of material points, applied to an arbitrary continuum volume in the pipeline there is one more fundamental physical law valid for any continuum – the law of total energy conservation or, as it is also called, the *first law of thermodynamics*. This law asserts that the energy does not appear from anywhere and does not disappear to anywhere. It changes in total quantity from one form into another. As applied to our case this law may be written as follows

$$\frac{d(E_{kin} + E_{in})}{dt} = \frac{dQ^{ex}}{dt} + \frac{dA^{ex}}{dt} \tag{1.35}$$

that is the change in *total energy* ($E_{kin} + E_{in}$) of an arbitrary volume of the transported medium happens only due to the exchange of energy with surrounding bodies owing to external inflow of heat dQ^{ex} and the work of external forces dA^{ex}.

In Eq. (1.35) E_{in} is the internal energy of the considered mass of transported medium, unrelated to the kinetic energy, that is the energy of heat motion, interaction between molecules and atoms and so on. In thermodynamics reasons are given as to why the internal energy is a function of state, that is at thermodynamic equilibrium of a body in some state the energy has a well-defined value regardless of the means (procedure) by which this state was achieved. At the same time the quantities dQ^{ex}/dt and dA^{ex}/dt are not generally derivatives with respect to a certain function of state but only represent the ratio of elementary inflows of heat energy (differential dQ^{ex}) and external mechanical energy (differential dA^{ex}) to the time dt in which these inflows happened. It should be kept in mind that these quantities depend on the process going on in the medium.

In addition to function E_{in} one more function e_{in} is often introduced, representing the internal energy of a unit mass of the considered body $e_{\text{in}} = E_{\text{in}}/m$, where m is the mass of the body.

We can write Eq. (1.35) for a movable volume of transported medium enclosed between cross-sections $x_1(t)$ and $x_2(t)$. The terms of this equation are

$$\frac{d(E_{\text{kin}} + E_{\text{in}})}{dt} = \frac{d}{dt}\left[\int_{x_1(t)}^{x_2(t)} \left(\alpha_k \frac{\rho v^2}{2} + \rho e_{\text{in}}\right) S\, dx\right],$$

$$\frac{dQ^{\text{ex}}}{dt} = \int_{x_1(t)}^{x_2(t)} \pi d \cdot q_n\, dx,$$

$$\frac{dA^{\text{ex}}}{dt} = -\int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial x}(pSv)\, dx - \int_{x_1(t)}^{x_2(t)} \rho g \sin\alpha \cdot v \cdot S\, dx + N_{\text{mech}}$$

where q_n is the heat flux going through the unit area of the pipeline surface per unit time (W m^{-2}); $\pi d \cdot dx$ is an element of pipeline surface area and d is the pipeline diameter.

Gathering all terms, we obtain

$$\frac{d}{dt}\left[\int_{x_1(t)}^{x_2(t)} \left(\alpha_k \cdot \frac{\rho v^2}{2} + \rho e_{\text{in}}\right) S\, dx\right] = \int_{x_1(t)}^{x_2(t)} \pi d \cdot q_n\, dx - \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial x}(pSv)\, dx - \int_{x_1(t)}^{x_2(t)} \rho g \sin\alpha \cdot v \cdot S\, dx + N_{\text{mech}}.$$

Differentiation of the left-hand side of this equation gives

$$\int_{x_1(t)}^{x_2(t)} \left\{\frac{\partial}{\partial t}\left[\left(\frac{\alpha_k v^2}{2} + e_{\text{in}}\right)\rho S\right] + \frac{\partial}{\partial x}\left[\left(\frac{\alpha_k v^2}{2} + e_{\text{in}}\right)\rho v S\right]\right\} dx = \int_{x_1(t)}^{x_2(t)} \pi d \cdot q_n\, dx - \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial x}\left(\frac{p}{\rho}\rho v S\right) dx - \int_{x_1(t)}^{x_2(t)} \rho v S g \frac{\partial z}{\partial x}\, dx + N_{\text{mech}}$$

or

$$\int_{x_1(t)}^{x_2(t)} \left\{ \frac{\partial}{\partial t}\left[\left(\frac{\alpha_k v^2}{2} + e_{\text{in}}\right)\rho S\right] + \frac{\partial}{\partial x}\left[\left(\frac{\alpha_k v^2}{2} + e_{\text{in}} + \frac{p}{\rho}\right)\rho v S\right]\right\} \mathrm{d}x$$
$$= \int_{x_1(t)}^{x_2(t)} \pi d \cdot q_{\text{n}}\, \mathrm{d}x - \int_{x_1(t)}^{x_2(t)} \rho v S g \frac{\partial z}{\partial x}\, \mathrm{d}x + N_{\text{mech}}. \tag{1.36}$$

If we assume that inside the region $[x_1(t), x_2(t)]$ the external sources of mechanical energy are absent, that is $N_{\text{mech}} = 0$, then it is possible to pass from integral equality (1.36) to the corresponding differential equation using, as before, the condition that this equation should be true for any volume of the transported medium, that is the limits of integration $x_1(t)$ and $x_2(t)$ in (1.36) are to be arbitrarily chosen. Then the sign of the integral can be omitted and the differential equation is

$$\frac{\partial}{\partial t}\left[\left(\frac{\alpha_k v^2}{2} + e_{\text{in}}\right)\rho S\right] + \frac{\partial}{\partial x}\left[\left(\frac{\alpha_k v^2}{2} + e_{\text{in}} + \frac{p}{\rho}\right)\rho v S\right]$$
$$= \pi d \cdot q_{\text{n}} - \rho v S g \frac{\partial z}{\partial x}. \tag{1.37}$$

Excluding from Eq. (1.37) the change in kinetic energy with the help of the Bernoulli equation with term by term subtraction of Eq. (1.16) from Eq. (1.37) we get one more energy equation

$$\rho S \frac{\partial}{\partial t}\left(\frac{\alpha_k v^2}{2}\right) + \rho v S \frac{\partial}{\partial x}\left(\frac{\alpha_k v^2}{2} + \int \frac{\mathrm{d}p}{\rho} + gz\right) = \rho v S g \cdot i$$

called the *equation of heat inflow*.

This equation could be variously written. First, it may be written through the internal energy e_{in}:

$$\frac{\partial}{\partial t}(e_{\text{in}} \cdot \rho S) + \frac{\partial}{\partial x}(e_{\text{in}} \cdot \rho v S) = \pi d \cdot q_n - p\frac{\partial v S}{\partial x} - \rho v S g \cdot i$$

or

$$\rho S\left(\frac{\partial e_{\text{in}}}{\partial t} + v\frac{\partial e_{\text{in}}}{\partial x}\right) = \pi d \cdot q_{\text{n}} - p \cdot \frac{\partial v S}{\partial x} - \rho v S g \cdot i. \tag{1.38}$$

This equation proved to be especially convenient for modeling flows of incompressible or slightly compressible fluids because the derivative $\partial(vS)/\partial x$ expressing the change in fluid volume in the pipeline cross-section is extremely small as is the work $p \cdot \partial(vS)/\partial x$ of the pressure forces. With this in mind Eq. (1.38) may be written in a particularly simple form:

$$\rho\frac{\mathrm{d}e_{\text{in}}}{\mathrm{d}t} \cong \frac{4}{d} \cdot q_{\text{n}} - \rho v g \cdot i. \tag{1.39}$$

This means that the rate of internal energy change of the transported medium is determined by the inflow of external heat through the pipeline surface and heat extraction due to conversion of mechanical energy into heat produced by friction between the continuum layers.

Second, the equation of heat inflow can be written using the function $J = e_{\text{in}} + p/\rho$ representing one of the basic thermodynamic functions, *enthalpy or heat content,* of the transported medium

$$\frac{\partial}{\partial t}(e_{\text{in}} \cdot \rho S) + \frac{\partial}{\partial x}\left[\left(e_{\text{in}} + \frac{p}{\rho}\right)\rho v S\right] = \pi d \cdot q_{\text{n}} + \rho v S g \cdot \left(\frac{1}{\rho g}\frac{\partial p}{\partial x} - i\right)$$

or

$$\frac{\partial}{\partial t}(e_{\text{in}} \cdot \rho S) + \frac{\partial}{\partial x}[J \cdot \rho v S] = \pi d \cdot q_{\text{n}} + \rho v S g \cdot \left(\frac{1}{\rho g}\frac{\partial p}{\partial x} - i\right). \tag{1.40}$$

If we take into account (as will be shown later) that the expression in parentheses on the right-hand side of this equation is close to zero, since for a relatively light medium, e.g. gas, the hydraulic slope is expressed through the pressure gradient by the formula $i = 1/\rho g \cdot \partial p/\partial x$, the equation of heat inflow can be reduced to a simpler form

$$\frac{\partial \rho S \cdot e_{\text{in}}}{\partial t} + \frac{\partial \rho v S \cdot J}{\partial x} \cong \pi d \cdot q_{\text{n}} \tag{1.41}$$

in which the dissipation of mechanical energy appears to be absent.

Temperature Distribution in Stationary Flow

The equation of heat inflow in the form (1.39) or (1.41) is convenient to determine the temperature distribution along the pipeline length in stationary flow of the transported medium.

1. For an *incompressible* or *slightly compressible* medium, e.g. dropping liquid: water, oil and oil product, this equation has the form

$$\rho v \cdot \frac{\mathrm{d}e_{\text{in}}}{\mathrm{d}x} \cong \frac{4}{d} \cdot q_{\text{n}} - \rho v g \cdot i. \tag{1.42}$$

The internal energy e_{in} depends primarily on the temperature of the fluid T, the derivative $\mathrm{d}e_{\text{in}}/\mathrm{d}T$ giving its specific heat C_v (J kg^{-1} K^{-1}). If we take $C_v = \text{const.}$ then $e_{\text{in}} = C_v \cdot T + \text{const.}$

To model the heat flux q_{n} the *Newton formula* is usually used

$$q_{\text{n}} = -\kappa \cdot (T - T_{\text{ex}}), \tag{1.43}$$

by which this flow is proportional to the difference between the temperatures T and T_{ex} in and outside the pipeline, with $q_{\text{n}} < 0$ when $T > T_{\text{ex}}$ and $q_{\text{n}} > 0$ when $T < T_{\text{ex}}$. The factor κ (W m^{-2} K^{-1}) in this formula characterizes the overall heat resistance of the materials through which the heat is transferred from the pipe to the surrounding medium (anticorrosive and heat insulation, ground, the boundary between ground and air and so on) or the reverse. This factor is called the *heat-transfer factor*.

The hydraulic gradient i can sometimes be considered constant $i = -i_0 \approx$ const., if the dissipation of mechanical energy in the stationary fluid flow in the pipeline with constant diameter is identical at all cross-sections of the pipeline.

With due regard for all the aforesaid Eq. (1.42) is reduced to the following ordinary differential equation

$$\rho C_v v \cdot \frac{dT}{dx} = -\frac{4\kappa}{d}(T - T_{ex}) + \rho v g i_0 \tag{1.44}$$

for temperature $T = T(x)$. From this equation in particular it follows that the heat transfer through the pipeline wall (the first term on the right-hand side) lowers the temperature of the transported medium when $T(x) > T_{ex}$ or raises it when $T(x) < T_{ex}$, whereas the dissipation of mechanical energy (the second term on the right-hand side) always implies an increase in the temperature of the transported medium.

The solution of the differential equation (1.44) with initial condition $T(0) = T_0$ yields

$$\frac{T(x) - T_{ex} - T_\otimes}{T_0 - T_{ex} - T_\otimes} = \exp\left(-\frac{\pi d\kappa}{C_v \dot{M}}x\right). \tag{1.45}$$

Where $T_\otimes = g i_0 \dot{M}/\pi d\kappa$ is a constant having the dimension of temperature; $\dot{M} = \rho v S$ is the mass flow rate of the fluid ($\dot{M} =$ const.). The formula thus obtained is called the *Shuchov formula*.

Figure 1.3 illustrates the distribution of temperature $T(x)$ along the pipeline length x in accordance with Eq. (1.45).

The figure shows that when the initial temperature T_0 is greater than $(T_{ex} + T_\otimes)$, the moving medium cools down, while when T_0 is less than $(T_{ex} + T_\otimes)$, the medium gradually heats up. In all cases with increase in the pipeline length the temperature $T \to (T_{ex} + T_\otimes)$.

In particular from Eq. (1.44) it follows that if the heat insulation of the pipeline is chosen such that at the initial cross-section of the pipeline $x = 0$

Figure 1.3 Temperature distribution along the pipeline length.

the condition of equality to zero of the right-hand side is obeyed

$$-\frac{4\kappa}{d}(T_0 - T_{\text{ex}}) + \rho v g i_0 = 0$$

that is the factor κ satisfies the condition

$$\kappa = \frac{\rho v d g i_0}{4(T_0 - T_{\text{ex}})} = \frac{g\dot{M} \cdot i_0}{\pi d \cdot (T_0 - T_{\text{ex}})}.$$

And the temperature of the transported medium would remain constant and equal to its initial value over the whole pipeline section. In such a case the heat outgoing from the pipeline would be compensated by the heat extracted by internal friction between the layers. Such an effect is used, for example, in oil transportation along the Trans-Alaska oil pipeline (USA, see the cover picture). Through good insulation of the pipeline the oil is pumped over without preheating despite the fact that in winter the temperature of the environment is very low.

From Eq. (1.45) follows the connection between the initial T_0 and final T_L temperatures of the transported medium. If in this formula we set $x = L$, where L is the length of the pipeline section, we obtain

$$\frac{T_L - T_{\text{ex}} - T_\otimes}{T_0 - T_{\text{ex}} - T_\otimes} = \exp\left(-\frac{\pi d \kappa L}{C_v \dot{M}}\right). \tag{1.46}$$

Expressing now from (1.46) the argument under the exponent and substituting the result in Eq. (1.45), we get the expression for the temperature distribution through the initial and final values

$$\frac{T(x) - T_{\text{ex}} - T_\otimes}{T_0 - T_{\text{ex}} - T_\otimes} = \left(\frac{T_L - T_{\text{ex}} - T_\otimes}{T_0 - T_{\text{ex}} - T_\otimes}\right)^{x/L}. \tag{1.47}$$

Exercise 1. The initial temperature of crude oil ($\rho = 870\ \text{kg m}^{-3}$, $C_v = 2000\ \text{J kg}^{-1}\ \text{K}^{-1}$, $Q = 2500\ \text{m}^3\ \text{h}^{-1}$), pumping over a pipeline section ($d = 800$ mm, $L = 120$ km, $i_0 = 0.002$) is 55 °C. The temperature of the surrounding medium is 8 °C. The heat insulation of the pipeline is characterized by the heat-transfer factor $\kappa = 2\ \text{W m}^{-2}\ \text{K}^{-1}$. It is required to find the temperature at the end of the section.

Solution. Calculate first the temperature $T_\otimes$:

$$T_\otimes = \frac{g i_0 \dot{M}}{\pi d \kappa} = \frac{9.81 \cdot 0.002 \cdot 870 \cdot (2500/3600)}{3.14 \cdot 0.8 \cdot 2} \cong 2.36\ \text{K}.$$

Using Eq. (1.46) we obtain

$$\frac{T_L - 8 - 2.36}{55 - 8 - 2.36} = \exp\left(-\frac{3.14 \cdot 0.8 \cdot 2 \cdot 120 \cdot 10^3}{2000 \cdot 870 \cdot (2500/3600)}\right),$$

from which follows $T_L \cong 37.5$ °C.

Exercise 2. By how much would the temperature of the oil ($C_v = 1950\ \mathrm{J\,kg^{-1}\,K^{-1}}$) be raised due to the heat of internal friction when the oil is transported by an oil pipeline ($L = 150$ km, $d = 500$ mm, $i_0 = 0.004$) provided with ideal heat insulation ($\kappa = 0$)?

Solution. In this case it is impossible to use at once Eq. (1.45) since $\kappa = 0$. To use Eq. (1.47) one should go to the limit at $\kappa \to 0$, therefore it would be better to use Eq. (1.44)

$$\rho C_v v \cdot \frac{\mathrm{d}T}{\mathrm{d}x} = \rho v g i_0 \quad \text{or} \quad C_v \cdot \frac{\mathrm{d}T}{\mathrm{d}x} = g i_0,$$

from which $\Delta T = g i_0 L / C_v = 9.81 \cdot 0.004 \cdot 150 \cdot 10^3/1950 \cong 3$ K.

Exercise 3. It is required to obtain the temperature of oil pumping over the pipeline section of length 150 km in cross-sections $x = 50$, 100 and 125 km, if the temperature at the beginning of the pipeline $T_0 = 60\,^\circ$C, that at the end $T_L = 30\,^\circ$C, and that of the environment $T_{ex} = 8\,^\circ$C. The extracted heat of internal friction may be ignored.

Solution. Using Eq. (1.46), one gets

$$\frac{T(x) - 8}{60 - 8} = \left(\frac{30 - 8}{60 - 8}\right)^{x/L} \quad \text{and} \quad T(x) = 8 + 52 \cdot (0.4231)^{x/150}.$$

Substitution in this formula of successive $x = 50$, 100 and 125 gives $T(50) \cong 47\,^\circ$C; $T(100) \cong 37.3\,^\circ$C; $T(125) \cong 33.4\,^\circ$C.

2. For stationary flow of a *compressible medium*, e.g. gas, the equation of heat inflow (1.41) takes the form

$$\rho v S \frac{\mathrm{d}J}{\mathrm{d}x} = \pi d \cdot q_n.$$

In the general case, the gas enthalpy J is a function of pressure and temperature $J = J(p, T)$, but for a *perfect gas*, that is a gas obeying the Clapeyron law $p = \rho R T$, where R is the gas constant, the enthalpy is a function only of temperature $J = C_p \cdot T + \text{const.}$, where C_p is the gas specific heat capacity at constant pressure ($C_p > C_v$; $C_p - C_v = R$). Regarding $C_p = \text{const.}$ and taking as before $q_n = -\kappa \cdot (T - T_{ex})$, we transform the last equation to

$$C_p \dot{M} \frac{\mathrm{d}T}{\mathrm{d}x} = -\pi d \kappa \cdot (T - T_{ex})$$

or

$$\frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{\pi d \kappa}{C_p \dot{M}} \cdot (T - T_{ex}).$$

The solution of this differential equation with initial condition $T(0) = T_0$ gives

$$\frac{T(x) - T_{\text{ex}}}{T_0 - T_{\text{ex}}} = \exp\left(-\frac{\pi d\kappa}{C_p\dot{M}}x\right), \tag{1.48}$$

which is similar to the solution (1.45) for temperature distribution in an incompressible fluid. The difference consists only in that instead of heat capacity C_v in the solution (1.47) we use heat capacity C_p and the temperature $T_\otimes$ taking into account the heat of internal friction is absent (for methane $C_p \cong 2230\ \text{J kg}^{-1}\ \text{K}^{-1}$; $C_v \cong 1700\ \text{J kg}^{-1}\ \text{K}^{-1}$).

The temperature T_{L} of the gas at the end of the gas pipeline section is found from

$$\frac{T_{\text{L}} - T_{\text{ex}}}{T_0 - T_{\text{ex}}} = \exp\left(-\frac{\pi d\kappa L}{C_p\dot{M}}\right) \tag{1.49}$$

with regard to which the distribution (1.47) takes the form

$$\frac{T(x) - T_{\text{ex}}}{T_0 - T_{\text{ex}}} = \left(\frac{T_{\text{L}} - T_{\text{ex}}}{T_0 - T}\right)^{x/\text{L}} \tag{1.50}$$

allowing us to express the temperature through the initial and final temperatures.

Note that for a real gas the enthalpy $J = J(p, T)$ of the medium depends not only on temperature but also on pressure, so the equation of heat inflow has a more complex form. By the dependence $J(p, T)$ is explained, in particular, the *Joule-Thomson effect*.

1.8
Complete System of Equations for Mathematical Modeling of One-Dimensional Flows in Pipelines

This system consists of the following equations.

1. Continuity equation (1.6)

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho v S}{\partial x} = 0;$$

2. Momentum (motion) equation (1.10)

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial x} - \frac{4}{d}\tau_{\text{w}} - \rho g \sin\alpha(x);$$

3. Equation of mechanical energy balance (1.15)

$$\frac{\partial}{\partial t}\left(\frac{\alpha_{\text{k}} v^2}{2}\right) + v\cdot\frac{\partial}{\partial x}\left(\frac{\alpha_{\text{k}} v^2}{2} + P(\rho) + gz\right) = vg\cdot i;$$

4. Equation of total energy balance (1.37)

$$\frac{\partial}{\partial t}\left[\left(\frac{\alpha_k v^2}{2} + e_{in}\right)\rho S\right] + \frac{\partial}{\partial x}\left[\left(\frac{\alpha_k v^2}{2} + J\right)\rho v S\right] = \pi d \cdot q_n - \rho v g S \frac{dz}{dx}.$$

The number of unknown functions in this equation is 10: ρ, v, p, S, e_{in}, T, τ_w, i, q_n, α_k, while the number of equations is 4. Therefore there are needed additional relations to *close* the system of equations. As *closing relations* the following relations are commonly used:

- equation of state $p = p(\rho, T)$, characterizing the properties of the transported medium;
- equation of pipeline state $S = S(p, T)$ characterizing the deformation ability of the pipeline;
- calorimetric dependences $e_{in} = e(p, T)$ or $J = J(p, T)$;
- dependence $q_n = -\kappa \cdot (T - T_{ex})$ or more complex dependences representing heat exchange between the transported medium and the environment;
- hydraulic dependence $\tau_w = \tau_w(\rho, v, \dot{v}, d, \nu, \ldots)$;
- dependences $\alpha_k = f(\rho, v, \nu, d, \ldots)$, or $i = \tilde{f}(\tau_w)$,

characterizing internal structure of medium flow.

To obtain closing relations a more detailed analysis of flow processes is needed. It is also necessary to consider mathematical relations describing properties of the transported medium and the pipeline in which the medium flows.

The division of mechanics in which properties of a transported medium such as viscosity, elasticity, plasticity and other more complex properties are studied is called *rheology*.