Annexure

Handbook of Ultra-Wideband Short-Range Sensing: Theory, Sensors Applications, First Edition. Jürgen Sachs.

Annex A: Mathematical Basics

A.1
Some useful Improper Integrals

Table A-1 Some useful improper integrals.

Fresnel integrals	$\int\limits_{-\infty}^{\infty} \sin(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{2}}$	$\int\limits_{-\infty}^{\infty} \cos(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{2}}$
Sinc-function $\operatorname{sinc} x = \dfrac{\sin \pi x}{\pi x}$	$\int\limits_{-\infty}^{\infty} \operatorname{sinc} x \, \mathrm{d}x = 1$	$\int\limits_{-\infty}^{\infty} \lvert\operatorname{sinc} x\rvert \mathrm{d}x = \infty$
	$\int\limits_{-\infty}^{\infty} \operatorname{sinc}^2 x \, \mathrm{d}x = 1$	
	$\int\limits_{-\infty}^{\infty} \operatorname{sinc}^3 x \, \mathrm{d}x = \dfrac{3}{4}$	$\int\limits_{-\infty}^{\infty} \operatorname{sinc}^4 x \, \mathrm{d}x = \frac{2}{3}$
Gauss function $y = \mathrm{e}^{-\pi x^2}$	$\int\limits_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2} \mathrm{d}x = 1$	$\int\limits_{-\infty}^{\infty} x^n \mathrm{e}^{-\pi x^2} \mathrm{d}x = \begin{cases} 0 & n \quad \text{odd} \\ \dfrac{(n-1)!!}{(2\pi)^{n/2}} & n \quad \text{even} \end{cases}$
	$\int\limits_{-\infty}^{\infty} \lvert x\rvert^n \mathrm{e}^{-\pi x^2} \mathrm{d}x = \begin{cases} \dfrac{(n-1)!!}{\pi(2\pi)^{(n-1)/2}} & n \quad \text{odd} \\ \dfrac{(n-1)!!}{(2\pi)^{n/2}} & n \quad \text{even} \end{cases}$	
Cauchy-Lorentz distribution $y = \dfrac{1}{\pi} \dfrac{a}{a^2 + x^2};\ a \in \mathbb{R}$	$\int\limits_{-\infty}^{\infty} \dfrac{1}{\pi} \dfrac{a}{a^2 + x^2} \, dx = 1$	$\lim\limits_{\xi \to \infty} \int\limits_{-\xi}^{\xi} \dfrac{1}{\pi} \dfrac{ax}{a^2 + x^2} \, dx = 0$
$\int\limits_{-\infty}^{\infty} \cos x \, e^{-\pi x^2} dx = \int\limits_{-\infty}^{\infty} e^{-\pi x^2 + jx} dx = e^{-\frac{1}{4\pi}}$		

$n!!$ - double factorial: It is like factorial but the product is taken either over even or odd numbers:

$$7!! = 1 \cdot 3 \cdot 5 \cdot 7$$
$$8!! = 2 \cdot 4 \cdot 6 \cdot 8$$

A.2
Dirac Delta Function and Doublets

The Dirac delta function is a generalised function since it cannot be described in the classical sense. An introduction into generalised functions is to be found in [1]. There are several definitions for the Dirac delta function in use. Many of them represent limits of "nascent" delta functions $\eta_\tau(t)$.

$$\delta(t) = \lim_{\tau \to +0} \eta_\tau(t) \Rightarrow \left\{ \begin{array}{c} \eta_\tau(t) = \dfrac{1}{\tau}\operatorname{rect}\dfrac{t}{\tau} \\[2ex] \eta_\tau(t) = \begin{cases} \dfrac{2\sqrt{\tau^2 - t^2}}{\pi\tau^2}, & |t| < \tau \\[2ex] 0, & |t| \geq \tau \end{cases} \\[2ex] \eta_\tau(t) = \dfrac{1}{\tau\sqrt{\pi}} e^{-\left(\frac{t}{\tau}\right)^2} \\[2ex] \eta_\tau(t) = \dfrac{1}{\tau}\operatorname{sinc}\dfrac{t}{\tau} \\[2ex] \eta_\tau(t) = \dfrac{1}{2\tau} e^{-\frac{|t|}{\tau}} \\[2ex] \vdots \end{array} \right\} \Rightarrow \begin{array}{l} \delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \\[2ex] \text{where at } \displaystyle\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1 \end{array} \tag{A.1}$$

Some properties of the Dirac delta function:
Symmetry:

$$\delta(t) = \delta(-t) \tag{A.2}$$

Fourier and Laplace transform:

$$\begin{gathered} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} \mathrm{d}t = 1 \\ \int_{-\infty}^{\infty} \delta(t - \tau_0) e^{-j2\pi f t} \mathrm{d}t = e^{-j2\pi f \tau_0} \\ \int_{0}^{\infty} \delta(t - \tau_0) e^{-st} \mathrm{d}t = e^{-s\tau_0} \end{gathered} \tag{A.3}$$

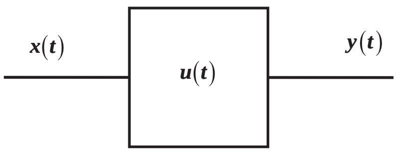
Sampling property:

$$g(t_s) = \int g(t)\,\delta(t - t_s)\,\mathrm{d}t \tag{A.4}$$

Convolution:

$$\begin{aligned} g(t-\tau_0) = g(t) * \delta(t-\tau_0) &= \int g(\tau)\,\delta(t-\tau_0-\tau)\,\mathrm{d}\tau \\ &= \int g(\tau)\,\delta(\tau-(t-\tau_0))\,\mathrm{d}\tau \quad \text{since} \quad \delta(t)=\delta(-t) \\ &= \int g(\tau)\,\delta(\tau-T)\,\mathrm{d}\tau \quad \text{with} \quad T = t-\tau_0 \\ &= g(T) \quad \text{due to sampling property} \end{aligned} \tag{A.5}$$

Derivation and Integration:



We are defining derivation and integration of a Dirac-function by convolution with an auxiliary function $u(t)$ symbolised by the transmission system in the figure above. Supposing it provides the first derivation of its input signal, then its output signal can be written as:

$$y(t) = \frac{\partial x(t)}{\partial t} = g(t) * x(t)$$

leading to the impulse response:

$$g(t) = u_1(t) = \frac{d\delta(t)}{dt} \quad \Rightarrow \quad y(t) = u_1(t) * x(t) \tag{A.6}$$

$u_1(t)$ is called the unit doublet. This procedure can be continued for higher order derivations and integrations too.

$$n\text{-fold derivation}: \quad x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \quad \Rightarrow \quad u_n(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{n \text{ times}} \tag{A.7}$$

$$\text{Integration}: \quad y(t) = \int_{-\infty}^{t} x(\tau)d\tau \quad \Rightarrow \quad y(t) = u_{-1}(t) * x(t) \tag{A.8}$$

$$u(t) = u_{-1}(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{A.9}$$

$u(t) = u_{-1}(t)$ represents the unit step or Heaviside function.

n-fold integration:

$$y(t) = \int_{-\infty}^{\tau_1} \cdots \int_{-\infty}^{\tau_n} x(\tau)\mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n$$

$$\Rightarrow \quad y(t) = u_{-n}(t) * x(t) \quad \text{with} \quad u_{-n}(t) = \underbrace{u_{-1}(t) * \cdots * u_{-1}(t)}_{n \text{ times}} = \frac{t^{n-1}}{(n-1)!}u_{-1}(t) \tag{A.10}$$

Generalisation:

$$\begin{aligned} &u_0(t) \stackrel{def}{=} \delta(t) \\ &u_n(t) * u_m(t) = u_{n+m}(t) \\ &\Rightarrow u_1(t) * u_{-1}(t) = u_0(t) \cdots \rightarrow \left(\frac{d}{dt}\int \delta(t)dt\right) = \delta(t) \end{aligned} \tag{A.11}$$

Product rules:

$$u_1(t) * (a(t)\, b(t)) = b(t)\, u_1(t) * a(t) + a(t)\, u_1(t) * b(t) \tag{A.12}$$

$$u_1(t) * (a(t)\, (u_{-1}(t) * b(t))) = (u_{-1}(t) * b(t))\, (u_1(t) * a(t)) + a(t)\, b(t) \tag{A.13}$$

$$a(t)\, b(t) = u_{-1}(t) * (a(t)\, u_1(t) * b(t)) + u_{-1}(t) * (b(t)\, u_1(t) * a(t)) \tag{A.14}$$

$$\frac{1}{2}a^2(t) = u_{-1}(t) * (a(t)\, u_1(t) * a(t)) \tag{A.15}$$

$$u_{-1}(t) * (a(t) * b(t)) = (u_{-1}(t) * a(t))\, (u_{-1}(t) * b(t)) \tag{A.16}$$

Multidimensional Dirac function:
It holds:

$$\int \delta^{(3)}(\mathbf{r} - \mathbf{r}')\, dV = 1 \tag{A.17}$$

$$\text{Cartesian coordinates :} \quad \delta^{(3)}(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z') \tag{A.18}$$

$$\text{Polar coordinates :} \quad \delta^{(3)}(\mathbf{r} - \mathbf{r}') = \frac{1}{r^2 \sin\vartheta}\delta(r - r')\delta(\vartheta - \vartheta')\delta(\varphi - \varphi') \tag{A.19}$$

A.3
Some Definitions and Calculation Rules for Statistic Variables

We suppose a set of random numbers originating e.g. from repeated measurements (sampled data; empirical values). Here, we will refer to the random variables $\underset{\sim}{x}$ and $\underset{\sim}{y}$:

$$\begin{aligned} \underset{\sim}{x} &= [\, x[1] \quad x[2] \quad \cdots \quad x[N]\,] \\ \underset{\sim}{y} &= [\, y[1] \quad y[2] \quad \cdots \quad y[N]\,] \end{aligned}$$

Probability Density Function (PDF)

- Random variables which are uniformly distributed between q and p, we assign as:

$$\underset{\sim}{x} \sim U(q,p) \quad \Rightarrow \quad p_{\underset{\sim}{x}}(x) = \begin{cases} 0 & x < q \\ \dfrac{1}{p-q} & q \le x \le p \\ 0 & x > p \end{cases} \tag{A.20}$$

Their mean value is $\mu = \frac{q+p}{2}$ and their variance is $\sigma_x^2 = \frac{(p-q)^2}{12}$. $p_{\underset{\sim}{x}}(x)$ is the PDF of the random variable $\underset{\sim}{x}$.

- Random variables of Gaussian distribution (normal distribution) are assigned as:

$$\underset{\sim}{x} \sim N(\mu, \sigma_x^2) \quad \Rightarrow \quad p_{\underset{\sim}{x}}(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2}. \tag{A.21}$$

- A multivariate process $\underset{\sim}{\mathbf{x}}$ covers a number of random variables $\underset{\sim}{x}_m$ which are typically summarised by vector notation:

$$\underset{\sim}{\mathbf{x}} = \begin{bmatrix} \underset{\sim}{x}_1 & \underset{\sim}{x}_2 & \underset{\sim}{x}_3 & \cdots & \underset{\sim}{x}_M \end{bmatrix}^T$$

Since every random variable $\underset{\sim}{x}_m$ may cover a set of data samples (empirical values), the random process $\underset{\sim}{\mathbf{x}}$ actually represents a [N, M] matrix.

We will call the matrix $\mathbf{\Sigma}$ as covariance matrix which is a generalisation of the variance (see below). For a real respectively complex valued random process, it is defined as (refer also to (A.29)):

$$\begin{aligned} \mathbf{\Sigma} &= (\underset{\sim}{\mathbf{x}} - \boldsymbol{\mu})\,(\underset{\sim}{\mathbf{x}} - \boldsymbol{\mu})^T \\ \underline{\mathbf{\Sigma}} &= (\underline{\underset{\sim}{\mathbf{x}}} - \underline{\boldsymbol{\mu}})\,(\underline{\underset{\sim}{\mathbf{x}}} - \underline{\boldsymbol{\mu}})^H \end{aligned} \tag{A.22}$$

A Gaussian multivariate process is assigned as:

$$\underset{\sim}{\mathbf{x}} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma}^2)$$

Its joint PDF is given by:

$$p_{\underset{\sim}{\mathbf{x}}}(\mathbf{x}) = \frac{1}{\sqrt{\det(2\pi\mathbf{\Sigma})}} \mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})} \tag{A.23}$$

- The PDF of the sum of two independent variables (for definition of independency see below) results from convolution of the individual PDFs:

$$\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y} \quad \Rightarrow \quad p_{\underset{\sim}{z}}(u) = p_{\underset{\sim}{x}}(u) * p_{\underset{\sim}{y}}(u) \tag{A.24}$$

Expected Value and Variance

Expected value:

$$\mu = \mathrm{E}\{\underset{\sim}{x}\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \underset{\sim}{x}[n] = \int_{-\infty}^{\infty} x\, p_{\underset{\sim}{x}}(x) dx \tag{A.25}$$

$$\boldsymbol{\mu} = \mathrm{E}\{\underset{\sim}{\mathbf{x}}\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \underset{\sim}{\mathbf{x}} = \int_{-\infty}^{\infty} \mathbf{x}\, p_{\underset{\sim}{\mathbf{x}}}(\mathbf{x}) d\mathbf{x} = \iint_{\infty} \cdots \int_{-\infty}^{\infty} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} p_{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots}(x_1, x_2, \cdots) dx_1 dx_2 \cdots \tag{A.26}$$

Expected value of function/linear function of sampled/empirical data:

$$\mathrm{E}\{f(\underset{\sim}{x})\} = \int_{-\infty}^{\infty} f(x)p_{\underset{\sim}{x}}(x)dx\mathrm{E}\{L(\underset{\sim}{x})\} = L(\mathrm{E}\{\underset{\sim}{x}\}) = L\left(\int_{-\infty}^{\infty} x\, p_{\underset{\sim}{x}}(x)dx\right) \quad \text{(A.27)}$$

Variance of sampled/empirical data:

$$\begin{aligned} \mathrm{var}\{\underset{\sim}{x}\} &= \sigma_x^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(\underset{\sim}{x}[n]-\mu)^2 = \mathrm{E}\left\{(\underset{\sim}{x}-\mathrm{E}\{\underset{\sim}{x}\})^2\right\} \\ &= \mathrm{E}\{\underset{\sim}{x}^2\} - (\mathrm{E}\{\underset{\sim}{x}\})^2 = \int_{-\infty}^{\infty}(x-\mu)^2 p_{\underset{\sim}{x}}(x)dx \end{aligned} \quad \text{(A.28)}$$

Covariance of sampled/empirical data:

$$\mathrm{cov}\{\underset{\sim}{x},\underset{\sim}{y}\} = \mathrm{E}\{(\underset{\sim}{x}-\mathrm{E}\{\underset{\sim}{x}\})(\underset{\sim}{y}-\mathrm{E}\{\underset{\sim}{y}\})\} \quad \text{(A.29)}$$

In the case of complex valued variables, variance and covariance are defined by:

$$\mathrm{var}\{\underline{\underset{\sim}{x}}\} = \mathrm{E}\{(\underline{\underset{\sim}{x}}-\mathrm{E}\{\underline{\underset{\sim}{x}}\})(\underline{\underset{\sim}{x}}-\mathrm{E}\{\underline{\underset{\sim}{x}}\})^*\} \quad \text{(A.30)}$$

$$\mathrm{cov}\{\underline{\underset{\sim}{x}},\underline{\underset{\sim}{y}}\} = \mathrm{E}\left\{(\underline{\underset{\sim}{x}}-\mathrm{E}\{\underline{\underset{\sim}{x}}\})(\underline{\underset{\sim}{y}}-\mathrm{E}\{\underline{\underset{\sim}{y}}\})^*\right\} \quad \text{(A.31)}$$

Some Rules

a and b are constant (deterministic) values.

$$\mathrm{E}\left\{(\underset{\sim}{x}-a)^2\right\} = \mathrm{var}\{\underset{\sim}{x}\} + (\mathrm{E}\{\underset{\sim}{x}\}-a)^2 \quad \text{(A.32)}$$

$$\mathrm{var}\{a+b\,\underset{\sim}{x}\} = b^2\mathrm{var}\{\underset{\sim}{x}\} \quad \text{(A.33)}$$

$$\mathrm{E}\{a\underset{\sim}{x}\pm b\underset{\sim}{y}\} = a\mathrm{E}\{\underset{\sim}{x}\}\pm b\mathrm{E}\{\underset{\sim}{y}\} \quad \text{(A.34)}$$

$$\mathrm{var}\{\underset{\sim}{x}\pm\underset{\sim}{y}\} = \mathrm{var}\{\underset{\sim}{x}\}+\mathrm{var}\{\underset{\sim}{y}\}\pm 2\mathrm{cov}\{\underset{\sim}{x},\underset{\sim}{y}\} \quad \text{(A.35)}$$

$$E\{\underset{\sim}{x}\cdot\underset{\sim}{y}\} = E\{\underset{\sim}{x}\}\cdot E\{\underset{\sim}{y}\}+\mathrm{cov}\{\underset{\sim}{x},\underset{\sim}{y}\} \quad \text{(A.36)}$$

$$\begin{aligned} &\mathrm{var}\{\underset{\sim}{x}\cdot\underset{\sim}{y}\} = \mathrm{var}\{\underset{\sim}{x}\}\cdot\mathrm{var}\{\underset{\sim}{y}\}+\mathrm{var}\{\underset{\sim}{x}\}\cdot(E\{\underset{\sim}{y}\})^2+\mathrm{var}\{\underset{\sim}{y}\}\cdot(E\{\underset{\sim}{x}\})^2 \\ &\qquad\qquad \text{if } \underset{\sim}{x} \text{ and } \underset{\sim}{y} \text{ are independent} \end{aligned} \quad \text{(A.37)}$$

$$\mathrm{var}\{\underset{\sim}{x}^2\} = E\{\underset{\sim}{x}^4\}-\left(E\{\underset{\sim}{x}^2\}\right)^2 = 2\sigma_x^2 \quad \text{for } \underset{\sim}{x}\sim N\left(0,\sigma_x^2\right), \quad \text{(A.38)}$$

since

$$E\{\underset{\sim}{x}^2\} = \sigma_x^2 \quad \text{for } \underset{\sim}{x}\sim N\left(0,\sigma_x^2\right) \text{ (results from (A.28), (A.21) and Table A-1)},$$

and from chapter 7.1 and (A.21), we get:

$$E\{\underset{\sim}{x}^4\} = \int_{-\infty}^{\infty} x^4 p_{\underset{\sim}{x}}(x)dx = 3\sigma_x^2 \quad \text{for } \underset{\sim}{x}\sim N\left(0,\sigma_x^2\right).$$

Definitions
Two random variables are called

- uncorrelated if: $\text{cov}\{\underset{\sim}{x}, \underset{\sim}{y}\} = 0$
- orthogonal if: $\text{E}\{\underset{\sim}{x}\,\underset{\sim}{y}\} = 0$, and
- independent if: $p_{\underset{\sim}{x},\underset{\sim}{y}}(u, v) = p_{\underset{\sim}{x}}(u) \cdot p_{\underset{\sim}{y}}(v)$. (Note if two processes are independent then they are also uncorrelated.)

We will call a process ergodic if the ensemble means equal the temporal means, i.e.

$$\bar{x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt = \text{E}\{\underset{\sim}{x}\} = \mu \tag{A.39}$$

$$x_{rms}^2 - \bar{x}^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (x(t) - \bar{x})^2 dt = \text{E}\left\{(\underset{\sim}{x} - \text{E}\{\underset{\sim}{x}\})^2\right\} = \text{var}\left\{\underset{\sim}{x}\right\} = \sigma_x^2 \tag{A.40}$$

A.4
Coordinate Systems

We define position and orientation of a rigid body in space by the location of an appropriately chosen reference point – expressed by the position vector **r** - and a set of angles referred to appropriately chosen reference directions of a local coordinate system (see Figure A.1).

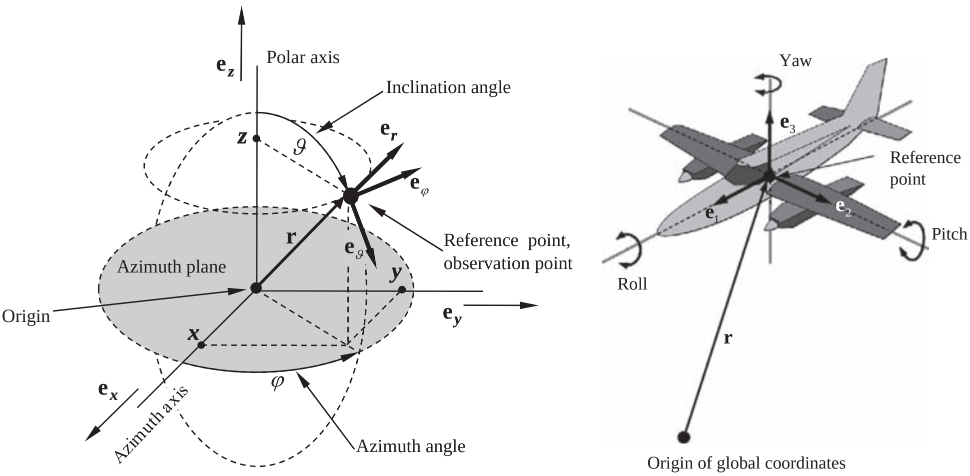


Figure A.1 Assigning the position of an object in space by Cartesian $[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$ or spherical coordinate system $[\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\varphi]$ (left) and its orientation by a set of angles (right) referred to a local coordinate system $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$. Note, that the spherical system used here is based on the inclination angle ϑ starting from the zenith. A different approach is to apply the elevation angle which starts at the azimuth plane.

$[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$, $[\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\varphi]$, $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ represent a set of unit direction vectors of an orthogonal right hand system, i.e. we have (see also annex 7.5):

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{e}_j &= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{with} \quad i,j = x,y,z \quad \text{or} \quad i,j = r,\vartheta,\varphi \quad \text{or} \quad i,j = 1,2,3 \\ \mathbf{e}_x \times \mathbf{e}_y &= \mathbf{e}_z \quad \text{and} \quad \mathbf{e}_r \times \mathbf{e}_\vartheta = \mathbf{e}_\varphi \quad \text{and} \quad \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \end{aligned} \tag{A.41}$$

Position vector:

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = r\mathbf{e}_r \tag{A.42}$$

In Cartesian coordinates, we also often find the notation:

$$\mathbf{r} = [x \quad y \quad z]^T$$

Conversion between Cartesian and spherical coordinates:

$$\text{Cartesian to spherical}: \quad r = \sqrt{x^2 + y^2 + z^2}; \quad \vartheta = \arctan\frac{\sqrt{x^2+y^2}}{z}; \quad \varphi = \arctan\frac{y}{x} \tag{A.43}$$

$$\text{Spherical to Cartesian}: \quad x = r\sin\vartheta\cos\varphi; \quad y = r\sin\vartheta\sin\varphi; \quad z = r\cos\vartheta \tag{A.44}$$

Table A-2 Conversion between unit vectors in Cartesian and spherical coordinates.

	$\mathbf{e}_x$	$\mathbf{e}_y$	$\mathbf{e}_z$
$\mathbf{e}_r$	$\sin\vartheta\cos\varphi$	$\sin\vartheta\sin\varphi$	$\cos\vartheta$
$\mathbf{e}_\vartheta$	$\cos\vartheta\cos\varphi$	$\cos\vartheta\sin\varphi$	$-\sin\vartheta$
$\mathbf{e}_\varphi$	$-\sin\varphi$	$\cos\varphi$	0

Examples:

$$\begin{aligned} \mathbf{e}_y &= \sin\vartheta\sin\varphi\,\mathbf{e}_r + \cos\vartheta\sin\varphi\,\mathbf{e}_\vartheta + \cos\varphi\,\mathbf{e}_\varphi \\ \mathbf{e}_r &= \sin\vartheta\cos\varphi\,\mathbf{e}_x + \sin\vartheta\sin\varphi\,\mathbf{e}_y + \cos\vartheta\,\mathbf{e}_z \\ \mathbf{r} &= r\mathbf{e}_r = r\sin\vartheta\cos\varphi\,\mathbf{e}_x + r\sin\vartheta\sin\varphi\,\mathbf{e}_y + r\cos\vartheta\,\mathbf{e}_z = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z \end{aligned}$$

Orientation and Rotation:

The orientation of a body is often given by three angles indicating the rotation of the local coordinates $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ with respect to the global coordinate system $[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$. These angles are usually referred as Euler angles. There are 12 possibilities to define these angles. Figure A.2 shows two options of them.

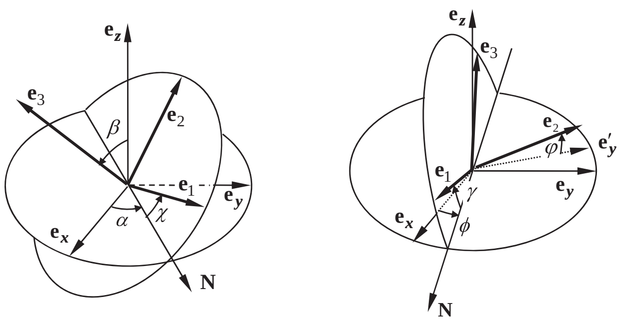


Figure A.2 Orientation of a rigid body in space by Euler (left) and nautical angles (right).

We consider two examples of Euler angle applications. In the first one, we like to express the vector $\mathbf{r}$ in coordinates of the system A defined by the unit vectors $\left[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\right]$ as well as in coordinates of a system B based on $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$:

$$\mathbf{r} = r_x\,\mathbf{e}_x + r_y\,\mathbf{e}_y + r_z\,\mathbf{e}_z = r_1\,\mathbf{e}_1 + r_2\,\mathbf{e}_2 + r_3\,\mathbf{e}_3 = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}_A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}_B .$$

The angles between the both coordinate systems (Figure A.2) are supposed to be known.

In the second example we keep the coordinate system but we like to rotate a vector $\mathbf{a}$ by a given set of angles resulting in a new vector $\mathbf{b}$. Both problems can be solved by the same operations where at we have 12 different options from which we will demonstrate two here.

Option 1 refers to rotations corresponding to Figure A.2 – left (z-x-z convention):

1) rotate about the z-axis by angle α
2) rotate about the line of nodes $\mathbf{N}$ (new x-axis) by angle β, and
3) rotate about the (new) z-axis ($\mathbf{e}_3$) by χ.

These three angles are usually assigned as the actual Euler angles in narrower sense. Our two examples involves following relations:

$$\left.\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}\right|_B = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{\vartheta}_{eu}(\alpha,\beta,\chi)} \left.\begin{bmatrix} r_x \\ r_y \\ r_y \end{bmatrix}\right|_A$$

In compressed notation, this takes the forms

$$\mathbf{r}|_B = \boldsymbol{\vartheta}_z(\alpha)\,\boldsymbol{\vartheta}_x(\beta)\,\boldsymbol{\vartheta}_z(\chi)\,\mathbf{r}|_A = \boldsymbol{\vartheta}_{eu}(\alpha,\beta,\chi)\,\mathbf{r}|_A \tag{A.45}$$

$$\mathbf{b} = \boldsymbol{\vartheta}_{eu}(\alpha,\beta,\chi)\,\mathbf{a} \tag{A.46}$$

Option 2 usually applied in nautics (see also Figure A.1) performs rotation corresponding to Figure A.2 right (z-x-y convention):

1) rotate about the z-axis by angle ϕ (yaw angle),
2) rotate about the (new) x-axis (line of nodes **N**) by angle φ (roll angle), and
3) rotate about the (new) y-axis by angle γ (pitch angle).

In this case the rotation matrix is given by:

$$\begin{aligned}\boldsymbol{\vartheta}_{na}(\phi,\varphi,\gamma) &= \begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\varphi & -\sin\varphi\\ 0 & \sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}\cos\gamma & 0 & \sin\gamma\\ 0 & 1 & 0\\ -\sin\gamma & 0 & \cos\gamma\end{bmatrix}\\ &= \boldsymbol{\vartheta}_z(\phi)\,\boldsymbol{\vartheta}_z(\varphi)\,\boldsymbol{\vartheta}_y(\gamma)\end{aligned} \tag{A.47}$$

For the properties of rotation matrices see Annex A.6.

Angle between two vectors:

$$\cos\gamma = \frac{\mathbf{r}_1\cdot\mathbf{r}_2}{r_1\cdot r_2} = \cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\cos(\varphi_1-\varphi_2) \tag{A.48}$$

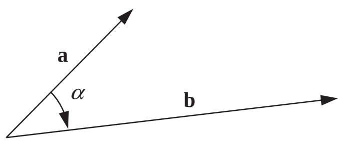
A.5
Some Vector Operations and useful Identities

Representation of three-dimensional vectors:

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z = \begin{bmatrix}a_x\\ a_y\\ a_z\end{bmatrix};\quad \mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z = \begin{bmatrix}b_x\\ b_y\\ b_z\end{bmatrix}$$

Dot product; inner product; scalar product:

$$\underbrace{\mathbf{a}\cdot\mathbf{b}}_{\text{vector notation}} = \underbrace{\mathbf{a}^T\mathbf{b}}_{\text{matrix notation}} = a_xb_x + a_yb_y + a_zb_z = \begin{bmatrix}a_x & a_y & a_z\end{bmatrix}\begin{bmatrix}b_x\\ b_y\\ b_z\end{bmatrix} = a\,b\cos\alpha \tag{A.49}$$



Outer product; dyadic product:

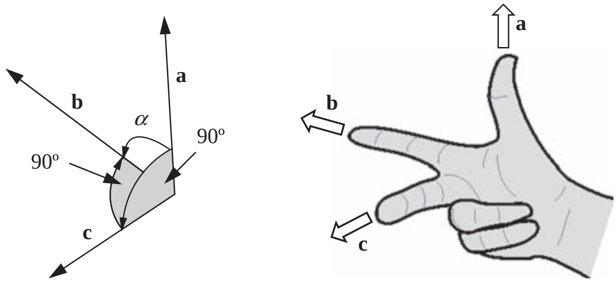
$$\underbrace{\mathbf{a}\otimes\mathbf{b}}_{\text{vector notation}} = \underbrace{\mathbf{a}\,\mathbf{b}^T}_{\text{matrix notation}} = \begin{bmatrix}\mathbf{a}b_x & \mathbf{a}b_y & \mathbf{a}b_z\end{bmatrix} = \begin{bmatrix}a_xb_x & a_xb_y & a_xb_z\\ a_yb_x & a_yb_y & a_yb_z\\ a_zb_x & a_zb_y & a_zb_z\end{bmatrix} \tag{A.50}$$

Cross-product; vector product; exterior product:

$$\mathbf{c} = \underbrace{\mathbf{a}\times\mathbf{b}}_{\text{vector notation}} = \underbrace{-\mathbf{b}\times\mathbf{a} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \mathbf{A}_\times\,\mathbf{b}}_{\text{matrix notation}}$$

$$= \underbrace{\begin{Vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{Vmatrix}}_{\text{determinate}} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} = a\,b\,\sin\alpha\,\mathbf{e} \tag{A.51}$$

The vector $\mathbf{c} = c\,\mathbf{e}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ corresponding to the right-hand rule.



For properties of the cross-product matrix see Annex A.6.

Algebraic Identities

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{A.52}$$

Special case : $\mathbf{e}\times(\mathbf{e}\times\mathbf{a}) = \mathbf{e}(\mathbf{e}\cdot\mathbf{a}) - \mathbf{a} = \left(\mathbf{e}\,\mathbf{e}^T - \mathbf{I}\right)\mathbf{a} = \mathbf{E}_\times^2\,\mathbf{a}$

$\mathbf{e}$-unit vector; for $\mathbf{E}_x$ see (A.164)-(A.167)

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} \tag{A.53}$$

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]\mathbf{c} - [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\mathbf{d} \tag{A.54}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{A.55}$$

Special case ($\mathbf{c} = \mathbf{a}$; $\mathbf{d} = \mathbf{b}$) : $a^2\,b^2 = |\mathbf{a}\cdot\mathbf{b}|^2 + |\mathbf{a}\times\mathbf{b}|^2$

Vector decomposition in perpendicular and parallel component ($\mathbf{a}_\parallel$ -parallel to $\mathbf{e}$; $\mathbf{a}_\perp$- perpendicular to $\mathbf{e}$):

$$\mathbf{a} = \underbrace{(\mathbf{e}\cdot\mathbf{a})\mathbf{e}}_{\mathbf{a}_\parallel} - \underbrace{\mathbf{e}\times(\mathbf{e}\times\mathbf{a})}_{\mathbf{a}_\perp} = \underbrace{\mathbf{e}\mathbf{e}^T\mathbf{a}}_{\mathbf{a}_\parallel} - \underbrace{\mathbf{E}_\times{}^2\mathbf{a}}_{\mathbf{a}_\perp} \tag{A.56}$$

Differential operators

Cartesian coordinates:

$$\text{Scalar field}: \quad \Phi = \Phi(x, y, z)$$

$$\text{Vector field}: \quad \mathbf{A} = A_x(x,y,z)\,\mathbf{e}_x + A_y(x,y,z)\,\mathbf{e}_y + A_z(x,y,z)\,\mathbf{e}_z = \begin{bmatrix} A_x(x,y,z) \\ A_y(x,y,z) \\ A_z(x,y,z) \end{bmatrix}$$

$$\text{Gradient}: \quad \nabla\Phi = \begin{bmatrix} \partial\Phi/\partial x \\ \partial\Phi/\partial y \\ \partial\Phi/\partial z \end{bmatrix} = \frac{\partial\Phi}{\partial x}\mathbf{e}_x + \frac{\partial\Phi}{\partial y}\mathbf{e}_y + \frac{\partial\Phi}{\partial z}\mathbf{e}_z \tag{A.57}$$

$$\text{Laplacian}: \quad \Delta\Phi = \nabla\cdot\nabla\Phi = \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \tag{A.58}$$

$$\text{Divergence}: \quad \nabla\cdot\mathbf{A} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{A.59}$$

$$\text{Curl (rotor)}: \quad \begin{aligned} \nabla\times\mathbf{A} &= \begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & -\partial/\partial x \\ -\partial/\partial y & \partial/\partial x & 0 \end{bmatrix} \cdot \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \\ &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{e}_z \end{aligned} \tag{A.60}$$

$$\begin{aligned} \nabla\times(\nabla\times\mathbf{A}) &= \begin{bmatrix} -\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial x\partial z} \\ \frac{\partial^2}{\partial x\partial y} & -\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y\partial z} \\ \frac{\partial^2}{\partial x\partial z} & \frac{\partial^2}{\partial y\partial z} & -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \end{bmatrix} \cdot \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \\ &= \left(-\frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z}\right)\mathbf{e}_x \\ &\quad + \left(\frac{\partial^2 A_x}{\partial x\partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_z}{\partial y\partial z}\right)\mathbf{e}_y \\ &\quad + \left(\frac{\partial^2 A_x}{\partial x\partial z} + \frac{\partial^2 A_y}{\partial y\partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2}\right)\mathbf{e}_z \end{aligned} \tag{A.61}$$

Spherical coordinates:

$$\text{Scalar field}: \quad \Phi = \Phi(r, \vartheta, \varphi)$$

$$\text{Vector field}: \quad \mathbf{A} = A_r(r,\vartheta,\varphi)\,\mathbf{e}_r + A_\vartheta(r,\vartheta,\varphi)\,\mathbf{e}_\vartheta + A_\varphi(r,\vartheta,\varphi)\,\mathbf{e}_\varphi$$

$$\text{Gradient}: \quad \nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\mathbf{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\Phi}{\partial\varphi}\mathbf{e}_\varphi \tag{A.62}$$

Laplacian : $$\Delta\Phi = \nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\varphi^2} \tag{A.63}$$

Divergence : $$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta A_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi} \tag{A.64}$$

Curl (rotor) : $$\begin{aligned}\nabla\times\mathbf{A} &= \frac{1}{r\sin\vartheta}\left(\frac{\partial\left(\sin\vartheta A_\varphi\right)}{\partial\vartheta} - \frac{\partial A_\vartheta}{\partial\varphi}\right)\mathbf{e}_r \\ &\quad + \frac{1}{r}\left(\frac{1}{\sin\vartheta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial\left(rA_\varphi\right)}{\partial r}\right)\mathbf{e}_\vartheta + \frac{1}{r}\left(\frac{\partial(rA_\vartheta)}{\partial r} - \frac{\partial A_r}{\partial\vartheta}\right)\mathbf{e}_\varphi\end{aligned} \tag{A.65}$$

$$\begin{aligned}(\mathrm{d}l)^2 &= dr^2 + r^2d\vartheta^2 + r^2\sin^2 d\varphi^2 \\ \mathrm{d}\mathbf{S} &= r^2\sin\vartheta\, d\vartheta\, d\varphi\,\mathbf{e}_r \\ \mathrm{d}\Omega &= \frac{\mathbf{e}_r\cdot\mathrm{d}\mathbf{S}}{r} = \sin\vartheta\, d\vartheta\, d\varphi \\ \mathrm{d}V &= r^2\sin\vartheta\, dr\, d\vartheta\, d\varphi\end{aligned} \tag{A.66}$$

Differential Identities

$$\nabla\times(\nabla\Phi) = \mathbf{0} \tag{A.67}$$

$$\nabla\cdot(\Phi\nabla\Psi) = \Phi\Delta\Psi + \nabla\Phi\cdot\nabla\Psi \tag{A.68}$$

$$\nabla\cdot(\Phi\nabla\Psi - \Psi\nabla\Phi) = \Phi\Delta\Psi - \Psi\Delta\Phi \tag{A.69}$$

$$\nabla\cdot(\Phi\mathbf{A}) = (\nabla\Phi)\cdot\mathbf{A} + \Phi\nabla\cdot\mathbf{A} \tag{A.70}$$

$$\nabla\times(\Phi\mathbf{A}) = (\nabla\Phi)\times\mathbf{A} + \Phi\nabla\times\mathbf{A} \tag{A.71}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \tag{A.72}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \tag{A.73}$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A} \tag{A.74}$$

We applied here $\nabla\cdot\nabla = \Delta$ which is also often expressed by $\nabla\cdot\nabla = \nabla^2$ (Laplacian).

Derivations of the position vector:

$$\mathbf{r} = r\mathbf{e}_r = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z \quad\Rightarrow\quad r = \sqrt{x + y^2 + z^2};\quad \mathbf{e}_r = \mathbf{r}/r$$

$$\nabla r = \mathbf{e}_r \tag{A.75}$$

$$\nabla r^2 = 2\mathbf{r} \tag{A.76}$$

$$\nabla r^{-1} = -\frac{\mathbf{e}_r}{r^2} \tag{A.77}$$

$$\nabla r^{-2} = -\frac{2\,\mathbf{e}_r}{r^3} \tag{A.78}$$

$$\nabla\cdot\mathbf{r} = 3 \tag{A.79}$$

$$\nabla\cdot\mathbf{e}_r = \frac{2}{r} \tag{A.80}$$

$$\nabla\times\mathbf{r} = \mathbf{0} \tag{A.81}$$

A.6
Some Matrix Operations and useful Identities

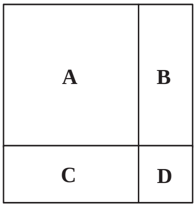
Some Matrix Inversion Identities

$$
\begin{aligned}
(\mathbf{I}+\mathbf{A})^{-1} &= (\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A}-\mathbf{A}) = \mathbf{I}-(\mathbf{I}+\mathbf{A})^{-1}\mathbf{A} \\
&= (\mathbf{I}+\mathbf{A}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} = \mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{A})^{-1}
\end{aligned}
\tag{A.82}
$$

$$
\begin{aligned}
&\mathbf{A}+\mathbf{ABA} = \mathbf{A}(\mathbf{I}+\mathbf{BA}) = (\mathbf{I}+\mathbf{AB})\mathbf{A} \\
&(\mathbf{I}+\mathbf{BA})^{-1}\mathbf{A} = \mathbf{A}(\mathbf{I}+\mathbf{AB})^{-1}
\end{aligned}
\tag{A.83}
$$

Inversion of a matrix block (Frobenius/Schur/Woodbury identity):
Consider the system of linear equations:

$$
\begin{aligned}
\mathbf{A}\,\mathbf{x}_1 + \mathbf{B}\,\mathbf{x}_2 &= \mathbf{y}_1 \\
\mathbf{C}\,\mathbf{x}_1 + \mathbf{D}\,\mathbf{x}_2 &= \mathbf{y}_2
\end{aligned}
\tag{A.84}
$$

$\mathbf{M} =$ 

in which $\mathbf{x}_i$, $\mathbf{y}_j$ are column vectors and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are matrices of appropriate dimensions ($\mathbf{A}$ and $\mathbf{D}$ must be square matrices). We can compress (A.84) in matrix block form:

$$\mathbf{M}\,\mathbf{X} = \mathbf{Y}$$

$$
\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}; \quad Y = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_1 \end{bmatrix}
\tag{A.85}
$$

In order to solve (A.85) for $\mathbf{X}$ we perform a block Gaussian elimination to reduce dimensionality of matrix inversion leading to:

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{S}_D & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1^{(r)} \\ \mathbf{y}_2 \end{bmatrix}; \quad \begin{aligned} \mathbf{S}_D &= \mathbf{A} - \mathbf{B}\,\mathbf{D}^{-1}\,\mathbf{C} \\ \mathbf{y}_1^{(r)} &= \mathbf{y}_1 - \mathbf{B}\,\mathbf{D}^{-1}\,\mathbf{y}_2 \end{aligned} \\
&\qquad\qquad \text{or} \\
&\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{S}_A \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2^{(r)} \end{bmatrix}; \quad \begin{aligned} S_A &= \mathbf{D} - \mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{B} \\ \mathbf{y}_2^{(r)} &= \mathbf{y}^2 - \mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{y}_1 \end{aligned}
\end{aligned}
\tag{A.86}
$$

Herein $\mathbf{S}_D$ and $\mathbf{S}_A$ are called the Schur complements of the block $\mathbf{D}$ and $\mathbf{A}$ respectively of the matrix $\mathbf{M}$. Taking e.g. the upper version of (A.86), one solves the equation first for $\mathbf{x}_1$ supposing the inverse matrices of $\mathbf{S}_D$ and $\mathbf{D}$ exist. Then one solves for $\mathbf{x}_2$. For the inverse of matrix $\mathbf{M}$, it is easy to show that

$$
\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{S}_D^{-1} & -\mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{S}_A^{-1} \\ -\mathbf{D}^{-1}\,\mathbf{C}\,\mathbf{S}_D^{-1} & \mathbf{S}_A^{-1} \end{bmatrix}
\tag{A.87}
$$

Using matrix inversion lemma:

$$
(\mathbf{A}+\mathbf{B}\,\mathbf{D}\,\mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\,\mathbf{B}\left(\mathbf{D}^{-1}+\mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{B}\right)^{-1}\mathbf{C}\,\mathbf{A}^{-1}
\tag{A.88}
$$

the inverse of Schur's complements gets:

$$\begin{aligned} \mathbf{S}_D^{-1} &= \left(\mathbf{A} - \mathbf{B}\,\mathbf{D}^{-1}\,\mathbf{C}\right)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{S}_A^{-1}\,\mathbf{C}\,\mathbf{A}^{-1} \\ \mathbf{S}_A^{-1} &= \left(\mathbf{D} - \mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{B}\right)^{-1} = \mathbf{D}^{-1} + \mathbf{D}^{-1}\,\mathbf{C}\,\mathbf{S}_D^{-1}\,\mathbf{B}\,\mathbf{D}^{-1} \end{aligned} \tag{A.89}$$

so that finally the inverse of **M** can be expressed as:

$$\begin{aligned} \mathbf{M}^{-1} &= \begin{bmatrix} \mathbf{A}^{-1}\left(\mathbf{I} + \mathbf{B}\,\mathbf{S}_A^{-1}\,\mathbf{C}\,\mathbf{A}^{-1}\right) & -\mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{S}_A^{-1} \\ -\mathbf{S}_A^{-1}\,\mathbf{C}\,\mathbf{A}^{-1} & \mathbf{S}_A^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{S}_D^{-1} & -\mathbf{S}_D^{-1}\,\mathbf{B}\,\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\,\mathbf{C}\,\mathbf{S}_D^{-1} & \mathbf{D}^{-1}\left(\mathbf{I} + \mathbf{C}\,\mathbf{S}_D^{-1}\,\mathbf{B}\,\mathbf{D}^{-1}\right) \end{bmatrix} \end{aligned} \tag{A.90}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{v} \\ \mathbf{u}^H & \alpha \end{bmatrix}$$

For the special case where **A** is a square matrix of dimension $[N, N]$, $\mathbf{B} = \mathbf{v}$ is a $[N, 1]$column vector, $\mathbf{C} = \mathbf{u}^H$ is a $[1, N]$ row vector and $\mathbf{D} = \alpha$ is a scalar, the matrix inversion lemma reads:

$$\left(\mathbf{A} + \mathbf{v}\,\mathbf{u}^H\right)^{-1} = \mathbf{A}^{-1} - \frac{\left(\mathbf{A}^{-1}\,\mathbf{v}\right)\left(\mathbf{u}^H\,\mathbf{A}^{-1}\right)}{\alpha^{-1} + \mathbf{u}^H\,\mathbf{A}^{-1}\,\mathbf{v}} \tag{A.91}$$

and the inverse of **M** gets:

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \dfrac{\left(\mathbf{A}^{-1}\,\mathbf{v}\right)\left(\mathbf{u}^H\,\mathbf{A}^{-1}\right)}{\alpha - \mathbf{u}^H\,\mathbf{A}^{-1}\,\mathbf{v}} & -\dfrac{\mathbf{A}^{-1}\,\mathbf{v}}{\alpha - \mathbf{u}^H\,\mathbf{A}^{-1}\,\mathbf{v}} \\ -\dfrac{\mathbf{u}^H\,\mathbf{A}^{-1}}{\alpha - \mathbf{u}^H\,\mathbf{A}^{-1}\,\mathbf{v}} & \dfrac{1}{\alpha - \mathbf{u}^H\,\mathbf{A}^{-1}\,\mathbf{v}} \end{bmatrix} \tag{A.92}$$

Least Squares Normal Equation:

Representing a (real or complex valued) sequence as column vector and consider the approximation (e.g. a signal model) $\hat{\mathbf{y}} = [\hat{y}_1 \;\; \hat{y}_2 \;\; \cdots \;\; \hat{y}_n \;\; \cdots \;\; \hat{y}_N]^T$ of the sequence (e.g. captured data samples) $\mathbf{y} = [y_1 \;\; y_2 \;\; \cdots \;\; y_n \;\; \cdots \;\; y_N]^T$ by a linear combination of the M (with $M < N$) sequences $\mathbf{x}_m = [x_{m1} \;\; x_{m2} \;\; \cdots \;\; x_{mn} \;\; \cdots x_{mN}]^T; \quad m \in [1, M]$:

$$\hat{y}[n] = \sum_{m=1}^{M} a_m x_{mn} \quad \Rightarrow \quad \hat{\mathbf{y}} = \mathbf{X}\,\mathbf{\Theta}$$

$$\text{with} \quad \mathbf{\Theta} = [a_1 \;\; a_2 \;\; \cdots \;\; a_m \;\; \cdots \;\; a_M]; \quad \mathbf{X} = [\mathbf{x}_1 \;\; \mathbf{x}_2 \;\; \cdots \;\; \mathbf{x}_m \;\; \cdots \;\; \mathbf{x}_M] \tag{A.93}$$

where at $\mathbf{\Theta}$ is the so called parameter vector. (A.93) represents an overdetermined set of linear equations.

The approximation error $\mathbf{e} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n & \cdots & e_N \end{bmatrix}^T$ is:

$$\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \mathbf{X}\boldsymbol{\Theta} \tag{A.94}$$

The approximation error depends on the selection of the parameters. The goal is to find a parameter vector $\boldsymbol{\Theta}$ which minimises the L_2-norm of the approximation error

$$\boldsymbol{\Theta} = \arg\min_{\boldsymbol{\Theta}} \|\mathbf{e}(\boldsymbol{\Theta})\|_2 \tag{A.95}$$

respectively

$$\frac{d}{d\boldsymbol{\Theta}}\left(\mathbf{e}^H(\boldsymbol{\Theta})\,\mathbf{e}(\boldsymbol{\Theta})\right) = 0 \tag{A.96}$$

which solves in:

$$\boldsymbol{\Theta} = \left(\mathbf{X}^H\mathbf{X}\right)^{-1}\mathbf{X}^H\mathbf{y} \tag{A.97}$$

The resulting approximation error is:

$$\|\mathbf{e}(\boldsymbol{\Theta})\|_2^2 = \mathbf{e}^H(\boldsymbol{\Theta})\,\mathbf{e}(\boldsymbol{\Theta}) = \mathbf{y}^H(\mathbf{y} - \mathbf{X}\boldsymbol{\Theta}) \tag{A.98}$$

The least squares is the simplest and most common method of linear regression.

Eigenvalue Decomposition:

Let $\mathbf{A}$ be a [MxM] square matrix and $\mathbf{x}$ a column vector of length M, then the scalar values λ which meet (A.99) are called the Eigenvalues of $\mathbf{A}$:

$$\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x} \tag{A.99}$$

The Eigenvalues are the solutions of the characteristic polynomial:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{A.100}$$

Eigenvalue decomposition factorises a square matrix by following transformation:

$$\mathbf{A} = \mathbf{Q}\,\boldsymbol{\Lambda}\,\mathbf{Q}^{-1} \tag{A.101}$$

$$\text{with} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_M \end{bmatrix}; \quad \mathbf{Q} = \begin{bmatrix} \mathbf{x}_{q1} & \mathbf{x}_{q2} & \cdots & \mathbf{x}_{qM} \end{bmatrix}$$

$\mathbf{x}_{qm}$ is the Eigenvector which solves (A.99) for the Eigenvalue λ_m. The Eigenvectors are often normalised $\left\|\mathbf{x}_{qm}\right\|_2 = 1$ but they need not to be. The Eigenvectors are linearly independent (i.e. $\det\mathbf{Q} \neq 0$). Eigenvalue decomposition may be used to diagonalise a matrix. Note, not all square matrices may be diagonalised.

Properties:

$$\text{Trace}: \quad \mathrm{tr}(\mathbf{A}) = \sum \lambda_m \tag{A.102}$$

$$\text{Determinate}: \quad \det(\mathbf{A}) = \prod \lambda_m \tag{A.103}$$

$$\text{Matrix power}: \quad \mathbf{A}^n = \mathbf{Q}\,\mathbf{\Lambda}^n\,\mathbf{Q}^{-1}; \quad n \in \mathbb{Z} \tag{A.104}$$

Singular Value Decomposition:
Let $\mathbf{A}$ be an [MxN] matrix. The factorisation

$$\mathbf{A} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H \tag{A.105}$$

is called singular value decomposition of $\mathbf{A}$, with

$\mathbf{U}\,\mathbf{U}^H = \mathbf{I}$-unitary[MxM] − matrix
$\mathbf{V}\,\mathbf{V}^H = \mathbf{I}$-unitary[NxN]-matrix
Σ − [MxN] diagonal matrix of the so called singular values

$$\text{e.g. if } M > N: \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}; \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N$$

Singular vectors:

$$\mathbf{A}\,\mathbf{v}_n = \sigma_n \mathbf{u}_n; \quad \mathbf{A}^H\,\mathbf{u}_n = \sigma_n \mathbf{v}_n \tag{A.106}$$

Left-singular vector : $\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n \quad \cdots \quad \mathbf{u}_M]$
Right-singular vector : $\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n \quad \cdots \quad \mathbf{v}_N]$

Relation to Eigenvalue decomposition:

$$\begin{aligned} \mathbf{A}^H\mathbf{A} &= \left(\mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H\right)^H \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H = \mathbf{V}\mathbf{\Sigma}^H\mathbf{\Sigma}\,\mathbf{V}^H = \mathbf{V}\mathbf{\Sigma}^H\mathbf{\Sigma}\,\mathbf{V}^{-1} \\ \mathbf{A}\,\mathbf{A}^H &= \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H\left(\mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H\right)^H = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{\Sigma}^H\mathbf{U}^H = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{\Sigma}^H\mathbf{U}^{-1} \end{aligned} \tag{A.107}$$

- $\mathbf{V}$ is composed from the Eigenvectors of the matrix $\mathbf{A}^H\mathbf{A}$
- $\mathbf{U}$ is composed from the Eigenvectors of the matrix $\mathbf{A}\,\mathbf{A}^H$
- Non-zero singular values and Eigenvalues of $\mathbf{A}^H\mathbf{A}$ or $\mathbf{A}\,\mathbf{A}^H$ relates by $\sigma_n = \sqrt{\lambda_n}$.

Pseudo-Inverse (Moore Penrose Inverse)

- Using singular value decomposition, the pseudo inverse of matrix $\mathbf{A}^+$ is given by:

$$\mathbf{A}^+ = \mathbf{V}\,\mathbf{\Sigma}^+\,\mathbf{U}^H \tag{A.108}$$

using

$$\mathbf{A} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H; \quad \mathbf{\Sigma}^+ = \begin{bmatrix} \sigma_1^{-1} & 0 & \cdots & 0 \\ 0 & \sigma_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

- Explicit solution for $M \geq N$ ($\mathbf{A}^+$ is left inverse of $\mathbf{A}$; $\mathbf{I}$ - [NxN] – matrix):

$$\mathbf{A}^+ = \left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H \quad \Rightarrow \quad \mathbf{A}^+\mathbf{A} = \mathbf{I} \tag{A.109}$$

It is related to the least square solution (i.e. minimum L_2-norm of the error) of a set of overdetermined equations (compare (A.97)).

- Explicit solution for $M \leq N$ ($\mathbf{A}^+$ is right inverse of $\mathbf{A}$; $\mathbf{I}$ - [MxM] – matrix):

$$\mathbf{A}^+ = \mathbf{A}^H\left(\mathbf{A}\,\mathbf{A}^H\right)^{-1} \quad \Rightarrow \quad \mathbf{A}\,\mathbf{A}^+ = \mathbf{I} \tag{A.110}$$

It represents the solution of a set of underdetermined equations which minimises the L_2-norm of the solution vector.

Some Special Matrices

Identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{A.111}$$

$$\begin{aligned} \mathbf{I}\,\mathbf{A} &= \mathbf{A}\,\mathbf{I} = \mathbf{A} \\ \det \mathbf{I} &= 1 \end{aligned} \tag{A.112}$$

Reflection matrix:

$$\mathbf{J} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & ⋰ & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \tag{A.113}$$

$$\begin{aligned} \mathbf{J} &= \mathbf{J}^T \\ \mathbf{J} &= \mathbf{J}^{-1}; \quad \mathbf{J}\,\mathbf{J} = \mathbf{I} \\ \mathbf{J}^n &= \begin{cases} \mathbf{I} & n \text{ even} \\ \mathbf{J} & n \text{ odd} \end{cases} \\ \operatorname{tr} \mathbf{J} &= \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \\ \det \mathbf{J} &= \pm 1 \end{aligned} \tag{A.114}$$

Eigenvalues $\quad \lambda_n = \pm 1$

(Assuming $\mathbf{J}$ is a $[N, N]$ matrix, half of the Eigenvalues are $+1$ and the other half is -1 if N is even. If N is odd, the number of negative Eigenvalues is one less than the number of positive ones.)

Reverses ordering of rows:

$$\mathbf{J}\,\mathbf{A} = \begin{bmatrix} a_{M1} & a_{M2} & \cdots & a_{MN} \\ \vdots & \vdots & \ddots & \vdots \\ a_{21} & a_{22} & \cdots & a_{2N} \\ a_{11} & a_{12} & \cdots & a_{1N} \end{bmatrix} \tag{A.115}$$

Reverses ordering of columns in matrix **A**:

$$\mathbf{A}\,\mathbf{J} = \begin{bmatrix} a_{1N} & \cdots & a_{12} & a_{11} \\ a_{2N} & \cdots & a_{22} & a_{21} \\ \vdots & \ddots & \vdots & \vdots \\ a_{Mn} & \cdots & a_{M2} & a_{M1} \end{bmatrix} \tag{A.116}$$

Correspondingly, $\mathbf{J}\,\mathbf{x}$ will reverse the element order of the column vector $\mathbf{x}$ and $\mathbf{x}^T\mathbf{J}$ is doing the same with the row vector $\mathbf{x}^T$.

Shift matrix (see also circulant matrix):
Shifts cyclically columns or rows of a matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{A.117}$$

$$\mathbf{P}^T\,\mathbf{P} = \mathbf{P}\,\mathbf{P}^T = \mathbf{I} \tag{A.118}$$

$$\det \mathbf{P} = 1 \tag{A.119}$$

$$\text{Eigenvalues} \quad \lambda_n = e^{j\frac{n}{N}}; \quad n \in [0, N-1]$$

$$\begin{array}{ll} \text{Shift } n \text{ columns left :} & \mathbf{A}\,\mathbf{P}^n \\ \text{Shift } n \text{ columns right :} & \mathbf{A}\left(\mathbf{P}^T\right)^n \\ \text{Shift } n \text{ rows down :} & \mathbf{P}^n\,\mathbf{A} \\ \text{Shift } n \text{ rows up :} & \left(\mathbf{P}^T\right)^n\mathbf{A} \end{array}$$

Symmetric and skew-symmetric matrix

$$\textit{Symmetric} : \quad \mathbf{A} = \mathbf{A}^T \tag{A.120}$$

Every symmetric matrix with real entries can be diagonalised. It has real valued Eigenvalues. The Eigen decomposition takes a simpler form, i.e.:

$$\mathbf{A} = \mathbf{Q}\,\mathbf{\Lambda}\,\mathbf{Q}^T \tag{A.121}$$

since the Eigenvectors are orthogonal ($\mathbf{Q}\,\mathbf{Q}^T = \mathbf{I}$). Symmetric matrices are normal matrices.

The matrices $\mathbf{A},\ \mathbf{B}$ are supposed to be symmetric. It holds:

$$\begin{array}{l} \text{Matrix sum :} \quad \mathbf{A}+\mathbf{B} = (\mathbf{A}+\mathbf{B})^T \\ \text{Matrix product} \quad \mathbf{A}\,\mathbf{B} = (\mathbf{A}\,\mathbf{B})^T \quad \text{if} \quad \mathbf{A}\,\mathbf{B} = \mathbf{B}\,\mathbf{A} \\ \text{Power of matrix} \quad \mathbf{A}^n = (\mathbf{A}^n)^T \quad \text{if} \quad \mathbf{A} = \mathbf{A}^T; \quad n \in \mathbb{Z} \end{array}$$

$$\textit{Skew-symmetric}: \quad \mathbf{A} = -\mathbf{A}^T \tag{A.122}$$

Let **A** be an [MxM] matrix:

$$\det(\mathbf{A}) = 0 \quad \text{if} \quad M \text{ is odd} \tag{A.123}$$

The Eigenvalues come in pairs. In the case where M is odd, one unpaired $\lambda_m = 0$ appears.

$$\mathbf{A} = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & \cdots & 0 \\ -\lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & -\lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{A.124}$$

Real skew-symmetric matrices are normal matrices.

A square matrix may be decomposed in a symmetric $\mathbf{A}_{sy}$ and skew-symmetric matrix $\mathbf{A}_{sk}$:

$$\begin{aligned} &\mathbf{A} = \mathbf{A}_{sy} + \mathbf{A}_{sk} \\ &\text{where at} \quad \mathbf{A}_{sy} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) \text{ and } \mathbf{A}_{sk} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right) \end{aligned} \tag{A.125}$$

Per- and centrosymmetric matrix:

Persymmetric: square matrix of symmetry about its cross diagonal.

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1N-2} & a_{1N-1} & a_{1N} \\ a_{21} & \cdots & a_{2N-2} & a_{2N-1} & a_{1N-1} \\ a_{31} & \cdots & a_{3N-2} & a_{2N-2} & a_{1N-2} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{N1} & \cdots & a_{31} & a_{21} & a_{11} \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}^T &= \mathbf{J}\,\mathbf{A}\,\mathbf{J} \\ \mathbf{A} &= \mathbf{J}\,\mathbf{A}^T\,\mathbf{J} \\ \mathbf{J}\,\mathbf{A}^T &= \mathbf{A}\,\mathbf{J} \end{aligned} \tag{A.126}$$

Centrosymmetric matrix: Square matrix which is symmetric about its centre, e.g.

$$\begin{bmatrix} c & d & b \\ e & a & e \\ b & d & c \end{bmatrix} \text{ or } \begin{bmatrix} c & f & e & d \\ h & a & b & g \\ g & b & a & h \\ d & e & f & c \end{bmatrix}$$

$$\mathbf{A}\,\mathbf{J} = \mathbf{J}\,\mathbf{A} \tag{A.127}$$

Hermitian and skew-Hermitian matrix:

$$\textit{Hermitian}: \quad \mathbf{A} = \mathbf{A}^H \tag{A.128}$$

A Hermitian matrix is often also assigned as adjoint matrix. The determinate of a Hermitian matrix is real:

$$\det(\mathbf{A}) = \det\left(\mathbf{A}^H\right) = (\det(\mathbf{A}))^* \tag{A.129}$$

Every Hermitian matrix is a normal matrix. Hermitian matrices have real valued Eigenvalues and orthogonal Eigenvectors. The Eigen decomposition simplifies to:

$$\mathbf{A} = \mathbf{Q}\,\boldsymbol{\Lambda}\,\mathbf{Q}^H \quad \text{where at} \quad \mathbf{Q}\,\mathbf{Q}^H = \mathbf{Q}^H\,\mathbf{Q} = \mathbf{I} \tag{A.130}$$

$$\textit{Skew-Hermitian}: \quad \mathbf{A} = -\mathbf{A}^H \tag{A.131}$$

$$\det(\mathbf{A}) = \begin{cases} 0 & M\,\text{odd} \\ \text{real} & M\,\text{even} \end{cases} \tag{A.132}$$

The Eigenvalues are purely imaginary. Skew-Hermitian matrices are normal. Their Eigenvectors are orthogonal.

Orthogonal Matrix:

$$\begin{aligned} \mathbf{Q}^T &= \mathbf{Q}^{-1} \\ \mathbf{Q}^T\mathbf{Q} &= \mathbf{Q}\,\mathbf{Q}^T = \mathbf{I} \end{aligned} \tag{A.133}$$

$$\det(\mathbf{Q}) = \pm 1 \tag{A.134}$$

Unitary Matrix:

$$\begin{aligned} \mathbf{U}^H &= \mathbf{U}^{-1} \\ \mathbf{U}^H\mathbf{U} &= \mathbf{U}\,\mathbf{U}^H = \mathbf{I} \end{aligned} \tag{A.135}$$

$$\det(\mathbf{U}) = \pm 1 \tag{A.136}$$

Eigenvalues are lying on the unit circle:

$$\lambda_n = e^{j\varphi_n} \tag{A.137}$$

Normal Matrix:

$$\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H \tag{A.138}$$

A matrix is normal if it can be factorised in a diagonal matrix $\boldsymbol{\Lambda}$ and a unitary matrix $\mathbf{U}$ by the equation:

$$\mathbf{A} = \mathbf{U}\,\boldsymbol{\Lambda}\,\mathbf{U}^H \tag{A.139}$$

Vandermonde matrix:

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_M & x_M^2 & \cdots & x_M^{N-1} \end{bmatrix} \tag{A.140}$$

$$\det \mathbf{X} = \prod_{\substack{n,m=1 \\ n<m}}^{M} (x_m - x_n); \quad \text{if} \quad M = N \tag{A.141}$$

Application in polynomial fitting: Supposing we have a data set of M samples $\mathbf{y} = [\,y_1 \quad y_2 \quad \cdots \quad y_M\,]$; $\quad \mathbf{x} = [\,x_1 \quad x_2 \quad \cdots \quad x_M\,]$ which should be fitted by a polynomial of order $N-1$:

$$\hat{y} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} \tag{A.142}$$

In order to determine the parameter vector $\mathbf{\Theta} = [\,a_0 \quad a_1 \quad \cdots \quad a_{N-1}\,]^T$, we apply least square estimation which minimises the L_2-norm of the fitting error $\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$ where at (A.142) may be expressed in matrix form by:

$$\hat{\mathbf{y}} = \mathbf{X}\,\mathbf{\Theta} \tag{A.143}$$

Herein, $\mathbf{X}$ represents the Vandermonde matrix (A.140). The polynomial coefficients result to (refer also to (A.97):

$$\mathbf{\Theta} = \left(\mathbf{X}^H \mathbf{X}\right)^{-1} \mathbf{X}^H \mathbf{y} \tag{A.144}$$

Toeplitz matrix: all elements along a diagonal are identical.

$$\mathbf{A}_T = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{A.145}$$

A square Toeplitz matrix is persymmetric:

$$\mathbf{A}_T = \mathbf{J}\,\mathbf{A}_T^T\,\mathbf{J} \tag{A.146}$$

The Toeplitz matrix allows writing the discrete convolution in matrix form:

$$\mathbf{y} = \mathbf{g}_T\,\mathbf{x} = \begin{bmatrix} g_1 & 0 & \cdots & 0 & 0 \\ g_2 & g_1 & \cdots & 0 & 0 \\ g_3 & g_2 & \cdots & 0 & 0 \\ \vdots & g_3 & \cdots & g_1 & 0 \\ g_{M-1} & \vdots & \cdots & g_2 & g_1 \\ g_M & g_{M-1} & \ddots & g_3 & g_2 \\ 0 & g_M & \vdots & \vdots & g_3 \\ 0 & 0 & \cdots & g_{M-1} & \vdots \\ 0 & 0 & \cdots & g_M & g_{M-1} \\ 0 & 0 & \cdots & 0 & g_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \tag{A.147}$$

$\mathbf{g}_T$ - Toeplitz matrix of discrete impulse response of M samples. $\mathbf{x}$ column vector of $N > M$ data samples. See [2] for more on Toeplitz matrix.

Hankel matrix: All elements along the cross diagonals are identical.

$$\mathbf{A}_H = \begin{bmatrix} \cdots & a_{-3} & a_{-2} & a_{-1} & a_0 \\ \cdots & a_{-2} & a_{-1} & a_0 & a_1 \\ \cdots & a_{-1} & a_0 & a_1 & a_2 \\ \cdots & a_0 & a_1 & a_2 & a_3 \\ ⋰ & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{A.148}$$

A square Hankel matrix is symmetric:

$$\mathbf{A}_H = \mathbf{A}_H^T \tag{A.149}$$

$\mathbf{J}\,\mathbf{A}_H$ respectively $\mathbf{A}_H\,\mathbf{J}$ is a Toeplitz matrix.

Circulant Matrix:

Circulant matrices[1] are special cases of the Toeplitz or Hankel matrices. Several versions are in use:

$$\text{down} - \text{circulant, Toeplitz type :} \quad \mathbf{A}_D = \begin{bmatrix} a_0 & a_{M-1} & a_{M-2} & \cdots & a_1 \\ a_1 & a_0 & a_{M-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{M-1} & a_{M-2} & a_{M-3} & \cdots & a_0 \end{bmatrix}$$

$$= \sum_{m=0}^{M-1} a_m\,\mathbf{P}^m \tag{A.150}$$

with $\mathbf{P}$ - shift matrix (A.117).

$$\text{Left-circulant; Hankel type} \quad \mathbf{A}_L = \mathbf{A}_D\,\mathbf{J}\,\mathbf{P}^T \tag{A.151}$$

$$\text{Right-circulant; Toeplitz type} \quad \mathbf{A}_R = \mathbf{J}\,\mathbf{A}_L\,\mathbf{P} = \mathbf{J}\,\mathbf{A}_D\,\mathbf{J} \tag{A.152}$$

Eigenvalues and vectors of $\mathbf{A}_D$ (corresponding for the other types):

$$\lambda_k = \sum_{m=0}^{M-1} a_m e^{-j2\pi\frac{k\,m}{M}}; \quad \mathbf{x}_{qk} = \begin{bmatrix} 1 \\ e^{j2\pi\frac{k}{M}} \\ e^{j2\pi\frac{2\,k}{M}} \\ \vdots \\ e^{j2\pi\frac{(M-1)\,k}{M}} \end{bmatrix}; \quad k, m \in [0, M-1] \tag{A.153}$$

1) Do not confuse with circular matrix: $\mathbf{A} = e^{j\mathbf{B}}$ if $\mathbf{B}$ is a real matrix.

Arranging the Eigenvectors in the matrix $\mathbf{F}$ and the Eigenvalues in the diagonal matrix $\mathbf{\Lambda}$, we get:

$$\begin{aligned}
\mathbf{F} &= \begin{bmatrix} \mathbf{x}_{q0} & \mathbf{x}_{q1} & \cdots & \mathbf{x}_{q(M-1)} \end{bmatrix} = e^{j2\pi \frac{\mathbf{z}\,\mathbf{z}^T}{M}}; \quad \mathbf{z} = \begin{bmatrix} 0 & 1 & 2 & \cdots & M-1 \end{bmatrix}^T \\
F_{k,m} &= e^{j2\pi \frac{k\,m}{M}}; \quad k, m \in [0, M-1] \\
\mathbf{\Lambda} &= \begin{bmatrix} \lambda_0 & 0 & \cdots \\ 0 & \lambda_1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}
\end{aligned}$$

From this, we can observe

$$\mathbf{F}^H\,\mathbf{F} = M\,\mathbf{I} \tag{A.154}$$

and the Eigenvalues in (A.153) may be expressed as:

$$\begin{aligned}
\lambda_k &= \sum_{m=0}^{M-1} a_m\, F^*_{k,m} \\
\operatorname{diag}(\mathbf{\Lambda}) &= \mathbf{F}^H\,\mathbf{a}
\end{aligned} \tag{A.155}$$

where at $\operatorname{diag}(\mathbf{\Lambda}) = \begin{bmatrix} \lambda_0 & \lambda_1 & \cdots \end{bmatrix}^T$ is a vector constituting the diagonal elements of $\mathbf{\Lambda}$ and $\mathbf{a}$ is a vector representing the first column of the circulant matrix $\mathbf{a} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{M-1} \end{bmatrix}^T$. Comparing (A.153) or (A.155) with (2.179), (2.181) and (2.182), we can state that the Eigenvalues represent the complex spectrum of the vector **a**. Finally, the circulant matrix can be factorised as:

$$\begin{aligned}
\mathbf{A}_D &= \mathbf{F}\,\mathbf{\Lambda}\,\mathbf{F}^{-1} \\
\mathbf{A}_D &= \frac{1}{M}\mathbf{F}\,\mathbf{\Lambda}\,\mathbf{F}^H \\
\mathbf{A}_D^H &= \mathbf{F}\,\mathbf{\Lambda}^*\,\mathbf{F}^{-1} = \frac{1}{M}\mathbf{F}\,\mathbf{\Lambda}^*\,\mathbf{F}^H
\end{aligned} \tag{A.156}$$

which is nothing but a Fourier decomposition of the circulant matrix $\mathbf{A}_D$.

The power of a circulant matrix is given by (applying (A.104):

$$\begin{aligned}
&\mathbf{A}_D^n = \mathbf{F}\,\mathbf{\Lambda}^n\,\mathbf{F}^{-1} \\
\text{where} \quad &\lambda_k^n = \left(\sum_{m=0}^{M-1} a_m\, e^{-j2\pi \frac{k\,m}{M}} \right)^n
\end{aligned} \tag{A.157}$$

which involves the calculation of the n^th^ power of a polynomial. Since a product of two polynomials may be written as discrete convolution (also assigned as Cauchy product)

$$\begin{aligned}
&\left(\sum_{n=0}^{N-1} a_n\right)\left(\sum_{n=0}^{N-1} b_n\right) = \sum_{m=0}^{M-1} c_m \quad \text{where} \quad c_m = \sum_k a_k b_{m-k}; \\
\text{or} \quad &\left(\sum_{n=0}^{N-1} a_n x^n\right)\left(\sum_{n=0}^{N-1} b_n x^n\right) = \sum_{m=0}^{M-1} c_m x^m \\
&M = 2N-1; \quad k = \max(0, m+1-N) : \min(m, N-1)
\end{aligned} \tag{A.158}$$

the Eigenvalues of $\mathbf{A}_D^n$ may also be expressed by a n-fold convolution. This observation coincides well with the product rule of the Fourier-transform from Table B-2.

Hadamard Matrix:

The entries of a [NxN] Hadamard matrix $\mathbf{H}$ are either 1 or -1. Their columns (and rows) are orthogonal:

$$\mathbf{H}^T\,\mathbf{H} = N\,\mathbf{I} \tag{A.159}$$

Hadamard matrixes exist only of order $N = 2^n$ or if N is a multiple of 4. We are only interested in Hadamard matrices whose order is $N = 2^n$. They can be built recursively by Sylvester's construction:

$$\begin{aligned} \mathbf{H}_1 &= 1 \\ \mathbf{H}_{2n} &= \begin{bmatrix} \mathbf{H}_n & \mathbf{H}_n \\ \mathbf{H}_n & -\mathbf{H}_n \end{bmatrix} \end{aligned} \tag{A.160}$$

The Eigenvalues of a Sylvester type Hadamard matrix are $\lambda_k = \pm\sqrt{N} = \pm 2^{n/2}$ with equal number of positive or negative Eigenvalues. From (A.102) to (A.104), we can state

$$\begin{aligned} \det(\mathbf{H}_{2n}) &= \left(\sqrt{N}\right)^N \\ \mathrm{tr}(\mathbf{H}_{2n}) &= 0 \\ \mathbf{H}_{2n}^k &= \begin{cases} \left(\sqrt{N}\right)^k \mathbf{I}; & k \quad \text{even} \\ \left(\sqrt{N}\right)^{k-1} \mathbf{H}_{2n}; & k \quad \text{odd} \end{cases} ; \quad k \in \mathbb{Z} \end{aligned} \tag{A.161}$$

Rotation Matrix; Euler angles
(refer also to Annex A.4)

$$\begin{aligned} \boldsymbol{\vartheta}_{eu}(\alpha,\beta,\chi) &= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \boldsymbol{\vartheta}_z(\alpha)\,\boldsymbol{\vartheta}_x(\beta)\,\boldsymbol{\vartheta}_z(\chi) \end{aligned} \tag{A.162}$$

Note that 12 different versions of rotation matrices may be created. They have all the same properties:

$$\boldsymbol{\vartheta}^T\,\boldsymbol{\vartheta} = \mathbf{I}$$

$$\boldsymbol{\vartheta}^T\,\boldsymbol{\vartheta} = \boldsymbol{\vartheta}\,\boldsymbol{\vartheta}^T$$

$$\det\,\boldsymbol{\vartheta} = 1$$

For the sub-matrices holds ($\zeta = x, y, z; \quad \theta = \alpha, \beta, \chi, \phi, \varphi, \gamma .....$):

$$\begin{aligned}
\boldsymbol{\vartheta}_\zeta^T \, \boldsymbol{\vartheta}_\zeta &= \mathbf{I} \\
\boldsymbol{\vartheta}_\zeta^T \, \boldsymbol{\vartheta}_\zeta &= \boldsymbol{\vartheta}_\zeta \, \boldsymbol{\vartheta}_\zeta^T \\
\det \; \boldsymbol{\vartheta}_\zeta &= 1 \\
\boldsymbol{\vartheta}_\zeta(-\theta) &= \boldsymbol{\vartheta}_\zeta^{-1}(\theta) = \boldsymbol{\vartheta}_\zeta^T(\theta) \\
\boldsymbol{\vartheta}_\zeta(\theta_1 + \theta_2) &= \boldsymbol{\vartheta}_\zeta(\theta_1) \, \boldsymbol{\vartheta}_\zeta(\theta_2)
\end{aligned}$$

Eigenvalues of $\boldsymbol{\vartheta}_\zeta(\theta)$: $\lambda_1 = 1; \quad \lambda_{2,3} = e^{\pm j\theta}$

Householder Matrix:

The Householder matrix performs a reflection about a plane having the normal vector **u.u** being an unitary column vector, i.e. $\mathbf{u}^H \mathbf{u} = 1$, the Householder matrix is defined as:

$$\mathbf{T}_H^{(\mathbf{u})} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^H \tag{A.163}$$

Properties:

$$\begin{aligned}
&\text{Normal}: \quad \mathbf{T}_H^{(\mathbf{u})} \left(\mathbf{T}_H^{(\mathbf{u})}\right)^H = \left(\mathbf{T}_H^{(\mathbf{u})}\right)^H \mathbf{T}_H^{(\mathbf{u})} \\
&\text{Hermitian}: \quad \mathbf{T}_H^{(\mathbf{u})} = \left(\mathbf{T}_H^{(\mathbf{u})}\right)^H \\
&\text{Unitary}: \quad \left(\mathbf{T}_H^{(\mathbf{u})}\right)^{-1} = \left(\mathbf{T}_H^{(\mathbf{u})}\right)^H \\
&\text{Involutary}: \quad \left(\mathbf{T}_H^{(\mathbf{u})}\right)^2 = \mathbf{I} \\
&\text{Eigenvalues}: \quad \lambda_n = \pm 1 \\
&\text{Determinate}: \quad \det \; \mathbf{T}_H^{(\mathbf{u})} = -1
\end{aligned}$$

Cross-Product Matrix

The cross-products of two respectively three vectors may be expressed by following matrix relation (refer also to (A.52)):

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \mathbf{A}_\times \, \mathbf{b} \tag{A.164}$$

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{A}_\times \, \mathbf{A}_\times \, \mathbf{b} = \left(\mathbf{a}\,\mathbf{a}^T - \left(\mathbf{a}^T \, \mathbf{a}\right) \mathbf{I}\right)\mathbf{b} = \mathbf{A}_\times^2 \, \mathbf{b} \tag{A.165}$$

Properties:

$$\begin{aligned}
\mathbf{A}_\times^T &= -\mathbf{A}_\times \\
\left(\mathbf{A}_\times^2\right)^T &= \mathbf{A}_\times^2
\end{aligned}$$

If **a** is a unit vector **e**, we get:

$$\mathbf{e} = \begin{bmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{bmatrix}; \quad \mathbf{E}_\times = \begin{bmatrix} 0 & -\cos\vartheta & \sin\vartheta\sin\varphi \\ \cos\vartheta & 0 & -\sin\vartheta\cos\varphi \\ -\sin\vartheta\sin\varphi & \sin\vartheta\cos\varphi & 0 \end{bmatrix};$$

$$\mathbf{E}_\times^2 = \mathbf{e}\,\mathbf{e}^T - \mathbf{I} = \begin{bmatrix} -\left(\cos^2\vartheta - \sin^2\vartheta\sin^2\varphi\right) & \sin^2\vartheta\cos\varphi\sin\varphi & \cos\vartheta\sin\vartheta\cos\varphi \\ \sin^2\vartheta\cos\varphi\sin\varphi & -\left(\cos^2\vartheta - \sin^2\vartheta\cos^2\varphi\right) & \cos\vartheta\sin\vartheta\sin\varphi \\ \cos\vartheta\sin\vartheta\cos\varphi & \cos\vartheta\sin\vartheta\sin\varphi & -\sin^2\vartheta \end{bmatrix}$$

$$\mathbf{e}\times\mathbf{b} = \mathbf{E}_\times\,\mathbf{b} \tag{A.166}$$

$$\mathbf{e}\times(\mathbf{e}\times\mathbf{b}) = \left(\mathbf{e}\,\mathbf{e}^T - \mathbf{I}\right)\mathbf{b} = \mathbf{E}_\times^2\,\mathbf{b} \tag{A.167}$$

(note the difference between $\mathbf{E}_\times^2$ to the Householder matrix (A.163))

Properties:

$$\begin{aligned} \mathbf{E}_\times^T &= -\mathbf{E}_\times \\ \mathbf{E}_\times\,\mathbf{E}_\times^T &= \mathbf{E}_\times^T\,\mathbf{E}_\times \\ \det\mathbf{E}_x &= 0 \\ \lambda_n &= 0, \pm j \end{aligned}$$

$$\begin{aligned} \left(\mathbf{E}_\times^2\right)^T &= \mathbf{E}_\times^2 \\ \mathbf{E}_\times^2\left(\mathbf{E}_\times^2\right)^T &= \left(\mathbf{E}_\times^2\right)^T\mathbf{E}_\times^2 \\ \det\ \mathbf{E}_\times^2 &= 0 \\ \lambda_n &= 0, \pm 1 \end{aligned}$$

A.7
Quadric Surfaces and Curves

A quadric defines the locus (given by position vector **r**) of zeros of a quadratic polynomial. The geometric interpretation of the loci is either a surface or a curve of second order (by restricting to three dimensions). It can be written in any of the three forms:

$$\begin{aligned} \mathbf{r}^T\,\mathbf{A}\,\mathbf{r} + \mathbf{B}^T\,\mathbf{r} &= C' \\ (\mathbf{r}-\mathbf{r}_0)^T\,\mathbf{A}\,(\mathbf{r}-\mathbf{r}_0) &= C \\ (\mathbf{r}-\mathbf{r}_0)^T\,\boldsymbol{\vartheta}\,\mathbf{A}_c\,\boldsymbol{\vartheta}^T\,(\mathbf{r}-\mathbf{r}_0) &= C \end{aligned} \tag{A.168}$$

$\boldsymbol{\vartheta}$ - rotation matrix; $\mathbf{r}_0$ - midpoint; $\mathbf{B}^T = -\mathbf{r}_0^T\left(\mathbf{A}+\mathbf{A}^T\right)$; $C = C' - \mathbf{r}_0^T\,\mathbf{A}\,\mathbf{r}_0$; $\mathbf{A}_c$ - diagonal matrix (see below).

Canonical quadric (main axes coincide with the axes of the Cartesian coordinate system):

$$\mathbf{r}^T\,\mathbf{A}_c\,\mathbf{r} = C \tag{A.169}$$

The Eigenvector decomposition results to $\mathbf{A} = \mathbf{Q}\,\boldsymbol{\Lambda}\,\mathbf{Q}^{-1} = \mathbf{Q}\,\boldsymbol{\Lambda}\,\mathbf{Q}^T$ since $\mathbf{A}$ is symmetric. The Eigenvalues provide the length of the major axes $\boldsymbol{\Lambda} = \mathbf{A}_c$ and the Eigenvectors define their directions $\mathbf{Q} = \boldsymbol{\vartheta}^T$.

Tangent plane and normal vector respectively [3]:

$$\mathbf{n} = \frac{\mathbf{A}(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{A}(\mathbf{r} - \mathbf{r}_0)|} \tag{A.170}$$

Some quadric surfaces:

$$\text{Sphere}: \quad \mathbf{A}_c = \frac{\mathbf{I}}{a^2}; \quad C = 1$$

$$\text{Spheroid}: \quad \mathbf{A}_c = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & b^{-2} \end{bmatrix}; \quad C = 1$$

$$\text{Ellipsoid}: \quad \mathbf{A}_c = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{bmatrix}; \quad C = 1$$

$$\text{Circular paraboloid}: \quad \mathbf{A}_c = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad C = 0$$

Circular hyperboloid of one (+1) or two (−1) sheets:

$$\mathbf{A}_c = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & -c^{-2} \end{bmatrix}; \quad C = \pm 1$$

Some quadric curves (conic sections):

$$\text{Circle}: \quad \mathbf{A}_c = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad C = 1$$

$$\text{Ellipse}: \quad \mathbf{A}_c = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad C = 1$$

$$\text{Hyperbole}: \quad \mathbf{A}_c = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & -b^{-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad C = \pm 1$$

Rotation of an ellipse in canonical position about the main axis (equals the x-axis) by the angle α:

$$\mathbf{A}_R = \boldsymbol{\vartheta}_x^T(\alpha)\, \mathbf{A}_c\, \boldsymbol{\vartheta}_x(\alpha) \tag{A.171}$$

Rotation of an ellipse of arbitrary orientation about its main axis by the angle α:

$$\mathbf{A}_R = \mathbf{Q}^2\, \boldsymbol{\vartheta}_x^T(\alpha)\, \mathbf{A}\, \boldsymbol{\vartheta}_x(\alpha)\, \left(\mathbf{Q}^T\right)^2 \tag{A.172}$$

Q represents the Eigenvectors of **A**.

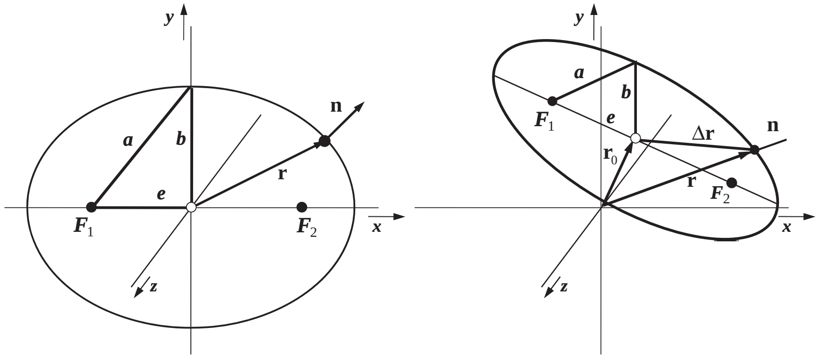


Figure A.3 Example of an elliptic curve in 3D-space at canonical position (left: focal points are placed at the x-axis and the curve is within the xy-plane) and at arbitrary position and orientation (right).

A.7.1 Ellipse

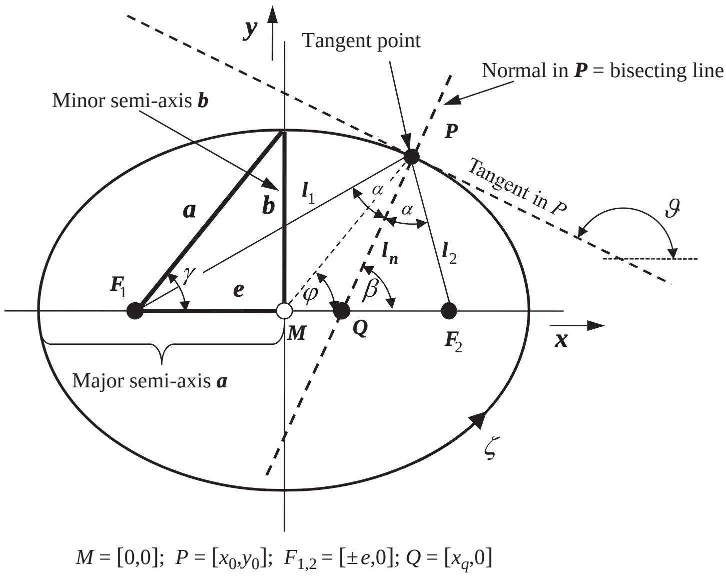


Figure A.4 Ellipse in canonical position.

By definition, an ellipse is the locus of all points P of the plane whose distances to two fixed points $F_{1,2} = [\pm e, 0]$ (the focal point) add to the same constant, i.e.

$$l_1 + l_2 = 2a \tag{A.173}$$

Ellipse in canonical position:

$$\text{Implicit form}: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{A.174}$$

$$\text{Parametric form}: \quad \begin{array}{l} x = a\cos\zeta \\ y = b\sin\zeta \end{array} \qquad \zeta = 0\cdots 2\pi \tag{A.175}$$

$$\text{where at} \quad a^2 = e^2 + b^2. \tag{A.176}$$

Eccentricity:

$$\varepsilon = \frac{e}{a} = \cos\gamma = \sqrt{1 - \frac{b^2}{a^2}} \tag{A.177}$$

Slope of the tangent in point $P = \left[x_0, y_0\right] = [a\cos\zeta_0,\ b\sin\zeta_0]$:

$$\tan\vartheta = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'}{x'} = -\frac{b}{a}cot\,\zeta_0. \tag{A.178}$$

Slope of the normal in point P:

$$\tan\beta = -\frac{1}{\tan\vartheta} = -\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{a}{b}\tan\zeta_0 \tag{A.179}$$

Length of bisecting line in point P:

$$l_n = \frac{y_0}{\sin\beta} = b\sqrt{1 - \varepsilon^2\cos^2\zeta_0} \tag{A.180}$$

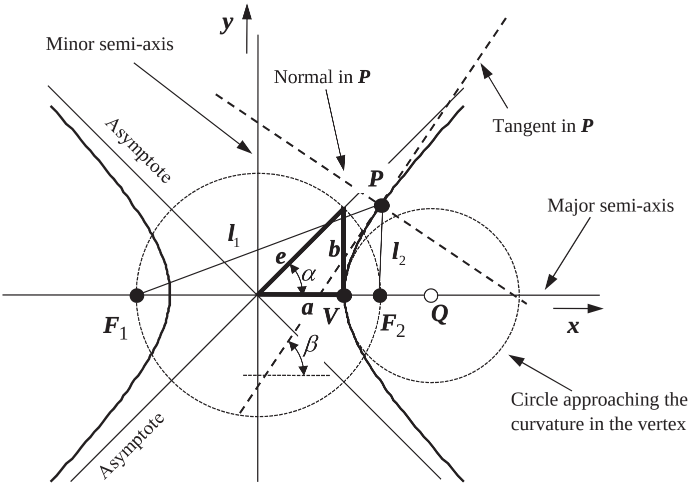
Intersection x_q of bisecting line with x-axis:

$$x_q = x_0 - \frac{y_0}{\tan\beta} = a\cos\zeta_0 - \frac{b^2}{a}\cos\zeta_0 = a\left(1 - \frac{b^2}{a^2}\right)\cos\zeta_0 = a\varepsilon^2\cos\zeta_0 \tag{A.181}$$

The relation between the central angle φ and the curve parameter ζ is given by:

$$\tan\varphi = \frac{y_0}{x_0} = \frac{b}{a}\tan\zeta = \sqrt{1 - \varepsilon^2}\ \tan\zeta \tag{A.182}$$

A.7.2
Hyperbola



$$\left.\begin{aligned} \boldsymbol{P} &= [\boldsymbol{x}_0, \boldsymbol{y}_0] \\ \boldsymbol{F}_{1,2} &= [\pm \boldsymbol{e}, 0] \\ \boldsymbol{V} &= [\boldsymbol{a}, 0] \\ \boldsymbol{Q} &= [\boldsymbol{r}_c, 0] \end{aligned}\right\} \quad \text{with} \quad \boldsymbol{x}_0 = \boldsymbol{a}\cosh\zeta_0;\ \boldsymbol{y}_0 = \boldsymbol{a}\sinh\zeta_0$$

Figure A.5 Hyperbola in canonical position.

By definition, a hyperbola is the locus of all points P of the plane whose distances to two fixed points F_1, F_2 (the focal points) differ in the same constant $2a$, i.e.

$$|l_1 - l_2| = 2a \tag{A.183}$$

Canonical forms of hyperbola:

$$\text{Implicit form}: \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{A.184}$$

$$\text{Parameter form}: \quad \begin{aligned} x &= a\cosh\zeta \qquad \zeta \in \mathbb{R} \\ y &= b\sinh\zeta \end{aligned} \tag{A.185}$$

with

$$e^2 = a^2 + b^2. \tag{A.186}$$

Eccentricity:

$$\varepsilon = \frac{e}{a} = \frac{1}{\cos\alpha} = \sqrt{1 + \tan^2\alpha} \tag{A.187}$$

Slope of the tangent in point P:

$$\tan\beta = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_P = \frac{y'}{x'} = \frac{b}{a}\coth\zeta \tag{A.188}$$

The asymptote represents the tangent with the lowest slope $\alpha = \|\beta\|_{-\infty}$:

$$y_a = \pm\frac{b}{a}x = \pm\tan\alpha\, x \tag{A.189}$$

Slope of the normal in point P:

$$-\frac{1}{\tan\beta} = -\left.\frac{\mathrm{d}x}{\mathrm{d}y}\right|_P = -\frac{x'}{y'} = -\frac{a}{b}\tanh\zeta \tag{A.190}$$

Radius r_v and midpoint $Q = [r_c, 0]$of the circle approaching the vertex curvature: The curvature κ of a hyperbola can be calculated via

$$\kappa = \left|\frac{x'y'' - y'x''}{\sqrt{\left(x'^2 + y'^2\right)^3}}\right|, \tag{A.191}$$

using the parametrically given curve. Insertion of (A.185) leads to

$$\kappa(\zeta) = \frac{ab}{\sqrt{\left(a^2\sinh^2\zeta + b^2\cosh^2\zeta\right)^3}}, \tag{A.192}$$

and consequently the "vertex circle" has a radius of

$$r_v = \frac{1}{\kappa_v} = \frac{1}{\kappa(\zeta = 0)} = \frac{b^2}{a}. \tag{A.193}$$

Hence, its midpoint Q is placed at

$$x = r_c = a + r_v = a\left(1 + \tan^2\alpha\right) = a\,\varepsilon^2. \tag{A.194}$$

A.7.3
Intersection of two Circles

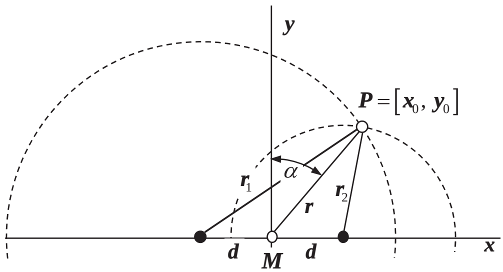


Figure A.6 Intersection of two circles

We are interested in the location of point P with respect to point M if radii r_1, r_2 and centre positions $[\pm d,\ 0]$ of the two circles are known. We get from Figure A.6:

$$\begin{aligned}(x_0 + d)^2 + y_0^2 &= r_1^2\\ (x_0 - d)^2 + y_0^2 &= r_2^2\end{aligned} \tag{A.195}$$

which yields:

$$r = \sqrt{x_0^2 + y_0^2} = \sqrt{\frac{1}{2}\left(r_1^2 + r_2^2\right) - d}$$
$$\sin\alpha = \frac{x_0}{r} = \frac{r_1^2 - r_2^2}{4\,d\,r} = \frac{r_1^2 - r_2^2}{4d\sqrt{\frac{1}{2}\left(r_1^2 + r_2^2\right) - d}} \tag{A.196}$$

Assuming the radii are affected by independent random errors of variance $\sigma_{r_1}^2 = \sigma_{r_2}^2 = \sigma^2$, the variances of range and angle are:

$$\sigma_r^2 = \left(\left(\frac{\partial r}{\partial r_1}\right)^2 + \left(\frac{\partial r}{\partial r_2}\right)^2\right)\sigma^2 = \frac{1}{2}\left(1 + \left(\frac{d}{r}\right)^2\right)\sigma^2 \tag{A.197}$$

$$\sigma_\alpha^2 = \left(\left(\frac{\partial \alpha}{\partial r_1}\right)^2 + \left(\frac{\partial \alpha}{\partial r_2}\right)^2\right)\sigma^2 = \frac{1}{\cos^2\alpha}\left(\left(\frac{\partial a}{\partial r_1}\right)^2 + \left(\frac{\partial a}{\partial r_2}\right)^2\right)\sigma^2 \tag{A.198}$$

with $a = \dfrac{r_1^2 - r_2^2}{4\,d\,r}$; $\quad \dfrac{\partial a}{\partial r_1} = \dfrac{r_1\left(4r^2 - r_1^2 + r_2^2\right)}{8\,d\,r^3}$; $\quad \dfrac{\partial a}{\partial r_2} = -\dfrac{r_2\left(4r^2 + r_1^2 - r_2^2\right)}{8\,d\,r^3}$

Figure A.7 depicts the dependency of range and angular error as function of the position of point P. Both, $(\sigma_r/\sigma)^2$ and $(\sigma_\alpha\, d\cos\alpha/\sigma)^2$ tend to 0.5 at large distance r.

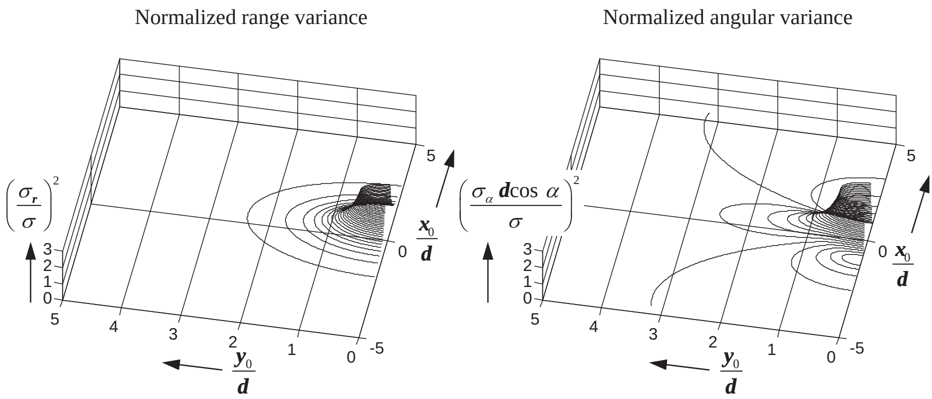


Figure A.7 Normalized range and angular variance of circle intersection.

In case of radar imaging or localisation, the value of σ may be given by the spatial extension $\delta_r \approx c/2\,\underset{\cdot\cdot}{B}$ of the sounding wave or the precision of pulse position estimation $\sigma \approx c\,\varphi/2$.

Annex B: Signals and Systems

B.1
Fourier and Laplace Transform

Table B-1 Fourier-Transform of selected elementary signals.

Name	Time domain *x(t)*	Frequency domain *X(f)*
Dirac-pulse (see also annex A.2)	$x(t)=\delta(t)=u_0(t)$	$\underline{X}(f)=1$
Cosine	$x(t)=\cos(2\pi f_0 t)$	$\underline{X}(f)=\frac{1}{2}\left(\delta(f-f_0)+\delta(f+f_0)\right)$
Sine	$x(t)=\sin(2\pi f_0 t)$	$\underline{X}(f)=\frac{j}{2}\left(\delta(f+f_0)-\delta(f-f_0)\right)$
Exponential	$\underline{x}(t)=e^{j2\pi f_0 t}$	$\underline{X}(f)=\delta(f-f_0)$
Exponential with sinusoidal phase modulation	$x(t)=e^{ja\sin 2\pi f_0 t}$	$\underline{X}(nf_0)=\mathrm{J}_n(a)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}e^{-j(n\tau-a\sin\tau)}d\tau$ $\mathrm{J}_n(a)$ – Bessel function of first kind and n^{th} order
Signum function	$x(t)=\operatorname{sgn} t=\begin{cases}1 & t>0\\ 0 & t=0\\ -1 & t<0\end{cases}$	$\underline{X}(f)=\frac{1}{j\pi f}$
Step function; Heaviside function (see also annex A.2)	$x(t)=u(t)=u_{-1}(t)$ $=\frac{1}{2}(\operatorname{sgn} t+1)=\begin{cases}1 & t>0\\ 1/2 & t=0\\ 0 & t<0\end{cases}$	$\underline{X}(f)=\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$
Exponential step	$x(t)=u_{-1}(t)\left(1-\mathrm{e}^{-t/\tau}\right)$ $=\begin{cases}1-\mathrm{e}^{-t/\tau} & t\geq 0\\ 0 & t<0\end{cases}$	$\underline{X}(f)=\frac{1}{j2\pi f(1+j2\pi f\tau)}+\frac{1}{2}\delta(f)$
Gaussian step	$x(t)=\frac{1}{\tau}\int\limits_{-\infty}^{t}e^{-\pi\left(\frac{\xi}{\tau}\right)^2}d\xi$ $=\frac{1}{\tau}u_{-1}(t)*e^{-\pi\left(\frac{\xi}{\tau}\right)^2}$	$\underline{X}(f)=\left(\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)\right)e^{-\pi(\tau f)^2}$ $=\frac{e^{-\pi(\tau f)^2}}{j2\pi f}+\frac{1}{2}\delta(f)$

Table B.1 (*Continued*)

Gaussian pulse	$x(t) = e^{-\pi\left(\frac{t}{\tau}\right)^2}$	$\underline{X}(f) = \tau e^{-\pi(\tau f)^2}$
Cosine square pulse	$x(t) = \cos^2\left(\frac{\pi t}{2\tau}\right), \quad \lvert t\rvert \le \tau$	$\underline{X}(f) = \tau \frac{\mathrm{sinc}(2\tau f)}{1-(2\tau f)^2}$
Rectangular pulse (normalised boxcar function)	$x(t) = \mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \lvert t\rvert < \tau/2 \\ 1/2 & \lvert t\rvert = \tau/2 \\ 0 & \lvert t\rvert > \tau/2 \end{cases}$	$\underline{X}(f) = \tau\, \mathrm{sinc}(f\tau)$
Sinc-pulse	$x(t) = \mathrm{sinc}\left(\frac{t}{\tau}\right) = \frac{\sin \pi t/\tau}{\pi t/\tau}$	$\underline{X}(f) = \tau\, \mathrm{rect}(\tau f)$
Triangular pulse	$x(t) = \mathrm{tri}\left(\frac{t}{\tau}\right) = \frac{1}{\tau}\mathrm{rect}\left(\frac{t}{\tau}\right) * \mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \lvert t\rvert/\tau & \lvert t\rvert \le \tau \\ 0 & \lvert t\rvert > \tau \end{cases}$	$\underline{X}(f) = \tau\, \mathrm{sinc}^2(\tau f)$
Exponential pulse	$x(t) = u_{-1}(t)\, \mathrm{e}^{-t/\tau} = \begin{cases} \mathrm{e}^{-t/\tau} & t \ge 0 \\ 0 & t < 0 \end{cases}$	$\underline{X}(f) = \frac{\tau}{1 + j2\pi f\tau}$
Decaying sine wave	$x(t) = u_{-1}(t) \sin 2\pi f_0 t\, \mathrm{e}^{-t/\tau} = \sin 2\pi f_0 t\, \mathrm{e}^{-t/\tau} \quad t \ge 0$	$\underline{X}(f) = \frac{j\tau}{2}\left(\frac{1}{1+j2\pi(f+f_0)\tau} \cdots - \frac{1}{1+j2\pi(f-f_0)\tau}\right)$
Dirac-comb	$\Delta_{\Delta t_P}(t) = x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nt_P)$	$\underline{X}(f) = \frac{1}{t_P}\sum_{m=-\infty}^{\infty} \delta(f - mf_0) = \sum_{m=-\infty}^{\infty} e^{-j2\pi f\, m\, t_p}; \quad f_0 = 1/t_P$

Note that complex time signals $\underline{x}(t)$ are not physical. They are usually applied to join two physically real signals into one (often applied in connection with IQ-modulation).

Table B-2 Useful properties and rules of the Fourier-Transform.

Description	Time domain	Frequency domain	Remarks
Real time function	$\underline{x}(t) = \underline{x}^*(t)$	$\underline{X}(-f) = \underline{X}^*(f)$	
Even time function	$x(t) = x(-t)$	$\underline{X}(f) = \underline{X}^*(f)$	Pure real spectrum
Odd time function	$x(t) = -x(-t)$	$\underline{X}(f) = -\underline{X}^*(-f)$	Pure imaginary spectrum
Linearity	$a\,x(t) + b\,y(t)$	$a\,\underline{X}(f) + b\,\underline{Y}(f)$	
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}\underline{X}\left(\frac{f}{a}\right)$	$a \neq 0,\ a \in \mathbb{R},$
Time shift	$x(t-\tau)$	$\underline{X}(f)\,\mathrm{e}^{-j2\pi f\tau}$	
Frequency shift (modulation)	$x(t)\cos\left(2\pi f_0 t\right)$	$\frac{1}{2}\left(\underline{X}(f+f_0) + \underline{X}(f-f_0)\right)$	Mixing or modulation with a cosine carrier
	$x(t)\sin\left(2\pi f_0 t\right)$	$\frac{j}{2}\left(\underline{X}(f+f_0) - \underline{X}(f-f_0)\right)$	Mixing or modulation with a sine carrier
	$\underline{x}(t)\,\mathrm{e}^{j2\pi f_0 t}$	$\underline{X}(f-f_0)$	IQ-mixing or modulation
Time reversal	$x(-t)$	$\underline{X}^*(f)$	
Differentiation	$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = u_1(t) * x(t)$	$j2\pi f\underline{X}(f)$	$u_1(t)$ unit doublet
Integration	$\int\limits_{-\infty}^{t} x(\tau)\mathrm{d}\tau = u_{-1}(t) * x(t)$	$\left(\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right)\underline{X}(f)$	
Moments	$\int\limits_{-\infty}^{\infty} t^n x(t)\mathrm{d}t$	$\frac{1}{(-j2\pi)^n}\left.\frac{\mathrm{d}^n\underline{X}(f)}{\mathrm{d}f^n}\right\rvert_{f=0}$	
	$\left.\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\right\rvert_{t=0}$	$(j2\pi)^n \int\limits_{-\infty}^{\infty} f^n\underline{X}(f)\mathrm{d}f$	

Table B.2 (*Continued*)

Convolution (product of spectra)	$\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\mathrm{d}\tau$	$\underline{X}(f)\, \underline{Y}(f)$	Short form: $x(t) * y(t)$
Correlation	$\int_{-\infty}^{\infty} x^*(t)\, y(t+\tau)\mathrm{d}t$	$\underline{X}^*(f)\, \underline{Y}(f)$	Short form: $x^*(t) * y(-t)$
Product (frequency domain convolution)	$x(t)\, y(t)$	$\int_{-\infty}^{\infty} \underline{X}(\phi)\, \underline{Y}(f-\phi)d\phi$	Short form: $\underline{X}(f) * \underline{Y}(f)$
Sampling	$x(t)\ \sum_n \delta(t-n\Delta t_s)$ $= \sum_n x(n\Delta t_s)$	$f_s \underline{X}(f) * \sum_n \delta(f-mf_s)$ $= f_s \sum_n \underline{X}(f-mf_s);\quad f_s \Delta t_s = 1$	
Parseval's theorem	$\int_{-\infty}^{\infty} \underline{x}(t)\, \underline{y}^*(t)\, \mathrm{d}t$	$\int_{-\infty}^{-\infty} \underline{X}(f)\, \underline{Y}^*(f)\, \mathrm{d}f$	
	$\int_{-\infty}^{\infty} x(t)^2 \mathrm{d}t$	$\int_{-\infty}^{\infty} \lvert\underline{X}(f)\rvert^2 \mathrm{d}f$	Energy conservation in time and frequency domain
Causality	$g(t) = \begin{cases} g(t) & \text{for } t > 0 \\ \equiv 0 & \text{for } t \le 0 \end{cases}$	$Re\{\underline{G}(f)\} = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{Im\{\underline{G}(\xi)\}}{f-\xi}\mathrm{d}\xi$ $\mathrm{Im}\{\underline{G}(f)\} = -\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{Re\{\underline{G}(\xi)\}}{f-\xi}\mathrm{d}\xi$	PV: Cauchy principal value; Hilbert transform

Table B-3 Some elementary properties of the Laplace-Transform ($s_0 = \sigma_0 + j2\pi f_0$).

Description	Time domain	s-domain
Delta pulse	$\delta(t)$	1
Sine	$u_{-1}(t)\,\sin 2\pi f_0 t$	$\frac{1}{s^2 + (2\pi f_0)^2}$
Cosine	$u_{-1}(t)\,\cos 2\pi f_0 t$	$\frac{s}{s^2 + (2\pi f_0)^2}$
Damped sine	$u_{-1}(t)\,\frac{t^{n-1}}{(n-1)!}\,e^{s_0 t}$	$\frac{1}{(s - s_0)^n}$
Time shift	$x(t-\tau)$	$\underline{X}(s)\,e^{-s\tau}$
Differentiation	$\frac{d}{dt}x(t)$	$s\,\underline{X}(s)$
Integration	$\int_0^t x(\xi)\,d\xi$	$\frac{1}{s}\underline{X}(s)$
Convolution	$x(t) * y(t)$	$\underline{X}(s)\,\underline{Y}(s)$
Product (mixing)	$x(t)\,y(t)$	$\underline{X}(s) * \underline{Y}(s)$

Table B.3 (*Continued*)

Periodic function	$x(t) = x_0(t - nt_p)$ $x_0(t) = \begin{cases} 0 & t < 0 \\ x_0(t) & 0 \leq x_0(t) \leq t_P \\ & 0 > t_P \end{cases}$	$\underline{X}(s) = \dfrac{\underline{X}_0(s)}{1 - e^{-s\,t_P}}$
Limit theorems	$\lim\limits_{t \to 0} x(t) = x(+0)$ $\lim\limits_{t \to \infty} x(t) = x(\infty)$	$\lim\limits_{s \to \infty} s\,\underline{X}(s)$ $\lim\limits_{s \to 0} s\,\underline{X}(s)$
Feedback loop (see also Annex B.7)	$g(t) = h(t) * [1 - h(t) * k(t) + h(t) * k(t) * h(t) * k(t) \cdots$ $- h(t) * k(t) * h(t) * k(t) * h(t) * k(t) + \cdots]$	$\underline{G}(s) = \dfrac{\underline{H}(s)}{1 - \underline{H}(s)\,\underline{K}(s)}$ open loop gain : $\lVert \underline{H}(s)\,\underline{K}(s) \rVert < 1$

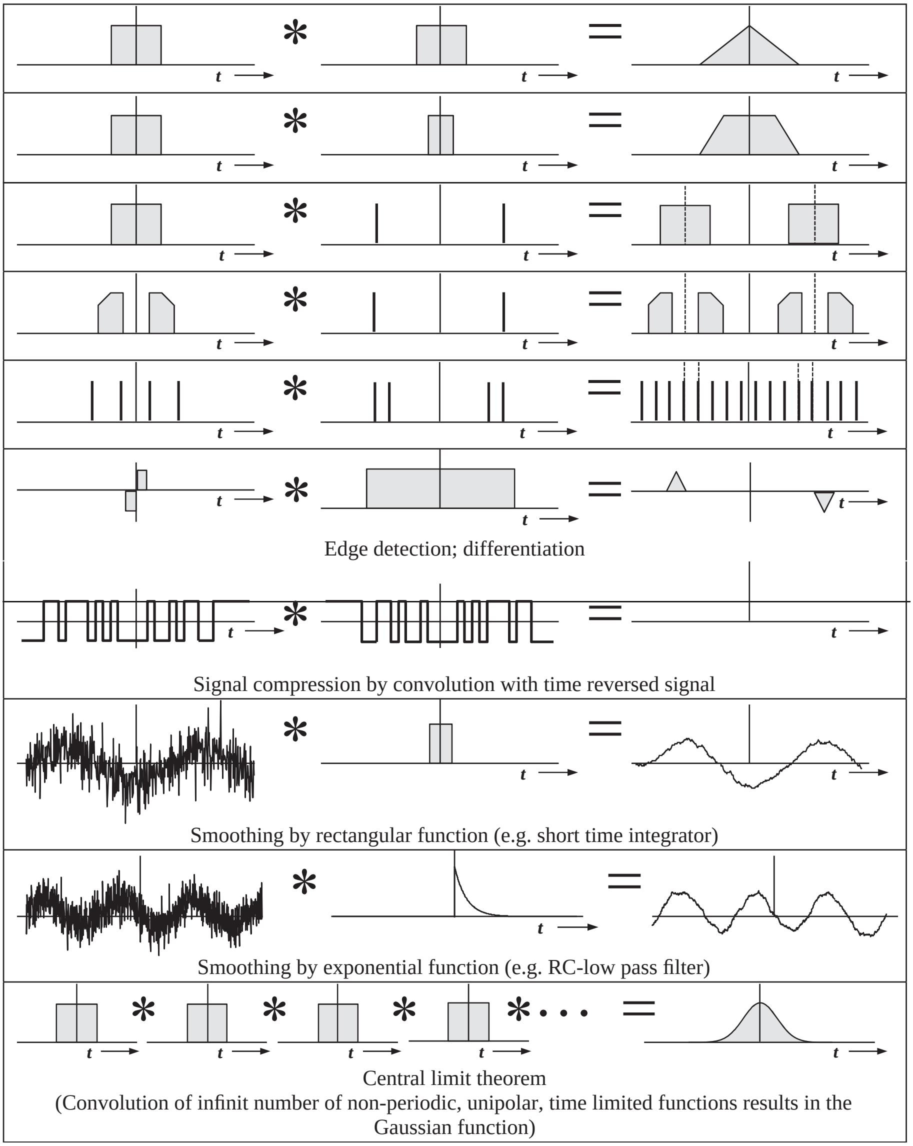
B.2
Properties of Convolution

Table B-4 Some properties of the convolution operation.

Description	Equation
Commutativity	$x(t) * y(t) = y(t) * x(t)$
Associativity	$x(t) * (y(t) * z(t)) = (x(t) * y(t)) * z(t) = (x(t) * z(t)) * y(t)$
Distributativity	$x(t) * (y(t) + z(t)) = x(t) * y(t) + x(t) * z(t)$
Multiplication with scalar	$a(x(t) * y(t)) = (a\, x(t)) * y(t) = x(t) * (a\, y(t))$
Convolution with Dirac function	$x(t) * \delta(t) = x(t)$
Inverse element	$x^{(-1)}(t) * x(t) = \delta(t)$
Integration[2)]	$\int\limits_T x(t) * y(t)\mathrm{d}t = \int\limits_T x(t)\mathrm{d}t \int\limits_T y(t)\mathrm{d}t$
Differentiation	$\frac{d}{dt}(x(t) * y(t)) = \frac{dx(t)}{dt} * y(t) = x(t) * \frac{dy(t)}{dt} = u_1(t) * (x(t) * y(t))$
Shift invariance	$x(t - t_0) * y(t) = x(t) * y(t - t_0) = \delta(t - t_0) * x(t) * y(t)$
Young's inequality	$\lVert x(t) * y(t)\rVert_r \leq \lVert x(t)\rVert_p \lVert y(t)\rVert_q \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \quad p, q, r \geq 1$

2) Results from Fubini's theorem: $\int\limits_{A\times B} f(x, y)dx\, dy = \int\limits_{A\times B} g(x)\, h(y)dx\, dy = \int\limits_A g(x)dx \int\limits_B h(y)dy$ if $f(x, y) = g(x)\, h(y)$

Table B-5 Illustration of convolution for idealised functions. Note the ordinate must not necessarily be the time axis. It also may relate to the frequency (convolution in frequency domain describes signal mixing); to magnitude values (superposition of random variables) or it may relate to space coordinates (e.g. in case of the point spread function of a sensor arrays).



B.3
Spectrum of Complex Exponential (FMCW-Signal)

Signal:

$$x_{expn}(t) = e^{j2\pi\left(f_0 t + \frac{at^2}{2}\right)} \tag{B.1}$$

Fourier-Transform:

$$\begin{aligned}\underline{X}_{expn}(f) &= \int_{-\infty}^{\infty} e^{j2\pi\left(f_0 t + \frac{at^2}{2}\right)} e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} e^{j\pi a\left(t^2 - \frac{2\Delta f}{a}\right)} dt \quad \text{with} \quad \Delta f = f - f_0 \\ &= e^{-j\pi \frac{\Delta f^2}{a}} \int_{-\infty}^{\infty} e^{j\pi a\left(t - \frac{\Delta f}{a}\right)^2} dt\end{aligned} \tag{B.2}$$

Substituting

$$\xi^2 = -j\pi a\left(t - \tfrac{\Delta f}{a}\right)^2; \quad \xi = (1-j)\sqrt{\frac{\pi a}{2}}\left(t - \frac{\Delta f}{a}\right); \quad d\xi = (1-j)\sqrt{\frac{\pi a}{2}}\, dt$$

we result in

$$\underline{X}_{expn}(f) = e^{-j\pi \frac{\Delta f^2}{a}} \int_{-\infty}^{\infty} e^{j\pi a\left(t - \frac{\Delta f}{a}\right)^2} dt = e^{-j\pi \frac{\Delta f^2}{a}} \frac{\sqrt{2}}{(1-j)\sqrt{\pi a}} \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi. \tag{B.3}$$

The remaining integral is related to the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\, d\xi$ for which holds $\operatorname{erf}(x \to \infty) = 1$. Thus, we finally we result in:

$$\underline{X}_{expn}(f) = e^{-j\pi \frac{\Delta f^2}{a}} \frac{(1+j)}{\sqrt{2a}} = \frac{e^{-j\pi\left(\frac{\Delta f^2}{a} - \frac{1}{4}\right)}}{\sqrt{a}}. \tag{B.4}$$

B.4
Product Detector

B.4.1
ACF of Band Limited White Gaussian Noise

We consider a product detector as depicted in Figure B.1 which is fed by two input signals $x(t)$ and $y(t)$ representing both Gaussian noise $n(t) \sim N\left(0, \sigma_n^2\right)$. We will distinguish two cases: (i) the noise has base-band spectrum; and (ii) the noise has band-pass spectrum. Both signals originate from the same source but one of them is delayed by τ; hence, we can write $x(t) = n(t); \quad y(t) = n(t+\tau)$. The noise has constant power spectral density $\underset{\cdot\cdot}{\Phi}_n = \underset{\cdot\cdot}{\Psi}_n / R_0$ distributed over the bandwidth $\underset{\cdot\cdot}{B}_n$. The auto spectrum $\underset{\cdot\cdot}{\Psi}_n$ and the variance σ_n^2 are linked by $\overline{n_{rms}^2} = \sigma_n^2 = \underset{\cdot\cdot}{\Psi}_n\, \underset{\cdot\cdot}{B}_n$. The

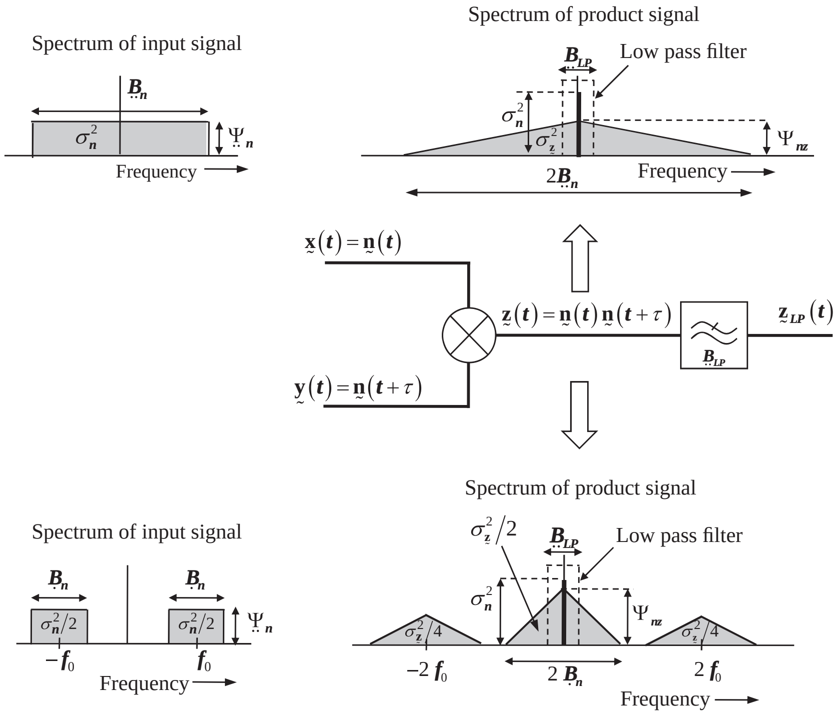


Figure B.1 Product detector with random noise input signals. The upper spectra relate to a low-pass signal. The spectra at the bottom refer to a band-pass signal of central frequency f_0.

average noise power (i.e. the variance) does not depend on time (stationary and ergodic signal).

In what follows, we will estimate the expected value and the variance of the signal product by a simple estimation. For that purpose, we consider the input signals as random processes. In a first step, we calculate the expected value and variance of the product $\underset{\sim}{z}(t)$, and then investigate the role of the low-pass filter (short-time integrator).

In order to simplify the situation, we will discuss only two cases:

- Both random processes are mutually delayed ($\tau \gg \tau_{coh}$) so that they are uncorrelated.
- Both random processes are perfectly correlated ($\tau = 0$)

The signals are not correlated ($\tau \gg \tau_{coh}$): Two wideband signals from the same source are not correlated if their mutual time shift is much larger than the coherence time. Since we have by assumption:

$$\mathrm{cov}\{\underset{\sim}{n}(t), \underset{\sim}{n}(t+\tau)\} = 0; \quad |\tau| \gg \tau_{coh}$$

$$\mathrm{E}\{\underset{\sim}{n}(t)\} = \mathrm{E}\{\underset{\sim}{n}(t+\tau)\} = 0$$

$$\mathrm{var}\{\underset{\sim}{n}(t)\} = \mathrm{var}\{\underset{\sim}{n}(t+\tau)\} = \sigma_n^2$$

the calculation rules (A.36) and (A.37) provide expected value and variance of the signal behind the multiplier:

$$\mathrm{E}\{\underset{\sim}{z}(t)\} = 0 \tag{B.5}$$

$$\mathrm{var}\{\underset{\sim}{z}(t)\} = \sigma_{\underset{\sim}{z}}^2 = \sigma_n^4. \tag{B.6}$$

That is, the signal mixture does not have a DC-component and its total AC-power is proportional to σ_n^4. The spectrum of $\underset{\sim}{z}(t)$ results roughly from the convolution of two rectangular spectra which approximately lead to a triangular spectrum of twice the width. These spectra are different for low- and band-pass signals (see Figure B.1).

Case 1 - low-pass spectrum: The area under the triangular spectrum represents the total AC-power of $\underset{\sim}{z}(t)$ which equals σ_n^4. Hence, the peak value of the spectral density is $\Psi_{nz}\, \underset{..}{B}_n \approx \sigma_{\underset{\sim}{z}}^2 = \sigma_n^4$. Supposing, the low-pass filter is implemented by a short-time integrator having a sufficiently long integration time t_I. Its IRF is shaped like a rectangular pulse $g(t) = \mathrm{rect}(t/t_I)/t_I$ so that its bandwidth is about $\underset{..}{B}_{LP} = \underset{..}{B}_{2,rect} = 1/t_I$ (compare Table 2-1 chapter 2.3.1). Within this small bandwidth, the spectral density of $\underset{\sim}{z}(t)$ can be considered as constant. Consequently, the AC-power of the output signal (representing nothing but the uncertainty of the measurement) is:

$$\mathrm{var}\left\{\underset{\sim}{z}_{LP}(t)\right\}\Big|_{\text{case 1}} = \sigma_{\underset{\sim}{z}_{LP}}^2 \approx \Psi_{nz}\, \underset{..}{B}_{LP} \approx \frac{\sigma_n^4}{\underset{..}{B}_n\, t_I} = \frac{n_{rms}^4}{\underset{..}{B}_n\, t_I} = \frac{n_{rms}^4}{TB}, \tag{B.7}$$

Case 2-- band-pass spectrum: The AC-power of $\underset{\sim}{z}(t)$within the central triangular spectrum is $\sigma_n^4/2$. The width of the triangle is $2\underset{..}{B}_n = \underset{..}{B}_n$, so that we get $\Psi_{nz}\underset{..}{B}_n/2 = \sigma_n^4/2$ which leads to an AC-power behind the integrator of:

$$\mathrm{var}\left\{\underset{\sim}{z}_{LP}(t)\right\}\Big|_{\text{case 2}} = \sigma_{\underset{\sim}{z}_{LP}}^2 \approx \Psi_{nz}\, \underset{..}{B}_{LP} \approx \frac{\sigma_n^4}{\underset{..}{B}_n\, t_I} = \frac{n_{rms}^4}{\underset{..}{B}_n\, t_I} = \frac{n_{rms}^4}{TB} \tag{B.8}$$

Obviously, the same result is obtained if we refer in both cases to the two-sided bandwidth. The product from signal bandwidth and integration time $\underset{..}{B}_n\, t_I$ is usually assigned as the time-bandwidth (TB)-product of the correlation. As the two equations (B.7) and (B.8) show, a stable measurement requires a large TB-product.

The signals are completely correlated ($\tau = 0$): Both signals are correlated if their mutual time shift is zero $\tau = 0$. Consequently, we have from eq. (A.28) and (A.38):

$$\mathrm{E}\{\underset{\sim}{z}(t)\} = \mathrm{E}\left\{\underset{\sim}{n}^2(t)\right\} = \mathrm{var}\{\underset{\sim}{n}(t)\} = \sigma_n^2 \tag{B.9}$$

$$\mathrm{var}\{\underset{\sim}{z}(t)\} = \mathrm{var}\left\{\underset{\sim}{n}^2(t)\right\} = 2\sigma_n^4. \tag{B.10}$$

After the multiplier, the DC-value of the signal $\underset{\sim}{z}(t)$ corresponds to the power $P \sim \sigma_n^2$ of the input signal and the AC-power of $\underset{\sim}{z}(t)$ is $2\sigma_n^4$, causing the uncertainties of the measurement. The spectrum of $\underset{\sim}{z}(t)$ is like that of the previous cases, but is has a DC-component and twice the spectral density of the AC-part as before, i.e. $\Psi_{nz}\, \underset{..}{B}_n \approx \sigma_{\underset{\sim}{z}}^2 = 2\sigma_n^4$. Hence, after the low-pass filter, we will get for both cases:

$$\mathrm{E}\{\underset{\sim}{z}(t)\}\Big|_{\substack{\text{case 1}\\ \text{case 2}}} = \mathrm{E}\left\{\underset{\sim}{z}_{LP}(t)\right\}\Big|_{\substack{\text{case 1}\\ \text{case 2}}} = \sigma_n^2 = n_{rms}^2, \text{ and} \tag{B.11}$$

$$\operatorname{var}\left\{\underset{\sim}{z}_{LP}(t)\right\}\Big|_{\substack{\text{case 1}\\ \text{case 2}}} = \sigma^2_{\underset{\sim}{z}_{LP}} \approx \Psi_{nz}\, \underset{\cdot\cdot}{B}_{LP} \approx \frac{2\,\sigma_n^4}{\underset{\cdot\cdot}{B}_n\, t_I} = \frac{2\,n_{rms}^4}{\underset{\cdot\cdot}{B}_n\, t_I} = \frac{2\,n_{rms}^4}{TB}. \tag{B.12}$$

B.4.2
CCF between a Perturbed and Unperturbed Version of the same Signal

We refer to the schematics depicted in Figure B.1, where the input signals are:

$$\begin{aligned} &\text{Perturbed signal:} \quad \underset{\sim}{x}(t) = x_0(t) + \underset{\sim}{n}(t); \quad \underset{\sim}{n}(t) \sim N\left(0, \sigma_n^2\right) \\ &\text{Reference signal:} \quad y(t) = x_0(t+\tau) \end{aligned}$$

Here, $x_0(t)$ shall represent a deterministic (periodic) signal. Both, $x_0(t)$ and $n(t)$ shall have the same bandwidth $\underset{\cdot\cdot}{B}_N$.

As is clear from the definition (2.49) (chapter 2.2.4) and the calculation rule (A.34), the expected value of the mixer output already represents the auto-correlation function of the unperturbed signal:

$$\mathrm{E}\{\underset{\sim}{z}(t)\} = \mathrm{E}\left\{\underset{\sim}{z}_{LP}(t)\right\} = \mathrm{E}\{x_0(t)x_0(t+\tau) + x_0(t+\tau)\underset{\sim}{n}(t)\} \triangleq C_{x_0x_0}(\tau) \tag{B.13}$$

since $\mathrm{E}\{x_0(t+\tau)\underset{\sim}{n}(t)\} = x_0(t+\tau)\mathrm{E}\{\underset{\sim}{n}(t)\} = 0$.

The variance of the random process after the mixer is (by applying rule (A.33)):

$$\begin{aligned} \operatorname{var}\{\underset{\sim}{z}(t)\} &= \operatorname{var}\{x_0(t)x_0(t+\tau) + x_0(t+\tau)\underset{\sim}{n}(t)\} \\ &= \operatorname{var}\{x_0(t+\tau)\underset{\sim}{n}(t)\} \\ &= \sigma_n^2 x_0^2(t+\tau) \end{aligned} \tag{B.14}$$

since $x_0(t)$ is a deterministic signal.

We can again approximately assume that the AC-components of the mixer output have a triangular spectrum comparable to that already depicted in Figure B.1. On average, this spectrum provides an AC-power of

$$\overline{P} \sim \sigma_n^2 \overline{x_0^2(t)} = \sigma_n^2 \|x_0(t)\|_2^2$$

and its width at the base is $2\,\underset{\cdot\cdot}{B}_n$. This leads to a maximum power spectral density at low frequencies to:

$$\Psi_{nz} = \sigma_n^2\, \|x_0(t)\|_2^2\, \underset{\cdot\cdot}{B}_n$$

so that we finally get at the integrator output:

$$\operatorname{var}\left\{\underset{\sim}{z}_{LP}(t)\right\} = \sigma^2_{\underset{\sim}{z}_{LP}} \approx \Psi_{nz}\, \underset{\cdot\cdot}{B}_{LP} \approx \frac{\sigma_n^2 \|x_0(t)\|_2^2}{\underset{\cdot\cdot}{B}_n\, t_I} = \frac{\sigma_n^2 \|x_0(t)\|_2^2}{TB} \tag{B.15}$$

It should be noted, that the signal product $x_0(t)\, x_0(t+\tau)$ also causes AC-components. Since by assumption, our signal $x_0(t)$ is periodic, the resulting spectrum is a line spectrum so that we can always exclude perturbing spectral lines by an appropriate choice of B_{LP} of the low-pass filter/integrator, i.e. $t_I > t_p$.

The signal-to-noise ratio (SNR) at the integrator output finally is:

$$SNR_{z_{LP}} = \frac{\|z_{LP}(t)\|_\infty^2}{\sigma^2_{z_{LP}}} = \frac{\max\left\{\mathrm{E}^2\left\{\underset{\sim}{z}_{LP}(t)\right\}\right\}}{\operatorname{var}\left\{\underset{\sim}{z}_{LP}(t)\right\}} = \frac{\|x_0(t)\|_2^4}{\sigma_n^2\, \|x_0(t)\|_2^2}\, \underset{\cdot\cdot}{B}_n t_I = \frac{\|x_0(t)\|_2^2}{\sigma_n^2}\, TB \tag{B.16}$$

B.4.3
ACF of a Perturbed Deterministic Signal

We again allude to Figure B.1, and we use the conditions and results from the two previous chapters. The two signals feeding the product detector, we write as:

$$\underset{\sim}{x}(t) = x_0(t) + \underset{\sim}{n}(t); \quad \underset{\sim}{y}(t) = x_0(t+\tau) + \underset{\sim}{n}(t+\tau) \quad \underset{\sim}{n}(t) \sim N\left(0, \sigma_n^2\right)$$

Hence, the output signal of the multiplier results as:

$$\begin{aligned} \underset{\sim}{z}(t) &= (x_0(t) + \underset{\sim}{n}(t))(x_0(t+\tau) + \underset{\sim}{n}(t+\tau)) \\ &= x_0(t)x_0(t+\tau) + \underset{\sim}{n}(t)x_0(t+\tau) + x_0(t)\underset{\sim}{n}(t+\tau) + \underset{\sim}{n}(t)\underset{\sim}{n}(t+\tau) \end{aligned} \tag{B.17}$$

The expected value of (B.17) leads to the superposition of the ACF of both signal components (note: $\mathrm{E}\{x_0(t)\underset{\sim}{n}(t+\tau)\} = x_0(t)\mathrm{E}\{\underset{\sim}{n}(t+\tau)\} = 0$):

$$\begin{aligned} \mathrm{E}\{\underset{\sim}{z}(t)\} &= \mathrm{E}\left\{\underset{\sim}{z}_{LP}(t)\right\} = \mathrm{E}\{x_0(t)x_0(t+\tau)\} + \mathrm{E}\{\underset{\sim}{n}(t)\underset{\sim}{n}(t+\tau)\} \\ &= C_{x_0x_0}(\tau) + C_{nn}(\tau) \end{aligned} \tag{B.18}$$

By using (B.6), (B.10) and (B.14), the variance of (B.17) can be approximated by ($\tau_{n,coh} \approx \underset{\cdot\cdot}{B}_n^{-1}$ - coherence time of the noise):

$$\mathrm{var}\{\underset{\sim}{z}(t)\} = \begin{cases} \sigma_n^2\left(x_0^2(t) + x_0^2(t+\tau) + \sigma_n^2\right); & \tau \gg \tau_{n,coh} \\ 2\sigma_n^2\left(x_0^2(t) + \sigma_n^2\right); & \tau = 0 \end{cases} \tag{B.19}$$

so that we can find for the remaining measurement variations behind the low-pass filter:

$$\mathrm{var}\left\{\underset{\sim}{z}_{LP}(t)\right\} \approx \frac{1}{TB}\begin{cases} 2\sigma_n^2\left(\|x_0(t)\|_2^2 + \dfrac{1}{2}\sigma_n^2\right); & \tau \gg \tau_{n,coh} \\ 2\sigma_n^2\left(\|x_0(t)\|_2^2 + \sigma_n^2\right); & \tau = 0 \end{cases} \tag{B.20}$$

B.4.4
IQ-demodulator

We consider two product detectors arranged in an IQ-demodulator as exhibited in Figure B.2. Both inputs are fed with pure sine waves. The signal $x(t)$ represents a reference signal which will not be affected by noise. Furthermore, we assume that both signals originate from the same source involving identical phase noise $\Delta\phi(t)$. The signal $y(t)$ represents a response of a LTI system i.e. it is subjected to propagation delay τ and additive random noise $n(t)$. In what follows, we will investigate the impact of additive random noise and phase jitter onto the output quantities I and Q.

Additive Random Noise

We model our input signals as:

$$\begin{aligned} x(t) &= X_0 \cos 2\pi f_0 t \\ y(t-\tau) &= Y_0 \cos 2\pi f_0 (t-\tau) + n(t) \end{aligned} \tag{B.21}$$

where at $n(t)$ is white Gaussian noise limited to a spectral band around the carrier, i.e. $n(t) \sim N\left(0, \sigma_n^2\right)$; $\sigma_n^2 = \underset{\cdot\cdot}{\Psi}_n\left(f \pm f_0\right) \underset{\cdot\cdot}{B}_n$ ($\underset{\cdot\cdot}{\Psi}_n$ - noise spectral density; $\underset{\cdot\cdot}{B}_n$- noise

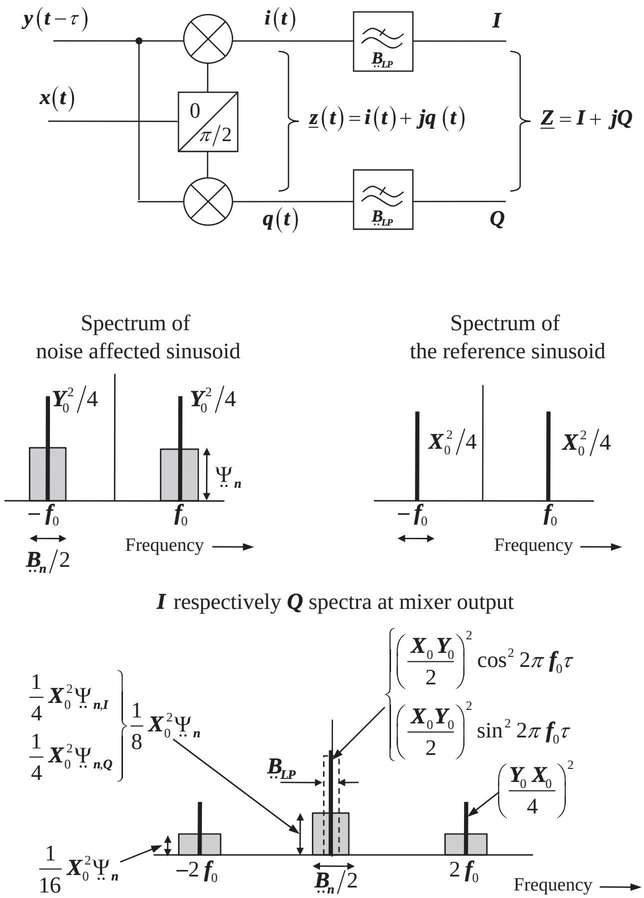


Figure B.2 IQ-demodulator and corresponding power spectra.

bandwidth; compare the figure). It can be decomposed in two components:

$$n(t) = \underline{n}_0(t)\, e^{j2\pi f_0 t} = \left(n_I(t) + jn_Q(t)\right) e^{j2\pi f_0 t} \tag{B.22}$$

where at $\underline{n}_0(t)$ represents a complex valued baseband noise having the spectral components $\Psi_{n,I}$ and $\Psi_{n,Q}$ for which hold:

$$\begin{aligned} \Psi_{n,I}(f) &= \Psi_{n,Q}(f) \\ \Psi_{n,I}(f) + \Psi_{n,Q}(f) &= \Psi_n(f \pm f_0) \end{aligned} \tag{B.23}$$

It is interesting to note, that the noisy parts of the I and Q components are decorrelated, even though they originate from the same source.

The output signals of the two mixers are:

$$\begin{aligned} \underline{z}(t) &= i(t) + jq(t) \\ i(t) &= X_0 \cos 2\pi f_0 t \left(Y_0 \cos 2\pi f_0(t-\tau) + n(t)\right) \\ q(t) &= X_0 \sin 2\pi f_0 t \left(Y_0 \cos 2\pi f_0(t-\tau) + n(t)\right) \end{aligned} \tag{B.24}$$

The spectra of the two mixing products are depicted in Figure B.2. The DC-components of these signals are:

$$
\begin{aligned}
\underline{Z}_0 &= I_0 + j\,Q_0 \\
I_0 &= \overline{i(t)} = \mathrm{E}\left\{\frac{1}{t_I}\int_{t_I} \underset{\sim}{i}(t)\,dt\right\} = \frac{X_0 Y_0}{2}\cos 2\pi f_0\tau \\
Q_0 &= \overline{q(t)} = \mathrm{E}\left\{\frac{1}{t_I}\int_{t_I} \underset{\sim}{q}(t)\,dt\right\} = \frac{X_0 Y_0}{2}\sin 2\pi f_0\tau
\end{aligned} \tag{B.25}
$$

They comply with the expected values of the averaged mixer products ($t_I\,\underset{\cdot\cdot}{B}_{LP} = 1$) if we consider the noise as ergodic random process.

The variances of I and Q represent the AC-power passing the low-pass filters. They are both identical and are simply estimated from Figure B.2 and (B.23):

$$
\begin{aligned}
\sigma_I^2 = \sigma_Q^2 &= \left\| \frac{1}{t_I}\int_{t_I} (i(t) - I_0)\,dt \right\|_2^2 = \left\| \frac{1}{t_I}\int_{t_I} \left(q(t) - Q_0\right) dt \right\|_2^2 \\
&= \mathrm{var}\left\{\frac{1}{t_I}\int_{t_I} \underset{\sim}{i}(t)\,dt\right\} = \mathrm{var}\left\{\frac{1}{t_I}\int_{t_I} \underset{\sim}{q}(t)\,dt\right\} = \frac{1}{4}X_0^2 \underset{\cdot\cdot}{\Psi}_{n,I;Q}\,\underset{\cdot\cdot}{B}_{LP} = \frac{1}{8}X_0^2\,\frac{\sigma_n^2}{\underset{\cdot\cdot}{B}_n\,t_I}
\end{aligned} \tag{B.26}
$$

The SNR-value[3)] of an IQ-pair is therefore:

$$
SNR_{IQ,n} = \frac{\mathrm{E}\{\underline{Z}\}(\mathrm{E}\{\underline{Z}\})^*}{\mathrm{var}\{\underline{Z}\}} = \frac{I_0^2 + Q_0^2}{\sigma_I^2 + \sigma_q^2} = \frac{Y_0^2}{\sigma_n^2}\underset{\cdot\cdot}{B}_n t_I \tag{B.27}
$$

The noise remaining after low-pass filtering forms a circular uncertainty area in the complex IQ-plane around the wanted IQ-value. Figure B.3 exhibits the IQ-demodulator output in the complex plane representing a point surrounded by an area of uncertainty.

Phase Noise

We suppose for simplicity that both signals originate from the same (phase noise affected) source and that the response signal is not additionally influenced by the measurement object except the propagation delay τ. The phase noise $\Delta\phi(t)$ is expected to be ergodic and of zero mean, i.e. $E\left\{\Delta\phi\right\} = \overline{\Delta\phi(t)} = 0$. We can write for the inputs signals

$$
\begin{aligned}
x(t) &= X_0\cos\left(2\pi f_0 t - \Delta\phi(t)\right) \\
y(t) &= Y_0\cos\left(2\pi f_0(t-\tau) - \Delta\phi(t-\tau)\right)
\end{aligned} \tag{B.28}
$$

3) This definition of SNR-value relates peak signal power to average noise power as defined by definition (2.42). Note that in connection with sinusoids, the SNR-value is defined very often by the ratio of average signal and noise power.

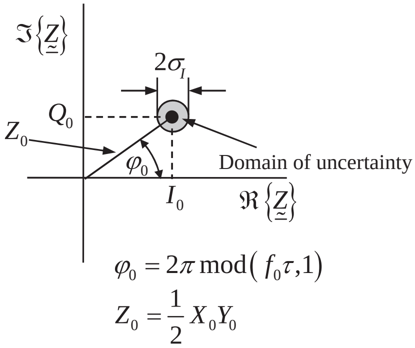


Figure B.3 IQ-demodulator output in the complex IQ-plan for additive random noise.

The output signals of the mixers are (RF-components omitted):

$$\begin{aligned}
\underline{z}(t) &= i(t) + jq(t) = \frac{X_0 Y_0}{2} e^{j\left(2\pi f_0\tau - \Delta\phi_c(t,\tau)\right)} = \underline{Z}_0\, e^{-j\Delta\phi_c(t,\tau)} \\
\underline{Z}_0 &= \frac{X_0 Y_0}{2} e^{j2\pi f_0\tau} = Z_0 e^{j2\pi f_0\tau} \\
i(t) &= \frac{X_0 Y_0}{2} \cos\left(2\pi f_0\tau - \Delta\phi_c(t,\tau)\right) \\
q(t) &= \frac{X_0 Y_0}{2} \sin\left(2\pi f_0\tau - \Delta\phi_c(t,\tau)\right)
\end{aligned} \tag{B.29}$$

where at (see also (2.7); (2.297))

$$\begin{aligned}
&\Delta\phi_c(t,\tau) = \Delta\phi(t-\tau) - \Delta\phi(t) = 2\pi \int_{t-\tau}^{t} f_n(t)dt = 2\pi\, \tau\, \Delta f_\tau(t) \\
&\text{with} \quad \Delta f_\tau(t) = \frac{1}{\tau} \int_{t-\tau}^{t} f_n(t)dt
\end{aligned} \tag{B.30}$$

$\Delta\phi_c(t,\tau)$ is called the cumulative phase noise and $\Delta f_\tau(t)$ represents the average frequency fluctuation within the time interval $[t-\tau, t]$ [4].

Figure B.4 illustrates (B.29) if we consider the IQ-output as random process. It depicts an ensemble of phase noise affected IQ-measurements. The individual

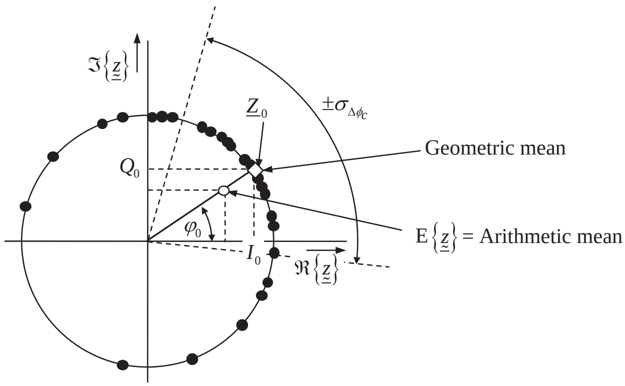


Figure B.4 IQ-demodulator output in the complex IQ-plan for random phase noise.

measurements are placed at the circumference of a circle of radius $Z_0 = .5\,X_0 Y_0$. They are concentrated around the angle $\varphi_0 = 2\pi f_0 \tau$. Their spreading depends on the strength of the phase noise. In the case of normal distributed phase noise, about 68% from all measurements will be located within the interval $\varphi_0 \pm \sigma_{\Delta\phi_c}$.

Intuitively, as depicted in Figure B.4, the mean respectively expected value $\overline{\underline{z}}(t) = \mathrm{E}\{\underset{\sim}{\underline{z}}\}$ of the IQ-output will be placed at the correct angle but with reduced magnitude due to the symmetric scattering of the measurements. According to (A.27), it is estimated as

$$\mathrm{E}\{\underset{\sim}{\underline{z}}\} = \mathrm{E}\left\{\underline{Z}_0\, e^{-j\Delta \underset{\sim}{\phi}_c}\right\} = \underline{Z}_0 \int\limits_{-\infty}^{\infty} e^{-j\varphi}\, p_{\Delta \underset{\sim}{\phi}_c}(\varphi) d\varphi \tag{B.31}$$

where at $p_{\Delta \underset{\sim}{\phi}_c}(\varphi)$ represents the PDF of the phase noise. Concerning to our assumption, the cumulative phase noise is normal distributed having the variance $\mathrm{var}\left\{\Delta \underset{\sim}{\phi}_c\right\} = \sigma^2_{\Delta\phi_c}$

$$p_{\Delta \underset{\sim}{\phi}_c}(\varphi) = \frac{1}{\sigma_{\Delta\phi_c}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\varphi}{\sigma_{\Delta\phi_c}}\right)^2} \tag{B.32}$$

so that (B.31) leads to:

$$\mathrm{E}\{\underset{\sim}{\underline{z}}\} = \frac{\underline{Z}_0}{\sigma_{\Delta\phi_c}\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{\varphi}{\sigma_{\Delta\phi_c}}\right)^2 - j\varphi} d\varphi. \tag{B.33}$$

Applying the substitution $\xi = \frac{\varphi}{\sqrt{2}\,\sigma_{\Delta\phi_c}} + j\frac{\sigma_{\Delta\phi_c}}{\sqrt{2}}$ and referring to Table A-1, we yield for (B.33):

$$\mathrm{E}\{\underset{\sim}{\underline{z}}\} = \overline{\underline{z}(t)} = \frac{\underline{Z}_0}{\sqrt{\pi}} e^{-\frac{\sigma^2_{\Delta\phi_c}}{2}} \int\limits_{-\infty}^{\infty} e^{-\xi^2} d\xi = \underline{Z}_0\, e^{-\frac{1}{2}\sigma^2_{\Delta\phi_c}} \tag{B.34}$$

The equation implies that the magnitude of the IQ-output will be reduced by increasing phase noise. Figure B.4 illustrates that behaviour. That is we get a biased estimation in case of the arithmetic mean. We can avoid this bias if magnitude and phase are averaged separately which represents nothing but the geometric mean of the data samples (see (2.288)). Hence, the application of the geometric mean is in favour to the classical use of the arithmetic mean if phase noise bestrides additive random noise.

The variance of the complex random process $\underset{\sim}{\underline{z}}$ is defined by (A.30), thus we get

$$\mathrm{var}\{\underset{\sim}{\underline{z}}\} = Z_0^2\, \mathrm{E}\left\{\left(e^{-j\Delta \underset{\sim}{\phi}_c}(t) - e^{-\frac{1}{2}\sigma^2_{\Delta\phi_c}}\right)\left(e^{-j\Delta \underset{\sim}{\phi}_c}(t) - e^{-\frac{1}{2}\sigma^2_{\Delta\phi_c}}\right)^*\right\} \tag{B.35}$$

So that we result in

$$\begin{aligned}
\operatorname{var}\{\underline{\underline{z}}\} &= Z_0^2\,\mathrm{E}\left\{1 - e^{-\frac{1}{2}\sigma^2_{\Delta\phi_c}} e^{j\Delta\underset{\sim}{\phi}_c}(t) - e^{-\frac{1}{2}\sigma^2_{\Delta\phi_c}} e^{-j\Delta\underset{\sim}{\phi}_c}(t) + e^{-\sigma^2_{\Delta\phi_c}}\right\} \\
&= Z_0^2\left(1 + e^{-\sigma^2_{\Delta\phi_c}} - e^{-\frac{1}{2}\sigma^2_{\Delta\phi_c}}\,\mathrm{E}\left\{e^{j\Delta\underset{\sim}{\phi}_c}(t) + e^{-j\Delta\underset{\sim}{\phi}_c}(t)\right\}\right) \\
&= Z_0^2\left(1 - e^{-\sigma^2_{\Delta\phi_c}}\right)
\end{aligned} \tag{B.36}$$

since $\mathrm{E}\left\{e^{j\Delta\underset{\sim}{\phi}_c}(t) + e^{-j\Delta\underset{\sim}{\phi}_c}(t)\right\} = 2\mathrm{E}\left\{\cos\Delta\underset{\sim}{\phi}_c(t)\right\} = \frac{2}{\sigma_{\Delta\phi_c}\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\cos\varphi\, e^{-\frac{1}{2}\left(\frac{\varphi}{\sigma_{\Delta\phi_c}}\right)^2}\,d\varphi$ $= 2\ e^{-\frac{1}{2}\sigma^2_{\Delta\phi_c}}$.

It still remains the estimation of the phase noise variance from measurable spectral quantities as the phase noise level (Figure 2.93, (2.300)).

$$\begin{aligned}
\sigma^2_{\Delta\phi_c} &= \operatorname{var}\left\{\Delta\underset{\sim}{\phi}_c(t,\tau)\right\} = \|\Delta\phi_c(t,\tau)\|_2^2 = \mathrm{E}\left\{\left(\Delta\underset{\sim}{\phi}_c(t-\tau) - \underset{\sim}{\phi}_c(t)\right)^2\right\} \\
&= 2\left(\mathrm{E}\left\{\Delta\underset{\sim}{\phi}^2(t)\right\} - \mathrm{E}\left\{\Delta\underset{\sim}{\phi}(t-\tau)\Delta\underset{\sim}{\phi}(t)\right\}\right) \\
&= 2\left(C_{\Delta\phi\Delta\phi}(0) - C_{\Delta\phi\Delta\phi}(\tau)\right) \quad \text{since} \quad \mathrm{E}\left\{\Delta\underset{\sim}{\phi}^2(t)\right\} = \mathrm{E}\left\{\Delta\underset{\sim}{\phi}^2(t-\tau)\right\}
\end{aligned} \tag{B.37}$$

$C_{\Delta\phi\Delta\phi}(\tau)$ is the auto-correlation function of the phase noise and the associated power spectrum is $\underset{\approx}{m}_\phi(f)$ so that holds $C_{\Delta\phi\Delta\phi}(\tau) = \int\limits_{-\infty}^{\infty}\underset{\approx}{m}_\phi(f)e^{j2\pi f\tau}df$. Respecting the symmetry of $\underset{\approx}{m}_\phi(f)$, equation (B.37) may also be expressed as:

$$\begin{aligned}
\sigma^2_{\Delta\phi_c} &= 2\int\limits_{-\infty}^{\infty}\underset{\approx}{m}_\phi(f)\left(1 - e^{j2\pi f\tau}\right)df = -4\int\limits_{-\infty}^{\infty}\underset{\approx}{m}_\phi(f)\sinh(j\pi f\tau)e^{j\pi f\tau}df \\
&= 4\int\limits_{-\infty}^{\infty}\underset{\approx}{m}_\phi(f)\sin^2\pi f\tau\,df
\end{aligned} \tag{B.38}$$

Assuming Cauchy-Lorentz phase noise spectrum (chapter 2.6.3. (2.302)), the equation leads to following expression[4]:

$$\sigma^2_{\Delta\phi_c} = \frac{4a}{\pi}\int\limits_{-\infty}^{\infty}\frac{\sin^2\pi f\tau}{a^2+f^2}\,df \approx 4a\pi\tau\int\limits_{-\infty}^{\infty}\frac{\sin^2\pi\xi}{(\pi\xi)^2}\,d\xi = 4a\pi\tau \tag{B.39}$$

The constant is taken from the phase noise spectrum according to $a = \pi\,\Delta f^2\underset{\approx}{m}_\phi(\Delta f)$ where at Δf is the offset frequency which should be selected sufficiently away from the carrier in order to be in the quadratic region of the Cauchy-Lorentz distribution. Thus, we can finally write for expected value and variance of the mixed signal $\underset{\sim}{z}(t)$:

$$\begin{aligned}
\mathrm{E}\{\underline{\underline{z}}\} &= \overline{z(t)} = \underline{Z}_0\,e^{-2\pi a\tau} = \underline{Z}_0\,e^{-\frac{1}{2}(2\pi\Delta f)^2\underset{\approx}{m}_\phi(\Delta f)\,\tau} \\
\operatorname{var}\{\underline{\underline{z}}\} &= Z_0^2\left(1 - e^{-4a\pi\tau}\right) = Z_0^2\left(1 - e^{-(2\pi\Delta f)^2\underset{\approx}{m}_\phi(\Delta f)\,\tau}\right)
\end{aligned} \tag{B.40}$$

4) The approximation in (B.39) causes errors smaller than 10% if $a\,\tau \le 3\cdot 10^{-2}$; smaller than 3% if $a\,\tau \le 8\cdot 10^{-3}$ or smaller than 1% if $a\,\tau \le 10^{-3}$.

The variance in (B.40) represents pure AC-power (signal components around $2f_0$ are not respected). The low-pass filter at the output of the IQ-demodulator will additionally suppress these AC-components as long as its bandwidth is very small or the signal source is very instable. This may happen if the rectangular bandwidth $\underset{\cdot\cdot}{B}_{2\phi,\mathrm{rect}}$ of the phase noise exceeds the bandwidth $\underset{\cdot\cdot}{B}_{LP}$ of the low-pass filter. According to (2.74), we yield $\underset{\cdot\cdot}{B}_{2\phi,\mathrm{rect}} = \pi\, a/2$. The expected value remains untouched by that filter. Due to low pass filtering, we have to modify (B.39) as follows:

$$\sigma_\varphi^2 = \frac{4a}{\pi} \int\limits_{-\underset{\cdot\cdot}{B}_{LP}/2}^{\underset{\cdot\cdot}{B}_{LP}/2} \frac{\sin^2 \pi f \tau}{a^2 + f^2}\, df = 4a\pi\tau \int\limits_{-\underset{\cdot\cdot}{B}_{LP}\,\tau/2}^{\underset{\cdot\cdot}{B}_{LP}\,\tau/2} \frac{\sin^2 \pi\xi}{(\pi\,\tau\, a)^2 + (\pi\xi)^2}\, d\xi \quad \text{if} \quad \underset{\cdot\cdot}{B}_{LP} < \frac{\pi\, a}{2} \tag{B.41}$$

The relation can be simplified for short range applications since τ only takes small values and bearing in mind that f represents here an offset frequency which is always rather small compared to the operational frequencies of an UWB-sensor. With the identity $\int\limits_{-b}^{b} \frac{\xi^2}{c^2+\xi^2}\, d\xi = b - 2\, c \arctan \frac{b}{c}$ we can finally find:

$$\sigma_\varphi^2 \approx 4a\pi\tau \left(\frac{1}{2}\tau \underset{\cdot\cdot}{B}_{LP} - \tau a \arctan \frac{\underset{\cdot\cdot}{B}_{LP}}{2a} \right) \tag{B.42}$$

In summary, we can write for the output signals of the IQ-modulator:

$$\begin{aligned} \mathrm{E}\{\underline{\underline{Z}}\} &= \underline{Z}_0\, e^{-\frac{1}{2}\sigma^2_{\Delta\phi_c}} \\ \mathrm{var}\{\underline{\underline{Z}}\} &= \begin{cases} Z_0^2 \left(1 - e^{-\sigma^2_{\Delta\phi_c}}\right); & \underset{\cdot\cdot}{B}_{LP} > \dfrac{\pi\, a}{2};\ \sigma^2_{\Delta\phi_c} = 4a\pi\tau \\ Z_0^2 \left(1 - e^{-\sigma^2_{\varphi}}\right); & \underset{\cdot\cdot}{B}_{LP} < \dfrac{\pi\, a}{2};\ \sigma^2_{\varphi} \approx 2\pi a\tau \left(\tau \underset{\cdot\cdot}{B}_{LP} - 2a\tau \arctan \dfrac{\underset{\cdot\cdot}{B}_{LP}}{2a} \right) \end{cases} \end{aligned} \tag{B.43}$$

where at the case $\underset{\cdot\cdot}{B}_{LP} < \pi\, a/2$ is practically unimportant. Thus, the phase noised affected SNR-value results mostly to:

$$SNR_{IQ,\Delta\phi} = \frac{\mathrm{E}\{\underline{\underline{Z}}\}(\mathrm{E}\{\underline{\underline{Z}}\})^*}{\mathrm{var}\{\underline{\underline{Z}}\}} = \frac{e^{-\sigma^2_{\Delta\phi_c}}}{1 - e^{-\sigma^2_{\Delta\phi_c}}} \approx \frac{1}{\sigma^2_{\Delta\phi_c}} = \frac{1}{4a\pi\tau} = \frac{1}{(2\pi\,\Delta f)^2 \underset{\cdot\cdot}{m}_\phi(\Delta f)\tau} \tag{B.44}$$

where at we assumed short range measurements causing only small τ values and hence a small phase noise variance, i.e. $e^{-\sigma^2_{\Delta\phi_c}} \approx 1 - \sigma^2_{\Delta\phi_c}$.

We can observe that the SNR-value depends on the quality of the frequency source (i.e. $\underset{\cdot\cdot}{m}_\phi$ @ Δf) as expected and it also depends on the DUT namely its delay time τ, but it is not affected by the signal levels.

B.5
Shape Factors

B.5.1
Generalised Shape Factors of Triangular Pulse

We consider the triangular pulse as simple signal model for the class of compact, short pulse waveforms. The results of the simple calculation below approve the relations depicted in chapter 4.7.3. However it is not expected, that this very simple model still holds for pulse signals which are composed from a couple of oscillations.

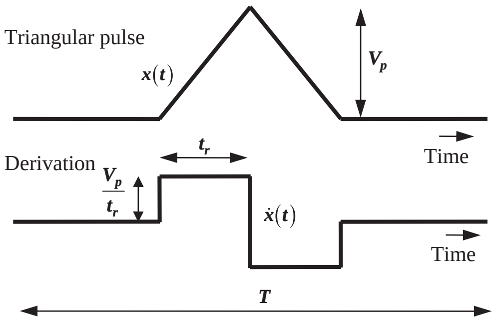


Figure B.5 Triangular pulse and its derivation.

The different L_p-norms result from Figure B.5 to:

$$\|x(t)\|_2^2 = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\mathrm{d}t = \frac{2V_p^2 t_r}{3T}, \tag{B.45}$$

$$x(t)_\infty = V_p \tag{B.46}$$

$$\|\dot{x}(t)x(t+\tau)\|_2^2 = \frac{1}{T}\int_{-T/2}^{T/2} (\dot{x}(t)x(t+\tau))^2\mathrm{d}t = \begin{cases} \dfrac{2V_p^4}{3T\,t_r} & \text{for} \quad \tau = 0 \\ 0 & \text{for} \quad |\tau| > 2t_r \end{cases} \tag{B.47}$$

$$\begin{aligned} &\|x(t)\|_2^2 = \|m(t)\|_2^2 - \frac{N}{2}\|\mathrm{ramp}(t)\|_2^2 = V_m^2\left(1 - \frac{t_r}{12\,t_c}\right) \\ &\text{with} \quad \|\mathrm{ramp}(t)\|_2^2 = \frac{1}{T}\int_{-t_r/2}^{t_r/2}\left(\tfrac{2V_m}{t_r}t\right)^2\mathrm{d}t = \frac{V_m^2 t_r}{12\,t_c} \end{aligned} \tag{B.48}$$

so that the shape factors defined in chapter 2.2.2 and radar chapter 4.7.3 can be expressed by

$$CF^2 = \frac{\|x(t)\|_\infty^2}{\|x(t)\|_2^2} = \frac{3T}{2t_r} = TB \tag{B.49}$$

$$SAF^2(\tau) = t_r^2 \frac{\|\dot{x}(t)x(t+\tau)\|_2^2}{\|x(t)\|_2^4} \approx \begin{cases} CF^2 = TB & \text{for} \quad \tau = 0 \\ 0 & \text{for} \quad |\tau| > 2t_r \end{cases} \tag{B.50}$$

B.5.2
Generalised Shape Factors of M-Sequence

In order to simplify the calculation, we will use M-sequence signals which are composed from linear ramps. Figure B.6 shows a small section of such a signal and its derivation. In what follows, we will consider the case in which the rising or falling edge of an individual chip covers half the chip duration, i.e. $t_c = 2t_r$.

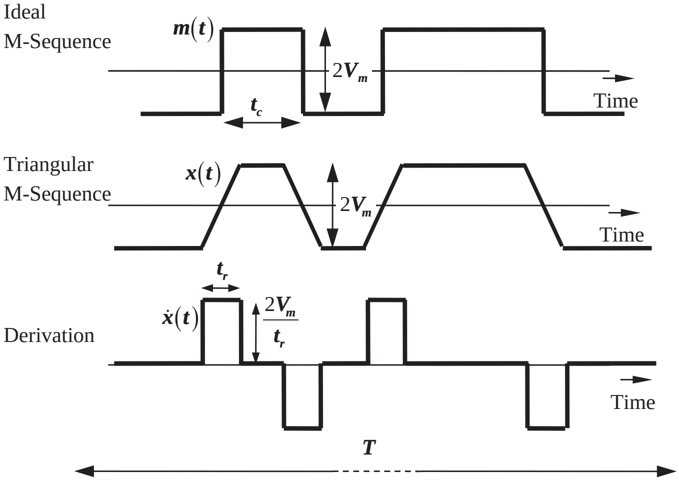


Figure B.6 Part of a simplified M-Sequence and its derivation if $t_c > 2t_r$.

An M-sequence of order n includes $N = 2^n - 1$ chips and $2^{n-1} \approx N/2$ transitions. It has a duration of $T = N\,t_c$. Its power (second order moment) can be calculated from the power of the ideal sequence minus the power of the missing triangles at the signal edges:

$$\begin{aligned} &\|x(t)\|_2^2 = \|m(t)\|_2^2 - \frac{N}{2}\|\text{ramp}(t)\|_2^2 = V_m^2\left(1 - \frac{t_r}{6\,t_c}\right) \\ &\text{with} \quad \|\text{ramp}(t)\|_2^2 = \frac{1}{T}\int_{-t_r/2}^{t_r/2}\left(\tfrac{2V_m}{t_r}t\right)^2 \mathrm{d}t = \frac{V_m^2 t_r}{3\,T} = \frac{V_m^2 t_r}{3\,N\,t_c} \end{aligned} \tag{B.51}$$

The remaining L_p-norms are

$$x(t)_\infty = V_m \tag{B.52}$$

$$\|\dot{x}(t)x(t+\tau)\|_2^2 = \frac{1}{T}\int_0^T (\dot{x}(t)x(t+\tau))^2 \mathrm{d}t \approx \frac{N}{2}\frac{1}{Nt_c}\int_{-t_r/2}^{t_r/2}\left(\tfrac{2V_m}{t_r}\,\tfrac{2V_m}{t_r}t\right)^2 \mathrm{d}t = \frac{2}{3}\frac{V_m^4}{t_r^2}\frac{t_r}{t_c} \tag{B.53}$$

and the considered shape factors result to:

$$CF^2 = \frac{V_m^2}{\|x(t)\|_2^2} = \frac{1}{1-\frac{t_r}{12t_c}} \approx 1 \tag{B.54}$$

$$SAF^2(\tau) = t_r^2 \frac{\|\dot{x}(t)x(t+\tau)\|_2^2}{\|x(t)\|_2^4} = \frac{2}{3}\frac{t_r}{t_c}\frac{V_m^4}{V_m^4\left(1-\frac{t_r}{12\,t_c}\right)^2} \approx \frac{1}{3} \quad \text{for} \quad \frac{t_r}{t_c} = \frac{1}{2} \tag{B.55}$$

As expected, all shape factors are independent from the signal length and they take values close to unity.

B.6
Conversions between N-Port Parameters

The N-port parameters are defined in chapter 2.5.2 and 2.5.3. Some conversion rules are summarised her. We apply following transformation matrices:

$$\mathbf{Q}_{(11)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{Q}_{(12)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{Q}_{(21)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{Q}_{(22)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{P}_1 = \begin{bmatrix} \sqrt{Z_0} & \sqrt{Z_0} \\ 1/\sqrt{Z_0} & -1/\sqrt{Z_0} \end{bmatrix}; \quad \mathbf{P}_2 = \begin{bmatrix} \sqrt{Z_0} & \sqrt{Z_0} \\ -1/\sqrt{Z_0} & 1/\sqrt{Z_0} \end{bmatrix}$$

Herein, Z_0 is the intrinsic impedance of the cable feeding the ports of the DUT.

$$\mathbf{Q}_V = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}; \quad \mathbf{Q}_I = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_1 = \frac{1}{2}\begin{bmatrix} 1/\sqrt{Z_b} & 0 & 0 & 0 \\ 0 & 1/\sqrt{Z_b} & 0 & 0 \\ 0 & 0 & 1/\sqrt{Z_u} & 0 \\ 0 & 0 & 0 & 1/\sqrt{Z_u} \end{bmatrix};$$

$$\mathbf{M}_2 = \frac{1}{2}\begin{bmatrix} \sqrt{Z_b} & 0 & 0 & 0 \\ 0 & \sqrt{Z_b} & 0 & 0 \\ 0 & 0 & \sqrt{Z_u} & 0 \\ 0 & 0 & 0 & \sqrt{Z_u} \end{bmatrix}$$

Herein, Z_b is the intrinsic impedance of the balanced mode of the feeding cable and Z_u is the intrinsic impedance of the unbalanced mode of the feeding cable.

Conversions between A-, Y-, Z-*parameters*:

$$\mathbf{Z} = \mathbf{Y}^{-1}$$

For two-ports only:

$$\mathbf{Y} = \frac{1}{\det(\mathbf{Z})}\begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{bmatrix}; \quad \mathbf{Z} = \frac{1}{\det(\mathbf{Y})}\begin{bmatrix} Y_{22} & -Y_{12} \\ -Y_{21} & Y_{11} \end{bmatrix}$$

$$\mathbf{A} = \left(\mathbf{Q}_{(11)} + \mathbf{Q}_{(21)}\mathbf{Y}\right)\left(\mathbf{Q}_{(12)} - \mathbf{Q}_{(22)}\mathbf{Y}\right)^{-1} = -\frac{1}{Y_{21}}\begin{bmatrix} Y_{22} & 1 \\ \det(\mathbf{Y}) & Y_{11} \end{bmatrix}$$

$$\mathbf{A}=\left(\mathbf{Q}_{(11)}\mathbf{Z}+\mathbf{Q}_{(21)}\right)\left(\mathbf{Q}_{(12)}\mathbf{Z}-\mathbf{Q}_{(22)}\right)^{-1}=\frac{1}{Z_{21}}\begin{bmatrix}Z_{11} & \det(\mathbf{Z})\\ 1 & Z_{22}\end{bmatrix}$$

$$\mathbf{Y}=\left(\mathbf{A}\,\mathbf{Q}_{(22)}+\mathbf{Q}_{(21)}\right)^{-1}\left(\mathbf{A}\,\mathbf{Q}_{(12)}-\mathbf{Q}_{(11)}\right)=\frac{1}{A_{21}}\begin{bmatrix}A_{22} & -\det(\mathbf{A})\\ -1 & A_{11}\end{bmatrix}$$

$$\mathbf{Z}=\left(\mathbf{A}\,\mathbf{Q}_{(12)}-\mathbf{Q}_{(11)}\right)^{-1}\left(\mathbf{A}\,\mathbf{Q}_{(22)}+\mathbf{Q}_{(21)}\right)=\frac{1}{A_{21}}\begin{bmatrix}A_{11} & \det(\mathbf{A})\\ 1 & A_{22}\end{bmatrix}$$

Conversion between S- *and* T-*parameters* (for two-ports only):

$$\mathbf{S}=\left(\mathbf{Q}_{(11)}-\mathbf{T}\mathbf{Q}_{(22)}\right)^{-1}\left(\mathbf{T}\mathbf{Q}_{(12)}-\mathbf{Q}_{(21)}\right)=-\frac{1}{T_{22}}\begin{bmatrix}T_{12} & \det(\mathbf{T})\\ 1 & -T_{21}\end{bmatrix}$$

$$\mathbf{T}=\left(\mathbf{Q}_{(11)}\mathbf{S}-\mathbf{Q}_{(21)}\right)\left(\mathbf{Q}_{(22)}\mathbf{S}+\mathbf{Q}_{12}\right)^{-1}=\frac{1}{S_{21}}\begin{bmatrix}-\det(\mathbf{S}) & S_{11}\\ -S_{22} & 1\end{bmatrix}$$

Conversion between Y-, Z- *and* S-*parameters*:

$$\begin{aligned}\mathbf{S}&=(\mathbf{Z}-Z_0\mathbf{I})(\mathbf{Z}+Z_0\mathbf{I})^{-1}\\ \mathbf{S}&=(\mathbf{I}-Z_0\mathbf{Y})(\mathbf{I}+Z_0\mathbf{Y})^{-1}\\ \mathbf{Z}&=Z_0(\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S})\\ \mathbf{Y}&=\frac{1}{Z_0}(\mathbf{I}+\mathbf{S})^{-1}(\mathbf{I}-\mathbf{S})\end{aligned}$$

Conversion between A- *and* T-*parameters*:

$$\begin{aligned}\mathbf{T}&=(\mathbf{A}\,\mathbf{P}_2)^{-1}\mathbf{P}_1\\ &=\frac{1}{2Z_0\det(\mathbf{Z})}\begin{bmatrix}Z_0(A_{22}-A_{11})+Z_0^2A_{21}-A_{12} & Z_0(A_{22}+A_{11})+Z_0^2A_{21}+A_{12}\\ Z_0(A_{22}-A_{11})-Z_0^2A_{21}-A_{12} & Z_0(A_{22}-A_{11})-Z_0^2A_{21}+A_{12}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A}&=\mathbf{P}_1(\mathbf{P}_2\,\mathbf{T})^{-1}\\ &=\frac{1}{2\det(\mathbf{T})}\begin{bmatrix}T_{11}-T_{12}-T_{21}+T_{22} & Z_0(T_{11}-T_{12}+T_{21}-T_{22})\\ (-T_{11}-T_{12}+T_{21}+T_{22})/Z_0 & -T_{11}-T_{12}-T_{21}-T_{22}\end{bmatrix}\end{aligned}$$

Modaldecomposition; mixed modeconversion:

- Z- and Y-parameters ($\mathbf{Z}_M,\ \mathbf{Y}_M$ - modal parameters):
  The notation of the port signals applied here refers to Figure 2.66 and 2.69 chapter 2.5.2.

$$\left.\begin{aligned}&\mathbf{V}=[\,V_{1A}\quad V_{1B}\quad V_{2A}\quad V_{2B}\,]^T;\quad \mathbf{I}=[\,I_{1A}\quad I_{1B}\quad I_{2A}\quad I_{2B}\,]^T\\ &\mathbf{V}_M=[\,V_{b1}\quad V_{b2}\quad V_{u1}\quad V_{u2}\,]^T;\quad \mathbf{I}_M=[\,I_{b1}\quad I_{b2}\quad I_{u1}\quad I_{u2}\,]^T\end{aligned}\right\}$$

$$\mathbf{V}_M=\mathbf{Q}_V\,\mathbf{V};\quad \mathbf{I}_M=\mathbf{Q}_I\,\mathbf{I}$$

$$\Rightarrow\quad \mathbf{V}=\mathbf{Z}\,\mathbf{I};\quad \mathbf{I}=\mathbf{Y}\,\mathbf{V}$$

$$\mathbf{V}_M=\mathbf{Z}_M\,\mathbf{I}_M;\quad \mathbf{I}_M=\mathbf{Y}_M\,\mathbf{V}_M$$

$$\Rightarrow\quad \mathbf{Z}_M=\mathbf{Q}_V\,\mathbf{Z}\,\mathbf{Q}_I^{-1}$$

$$\Rightarrow\quad \mathbf{Y}_M=\mathbf{Q}_I\,\mathbf{Y}\,\mathbf{Q}_V^{-1}$$

- *Scattering parameters* ($\mathbf{S}_M$- modal parameters)

$$\left.\begin{aligned}\mathbf{A}_M &= [\,a_{b1} \quad a_{b2} \quad a_{u1} \quad a_{u2}\,]^T; \quad \mathbf{B}_M = [\,b_{b1} \quad b_{b2} \quad b_{u1} \quad b_{u2}\,]^T \\ \mathbf{V}_M &= [\,V_{b1} \quad V_{b2} \quad V_{u1} \quad V_{u2}\,]^T; \quad \mathbf{I}_M = [\,I_{b1} \quad I_{b2} \quad I_{u1} \quad I_{u2}\,]^T\end{aligned}\right\}$$

$$\begin{aligned}&\mathbf{A}_M = \mathbf{M}_1\mathbf{V}_M + \mathbf{M}_2\mathbf{I}_M \\ &\mathbf{B}_M = \mathbf{M}_1\mathbf{V}_M - \mathbf{M}_2\mathbf{I}_M \\ \Rightarrow\quad &\mathbf{B}_M = \mathbf{S}_M\mathbf{A}_M \\ &\mathbf{V}_M = \mathbf{Z}_M\,\mathbf{I}_M \\ \Rightarrow\quad &(\mathbf{I}-\mathbf{S}_M)\mathbf{M}_1\mathbf{Z}_M = (\mathbf{S}_M+\mathbf{I})\mathbf{M}_2\end{aligned}$$

Insertion of conversion rule from above, leads finally to:

$$\mathbf{S}_M = \left(\mathbf{N}_1\,(\mathbf{I}-\mathbf{S})^{-1} - \mathbf{N}_2(\mathbf{I}+\mathbf{S})^{-1}\right)\left(\mathbf{N}_2(\mathbf{I}+\mathbf{S})^{-1} + \mathbf{N}_1\,(\mathbf{I}-\mathbf{S})^{-1}\right)^{-1}$$
$$\text{with}\quad \mathbf{N}_1 = \sqrt{Z_0}\mathbf{M}_1\mathbf{Q}_V; \quad \mathbf{N}_2 = \frac{\mathbf{M}_2\mathbf{Q}_I}{\sqrt{Z_0}}$$

For the practical relevant case of uncoupled feeding lines; i.e. $k = 0$ from which follows $Z_b = 2Z_0;\ Z_u = Z_0/2$ (see chapter 2.5.3), the conversion rule simplifies to (exploiting matrix inversion identity (A.82)):

$$\mathbf{N} = \mathbf{N}_1 = \mathbf{N}_2 = \frac{\sqrt{2}}{4}\begin{bmatrix}1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1\end{bmatrix}$$
$$\begin{aligned}\mathbf{S}_M &= \mathbf{N}\left((\mathbf{I}-\mathbf{S})^{-1} - (\mathbf{I}+\mathbf{S})^{-1}\right)\left((\mathbf{I}-\mathbf{S})^{-1} + (\mathbf{I}+\mathbf{S})^{-1}\right)^{-1}\mathbf{N}^{-1} \\ &= \mathbf{N}\,\mathbf{S}\,\mathbf{N}^{-1}\end{aligned}$$

B.7 Mason Graph

The Mason-graph (respectively signal flow graph) is an illustrative mean to symbolise graphically linear equations and relations. We will apply it to represent the signal flow in a physical system or numerical algorithm. It descriptively represents the relations of cause and effect.

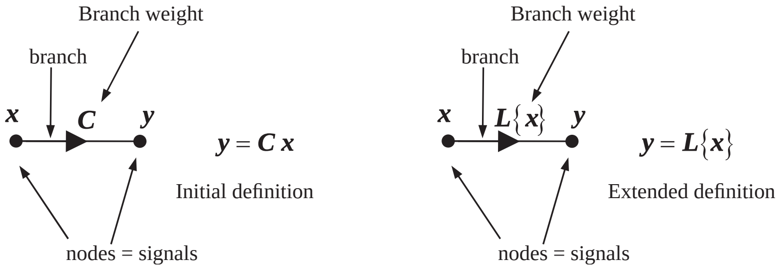


Figure B.7 Basic element of a Mason-graph: initial and extended definition.

The basic elements of a Mason-graph as depicted in Figure B.7 consist of nodes and a branch. The nodes are the variables and the branch represents a weighting. In our case, the nodes symbolise signals as e.g. x, y, which are affected by the branch (transmission path) following the relation:

$$y = C\,x \tag{B.56}$$

By definition, the Mason-graph can only respect multiplications and additions. We will extend this approach by allowing arbitrary linear operations:

$$y(t) = L\{x(t)\} \tag{B.57}$$

which covers the classical operations defined by (B.56) as e.g.

$$x(t) \xrightarrow{\;C\;} y(t) \qquad y(t) = C\,x(t)$$

$$\underline{X}(f) \xrightarrow{\;\underline{G}(f)\;} \underline{Y}(f) \qquad \underline{Y}(f) = \underline{G}(f)\,\underline{X}(f)$$

but also operations like:

$$x(t) \xrightarrow{\;\frac{d}{dt}\;} y(t) \qquad y = \frac{dx(t)}{dt} = u_1(t) * x(t)$$

$$x(t) \xrightarrow{\;\int dt\;} y(t) \qquad y = \int x(t)\,dt = u_{-1}(t) * x(t)$$

$$x(t) \xrightarrow{\;g(t)\;} y(t) \qquad y = g(t) * x(t)$$

Since the last three examples deviate from pure multiplication, attention has to be paid for the correct handling of a graph network in time domain. Here, multiplications have to be replaced by convolution and any division by time functions must be avoided. Table B-6 summarises some examples and calculation rules for fundamental Mason-graphs. These rules can be used to reduce successively complicated structured graphs to simpler ones.

Mason Rule

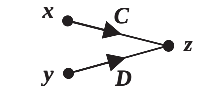
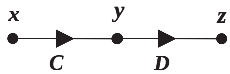
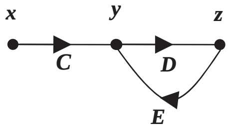
The transfer function between the arbitrary nodes x and y in a complex Mason graph can be calculated by [5]:

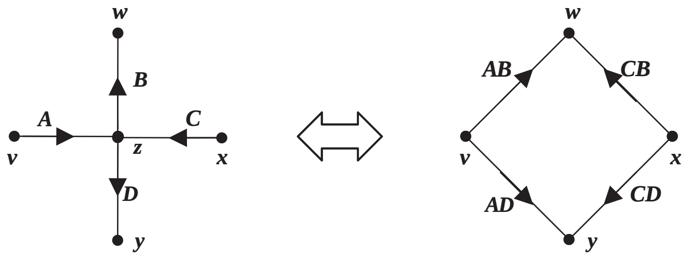
$$S_{yx} = \frac{\sum\limits_{\mu}\left(P_\mu\left(1 - \sum\limits_{k_\mu} L_{k_\mu}^{(1)} + \sum\limits_{l_\mu} L_{l_\mu}^{(2)} - \sum\limits_{m_\mu} L_{m_\mu}^{(3)} + \cdots\right)\right)}{1 - \sum\limits_{k} L_k^{(1)} + \sum\limits_{l} L_l^{(2)} - \sum\limits_{m} L_m^{(3)} + \cdots} \tag{B.58}$$

With

P_μ - FRF of the μ^{th} forward path between x and y

Table B-6 Fundamental structural elements of Mason-graphs and their algebraic expression in frequency and time domain.

Mason-graph	Frequency domain	Time domain
	$\underline{Z}(f)=\underline{C}(f)\,\underline{X}(f)+\underline{D}(f)\,\underline{Y}(f)$	$z(t)=C(t)*x(t)+D(t)*y(t)$
	$\underline{Z}(f)=\underline{C}(f)\,\underline{D}(f)\,\underline{Y}(f)$	$z(t)=C(t)*D(t)*x(t)$
	$\underline{Z}(f)=\dfrac{\underline{C}(f)\,\underline{D}(f)\,\underline{X}(f)}{1-\underline{D}(f)\,\underline{E}(f)}$ $=\underline{C}(f)\,\underline{D}(f)\,\underline{X}(f)\cdots$ $\cdots\sum_{n=0}^{\infty}\left(\underline{D}(f)\,\underline{E}(f)\right)^{n}$ See footnote[5]	$z(t)=C(t)*D(t)*x(t)\cdots$ $\cdots*\left[1+D(t)*E(t)+\cdots\right.$ $\left.\cdots D(t)*E(t)*D(t)*E(t)+\cdots\right]$



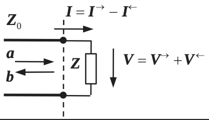
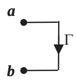
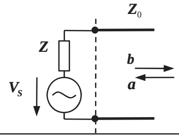
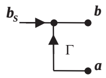
$L_k^{(1)},\ L_l^{(2)},\ L_m^{(3)},\cdots$ - FRF of all loops of first, second[6] and higher order.
$L_{k_\mu}^{(1)},\ L_{l_\mu}^{(2)},\ L_{m_\mu}^{(3)},\cdots$ - FRF of all loops which does not touch path μ

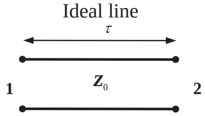
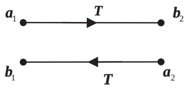
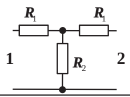
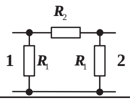
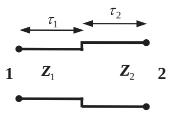
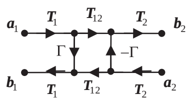
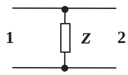
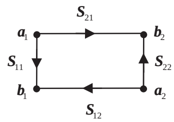
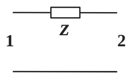
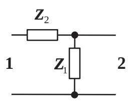
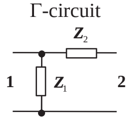
An example is given in chapter 4.9 Figure 4.84 and (4.293) and (4.294).

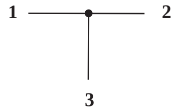
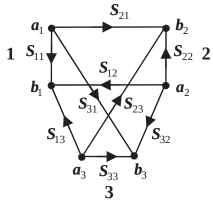
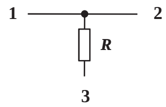
B.8
S-Parameters of Basic circuits

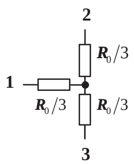
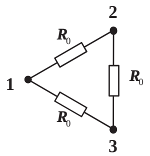
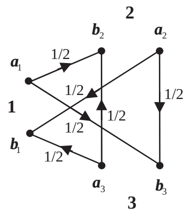
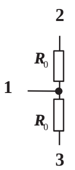
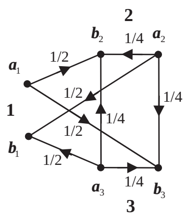
The scattering matrices are given in frequency domain notation. If the matrix entries are purely real, the notation is valid for both time and frequency domain.

5) The conversion exploits the identity $(1-x)^{-1}=\sum_{n=0} x^n;\ |x|<1$ which can be applied for open loop gain smaller one: $|\underline{D}(f)\,\underline{E}(f)|<1$.
6) A second order loop is a product of two first order loops which does not touch. A third order loop is a product of three non-touching loops, etc.

One port devices		
Sink, load 		$\underline{Z} = \frac{\underline{V}}{\underline{I}} = Z_0 \frac{\underline{a}+\underline{b}}{\underline{a}-\underline{b}}$ $\underline{\Gamma} = \frac{\underline{b}}{\underline{a}} = \frac{\underline{Z}-Z_0}{\underline{Z}+Z_0}$
Source 		$\underline{b} = \underline{b}_S + \underline{\Gamma}\,\underline{a}$ $\underline{b}_S = \frac{Z_0}{\underline{Z}+Z_0}\frac{\underline{V}_S}{\sqrt{Z_0}} = \frac{\underline{V}_S}{2\sqrt{Z_0}}(1-\underline{\Gamma})$ $\underline{\Gamma} = \frac{\underline{b}}{\underline{a}} = \frac{\underline{Z}-Z_0}{\underline{Z}+Z_0}$

Two port devices		
Ideal line 		$S = \begin{bmatrix} 0 & T \\ T & 0 \end{bmatrix}$ $\underline{T}(f) = e^{-j2\pi f\tau}$ - frequency domain $T(t) = \delta(t-\tau)$ - time domain $\tau = l/c$; c - propagation speed of the cable
Attenuator: T-circuit 		$S = \begin{bmatrix} 0 & T \\ T & 0 \end{bmatrix}$ $R_1 = Z_0 \frac{1-T}{1+T} \quad R_2 = Z_0 \frac{2T}{1-T^2}$
Attenuator: π-circuit 		$S = \begin{bmatrix} 0 & T \\ T & 0 \end{bmatrix}$ $R_1 = Z_0 \frac{1+T}{1-T} \quad R_2 = Z_0 \frac{1-T^2}{2T}$
Impedance step 		$S = \begin{bmatrix} T_1^2\Gamma & T_1T_{12}T_2 \\ T_1T_{12}T_2 & -T_2^2\Gamma \end{bmatrix}$ $\underline{T}_n(f) = e^{-j2\pi f\tau_n}; \quad n = 1,2$ $\underline{\Gamma} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$ $T_{12} = \frac{2\sqrt{Z_2\,Z_1}}{Z_2 + Z_1} = \sqrt{1-\Gamma^2}$
Shunt impedance 		$S = \frac{1}{2Z+Z_0}\begin{bmatrix} -Z_0 & 2Z \\ 2Z & -Z_0 \end{bmatrix}$
Series impedance 		$S = \frac{1}{2Z+Z_0}\begin{bmatrix} Z & 2Z_0 \\ 2Z_0 & Z \end{bmatrix}$
Γ^{-1}-circuit 		$S = \frac{1}{A}\begin{bmatrix} Z_1Z_2 + Z_0(Z_2 - Z_0) & 2Z_1Z_0 \\ 2Z_1Z_0 & Z_1Z_2 - Z_0(Z_2 + Z_0) \end{bmatrix}$ $A = (Z_1 + Z_0)(Z_2 + Z_0) + Z_1Z_0$
Γ-circuit 		$S = \frac{1}{A}\begin{bmatrix} Z_1Z_2 - Z_0(Z_2 + Z_0) & 2Z_1Z_0 \\ 2Z_1Z_0 & Z_1Z_2 + Z_0(Z_2 - Z_0) \end{bmatrix}$ $A = (Z_1 + Z_0)(Z_2 + Z_0) + Z_1Z_0$

<table>
<tr><th colspan="3">Three port devices</th></tr>
<tr>
<td>Tee-connection
</td>
<td rowspan="2"></td>
<td>$\mathbf{S} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$</td>
</tr>
<tr>
<td>Pick-off-Tee
</td>
<td>$\mathbf{S} = \frac{1}{3+2a}\begin{bmatrix} -1 & 2a+1 & 2 \\ 2a+1 & -1 & 2 \\ 2 & 2 & 2a-1 \end{bmatrix}$
with $a = R/Z_0$. Typical value
$S_{13} = S_{31} = S_{23} = S_{32} = 0{,}1$
$\Rightarrow \quad a = 17/2; \quad R = 425\ \Omega$

$\mathbf{S} = \begin{bmatrix} -0.05 & 0.9 & 0.1 \\ 0.9 & -0.05 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \approx \begin{bmatrix} 26 & 0.9 & 20 \\ 0.9 & 26 & 20 \\ 20 & 20 & 2 \end{bmatrix} \text{dB}$</td>
</tr>
<tr>
<td>Power divider – star circuit

Power divider – triangular circuit
</td>
<td></td>
<td>$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$</td>
</tr>
<tr>
<td>Power splitter
</td>
<td></td>
<td>$\mathbf{S} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$</td>
</tr>
</table>

Four port devices

A pair of coupled lines:

Supposing, the electric and magnetic fields of two lines are able to interact within a couple section of length l (see Figure B.8).

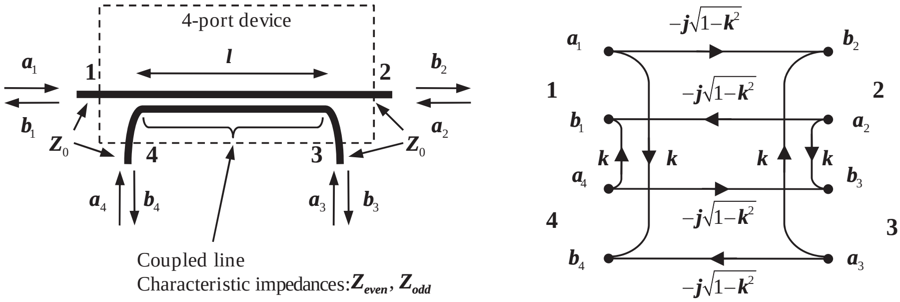


Figure B.8 Two coupled lines as four port device. Left: schematics. Right: idealised Mason-graph for a couple section of quarter wavelength.

Within that section the lines are characterised by two characteristic impedances depending on the type of feeding:

$$\begin{array}{rl} \text{Even mode impedance; common mode :} & a_1 = a_4 \quad \text{or} \quad a_2 = a_3 \quad \Rightarrow \quad Z_{even} \\ \text{Odd mode impedance; differential mode :} & a_1 = -a_4 \quad \text{or} \quad a_2 = -a_3 \quad \Rightarrow \quad Z_{odd} \end{array}$$

The four lines which feed the coupling section have the characteristic impedance Z_0

These three characteristic impedances are related by:

$$\begin{aligned} Z_0^2 &= Z_{even}\, Z_{odd} \\ k &= \frac{Z_{even} - Z_{odd}}{Z_{even} + Z_{odd}} \end{aligned} \tag{B.59}$$

where at k represents the coupling factor expressing the strength of mutual interaction.

If the length of the couple section equals a quarter wavelengths, the 4-port scattering matrix for an ideally symmetric coupled line may be written as:

$$\mathbf{S} = \begin{bmatrix} 0 & -j\sqrt{1-k^2} & 0 & k \\ j\sqrt{1-k^2} & 0 & k & 0 \\ 0 & k & 0 & j\sqrt{1-k^2} \\ k & 0 & j\sqrt{1-k^2} & 0 \end{bmatrix} \tag{B.60}$$

The corresponding Mason-graph is depicted in Figure B.8. Here, we will not further penetrate into the theory of coupled lines. More on this topic can be found e.g. in [6]. We are mainly interested in the specific structure of the scattering matrix which we will exploit in what follows.

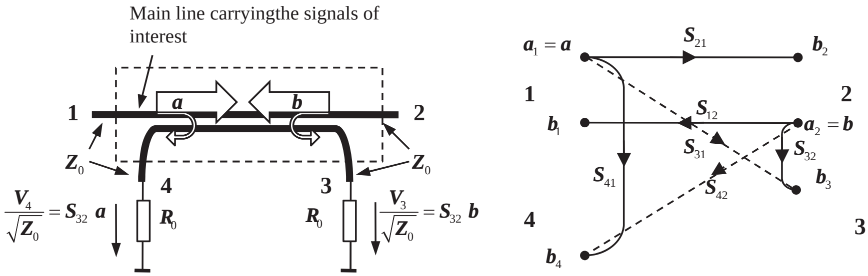


Figure B.9 Coupled line as directional coupler.

Coupled line as directional device:

As obvious from (B.60) or the Mason-graph of Figure B.8, the diagonal ports 1-3 and 2-4 are decoupled. We can take advantage of this behaviour to separate the waves at one of the lines. Supposing the line which connects port 1 and 2 is the main line from which we want to know the waves travelling toward right $a = a_1$ or toward left $b = a_2$.

If the ports 3 and 4 are matched with $R_0 = Z_0$, the Mason-graph degenerates to the structure depicted in Figure B.9 (bold lines only). The voltages across these resistors are proportional to the waves injected either in port 1 or 2. Typically one applies however only one receiver per directional coupler in order to reduces error signals due to mismatches at the ports 3 and 4, i.e. one of these ports is matched with a high quality resistor R_0 while the other one feeds the more imperfectly matched voltmeter. Consequently, two directional couplers are required to measure both waves. A further imperfection relates to the cross-talk of the unwanted wave to the measurements port. These signal paths are indicated by dashed lines in the Mason-graph of Figure B.9. Finally, Table B-7 summarises the definition of the most important parameter of a directional coupler. Here, we supposed that the measurement device is connected to port 4.

Table B-7 Important parameters of a directional coupler (port 4 is the measurement port).

	Definition	**Ideal device**
Coupling factor	$CF = -20lg\|S_{41}\|$	$CF = -20lg\,k$
Insertion loss	$IL = -20lg\|S_{21}\|$	$IL = -10lg(1-k^2)$
Return loss	$RL = -20lg\|S_{ii}\|$	∞
Isolation	$IS = -20lg\|S_{42}\|$	∞
Directivity	$D = -20lg\left\|\frac{S_{42}}{S_{21}S_{41}}\right\| = IS - IL - CF$	∞

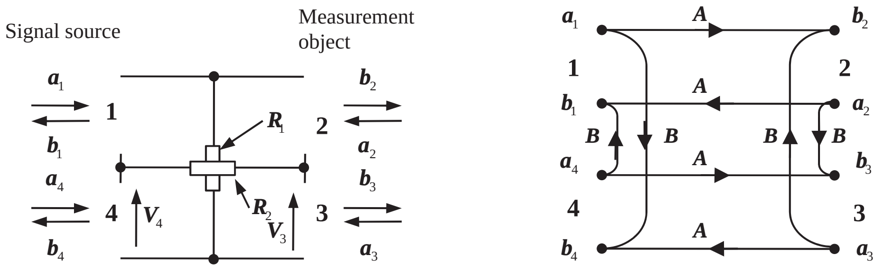


Figure B.10 Circuit schematic and Mason-graph of a resistive coupler. See (B.61) for definition of the S-parameters A and B.

The scattering matrix (B.60) is exactly valid for only one frequency. However, a simple line coupler can be applied over approximately one octave without dramatic loss of its performance. For wideband operation, the coupler must have several couple sections. The lower cut-off frequency determines its overall mechanical dimensions.

Resistive coupler:

The resistive coupler as shown in Figure B.10 actually represents a Wheatstone bridge. This can easily be seen by adding the external circuit elements and redrawing the schematic (compare Figure B.11). The resistive coupler is not frequency selective by principle (parasitic effects neglected) since it contains only ohmic resistors. Though it requires a floating ground measurement of V_3 and V_4. Recently, this is done via transformers which finally yet limit the bandwidth of the overall device.

For $Z_0^2 = R_1 R_2$, the scattering matrix results to:

$$\mathbf{S} = \frac{1}{Z_0 + R_2}\begin{bmatrix} 0 & Z_0 & 0 & R_2 \\ Z_0 & 0 & R_2 & 0 \\ 0 & R_2 & 0 & Z_0 \\ R_2 & 0 & Z_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A & 0 & B \\ A & 0 & B & 0 \\ 0 & B & 0 & A \\ B & 0 & A & 0 \end{bmatrix} \tag{B.61}$$

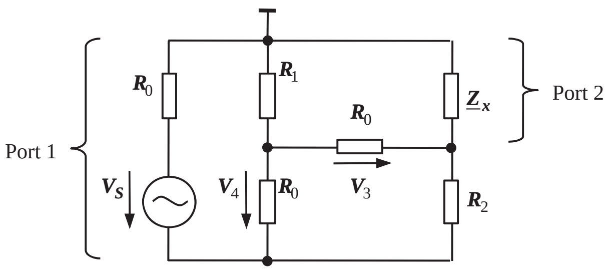


Figure B.11 Resistive coupler drawn as Wheatstone bridge.

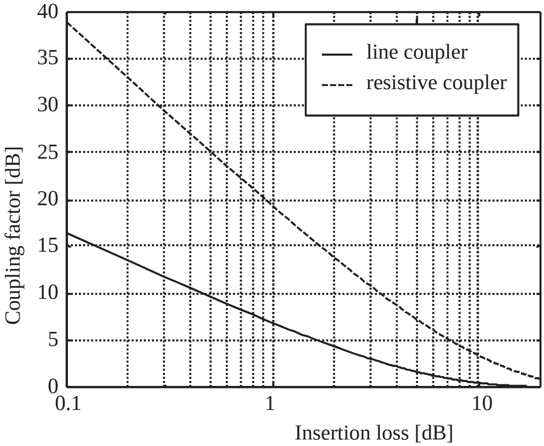


Figure B.12 Coupling factor versus insertion loss for line and resistive couplers.

Compared with the line coupler, the coupling factor of the resistive coupler is worsening due to its internal losses. Figure B.12 compares both types of couplers.

B.9
M-Sequence and Golay-Sequence

B.9.1
M-Sequence

Table B-8 summarises the feedback taps for shift registers of different length which provide an M-sequence. The whole set of possible feedback structures are only given till M-sequences of order nine. For the orders above, only one version (typically with the lowest number of taps) is indicated.

Table B-8 Selection of feedback structure for M-sequence shift register.

Order	Number of chips	Prime factorisation	Number of sets	Feedback taps
3	7	7	1	3, 2
4	15	3·5	1	4, 3
5	31	31	3	5, 3
				5, 4, 3, 2
				5, 4, 3, 1
6	63	3·3·7	3	6, 5
				6, 5, 4, 1
				6, 5, 3, 2
7	127	127	9	7, 6
				7, 4

(*continued*)

Table B.8 (*Continued*)

Order	Number of chips	Prime factorisation	Number of sets	Feedback taps
				7, 6, 5, 4
				7, 6, 5, 2
				7, 6, 4, 2
				7, 6, 4, 1
				7, 5, 4, 3
				7, 6, 5, 4, 3, 2
				7. 6, 5, 4, 2, 1
8	255	3·5·17	8	8, 7, 6, 1
				8, 7, 5, 3
				8, 7, 3, 2
				8, 6, 5, 4
				8, 6, 5, 3
				8, 6, 5, 2
				8, 7, 6, 5, 4, 2
				8, 7, 6, 5, 2, 1
9	511	7·31·73	24	9, 5
				9, 8, 7, 2
				9, 8, 6, 5
				9, 8, 5, 4
				9, 8, 5, 1
				9, 8, 4, 2
				9, 7, 6, 4
				9, 7, 5, 2
				9, 6, 5, 3
				9, 8, 7, 6, 5, 3
				9, 8, 7, 6, 5, 1
				9, 8, 7, 6, 4, 3
				9, 8, 7, 6, 4, 2
				9, 8, 7, 6, 3, 2
				9, 8, 7, 6, 3, 1
				9, 8, 7, 6, 2, 1
				9, 8, 7, 5, 4, 3
				9, 8, 7, 5, 4, 2
				9, 8, 6, 5, 4, 1
				9, 8, 6, 5, 3, 2
				9, 8, 6, 5, 3, 1
				9, 7, 6, 5, 4, 3
				9, 7, 6, 5, 4, 2
				9, 8, 7, 6, 5, 4, 3, 1
10	1,023	3·11·31	30	e.g. 10, 7
11	2,047	23·89	88	e.g. 11, 9
12	4,095	3·3·5·7·13	72	e.g. 12, 11, 10, 2
13	8,191	8,191	315	e.g. 13, 12, 11, 8

(continued)

Table B.8 (*Continued*)

Order	Number of chips	Prime factorisation	Number of sets	Feedback taps
14	16,383	3·43·127	376	e.g. 14, 13, 12, 2
15	32,767	7·31·151	900	e.g. 15, 14
16	65,535	3·5·17·257	1032	e.g. 16, 15, 13, 4
17	131,071	131,071	1941	e.g. 17, 14
18	262,143	3·3·3·7·19·73	1544	e.g. 18, 11
19	524,287	524,287	4314	e.g. 19, 18, 17, 14
20	1,048,575	3·5·5·11·31·41	2864	e.g. 20, 17

Supposing the stimulation of a LTI-system having the IRF $g(t)$ with the M-sequence $m(t)$ leads to the system reaction

$$y(t) = g(t) * m(t). \tag{B.62}$$

Involving correlation functions, the corresponding relation reads:

$$C_{ym}(t) = g(t) * C_{mm}(t) \tag{B.63}$$

where at we can find from Figure 2.33 that

$$C_{mm}(t) = V^2\left(\frac{N+1}{N}\operatorname{tri}\left(\frac{t}{t_c}\right) - \frac{1}{N}\right). \tag{B.64}$$

Hence, we get for the cross-correlation from stimulus and system reaction

$$C_{ym}(t) = V^2\frac{N+1}{N}\, g(t) * \operatorname{tri}\left(\frac{t}{t_c}\right) - \frac{V^2}{N}\,\overline{g(t)}. \tag{B.65}$$

It consists of two terms. The second one comes from the convolution with a constant. It provides a constant value which is proportional to the mean value of the IRF. It will however be mostly negligible since we are considering usually AC-coupled systems (e.g. any DC transmission between two antennas) for which hold $\overline{g(t)} = 0$. Furthermore, the DC-component of a measurement is often affected by an offset-value so that its quantity is less reliable.

We are mainly interested in the first term since it contains the wanted IRF $g(t)$ of the system under test. Therefore, we will investigate the influence of the triangular function onto the convolution product:

$$z(t) = g(t) * \operatorname{tri}\left(\frac{t}{t_c}\right) = \int g(t+\tau)\operatorname{tri}\left(\frac{\tau}{t_c}\right) d\tau. \tag{B.66}$$

For that purpose, we decompose the IRF in a Taylor-series

$$g(t+\tau) = \sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^n g(\xi)}{d\xi^n}\right|_{\xi=t}\tau^n \tag{B.67}$$

so that we yield:

$$z(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n g(\xi)}{d\xi^n}\bigg|_{\xi=t} \int \tau^n \operatorname{tri}\left(\frac{\tau}{t_c}\right) d\tau = \sum_{n=0}^{\infty} \frac{2}{(n+2)!} \frac{d^n g(\xi)}{d\xi^n}\bigg|_{\xi=t} t_c^{n+1} \tag{B.68}$$

for $n =$ even. Components with $n =$ odd do not contribute to $z(t)$. If the maximum curvature of $g(t)$ within the duration $2\,t_c$ of the triangular function is weak (i.e. the spectrum of $g(t)$ must be narrower than that of the triangular function), the higher order terms in (B.68) may be neglected so that we can finally write for the correlation function of an AC-coupled system:

$$C_{ym}(t) \approx V^2\, t_c\, g(t) \quad \text{for} \quad n = 0 \tag{B.69}$$

B.9.2
Complementary Golay-Sequence

Golay-sequences appear in pairs. Assuming $a(t)$ and $b(t)$ represent a pair of such sequences, then the sum of their auto-correlation functions $C_{aa}(t)$ and $C_{bb}(t)$ results by definition in a triangular function, i.e.:

$$C_{GG}(t) = \frac{C_{aa}(t) + C_{bb}(t)}{2} = V^2 \operatorname{tri}\left(\frac{t}{t_c}\right) \tag{B.70}$$

that is the main lobe of both auto-correlation functions superimpose where at the sidelobes mutually cancel out.

Figure B.13 depicts an example for a Golay-sequence of length 2^5.

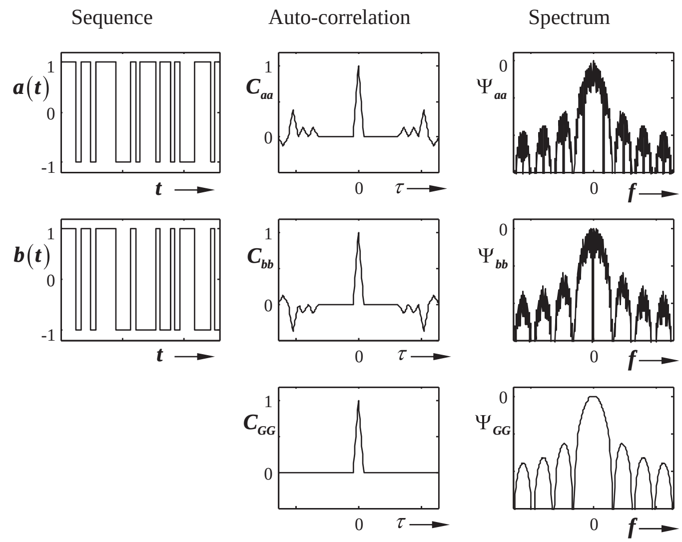


Figure B.13 Example of periodic Golay-sequences. Only one period is shown.

Starting from a pair of primitive Golay-sequences $a_1(t)$, $b_1(t)$, higher order sequences may be constructed via the recursive rule:

$$\begin{aligned} a_{n+1}(t) &= [\, a_n(t) \quad b_n(t) \,] \\ b_{n+1}(t) &= [\, a_n(t) \quad -b_n(t) \,] \end{aligned} \tag{B.71}$$

where at the simplest primitives are given by:

$$\begin{aligned} a_1(t) &= [\, 1 \quad 1 \,] \\ b_1(t) &= [\, 1 \quad -1 \,] \end{aligned} \tag{B.72}$$

Other primitives and more details on construction rules can be found in [7], [8], [9].

The measurement of the IRF by Golay-sequences is a two step procedure. First, the system under test is stimulated by the sequence $a(t)$ which can also be done repetitively in order to perform averaging. This gives us

$$C_{ya}(t) = g(t) * C_{aa}(t). \tag{B.73}$$

The second measurement is done with the sequence $b(t)$ by the same way, i.e.

$$C_{yb}(t) = g(t) * C_{bb}(t) \tag{B.74}$$

so that we can write finally:

$$\begin{aligned} C_{yG}(t) &= \frac{C_{ya}(t) + C_{yb}(t)}{2} = g(t) * C_{GG}(t) \\ &= V^2\, g(t) * \operatorname{tri}\left(\frac{t}{t_c}\right) \approx t_c\, V^2\, g(t) \end{aligned} \tag{B.75}$$

Disregarding the DC-component, (B.75) and (B.65) lead to the same results for sequences of comparable length where at the Golay-approach will (at least theoretically neglecting any DC-offset of the measurements) provide also the correct DC-behaviour of the DUT.

Since the measurement with Golay-sequences takes twice the time of an M-sequence, we have normalised the sum functions by the factor $1/2$ in order to be comparable. Basically, the considerations in chapter 3.3 with respect to M-sequences also hold for Golay-sequences. However, Golay-sequences cannot be generated by simple feedback-shift registers so that they have to be readout from a fast memory and their number of chips is always even. Thus in case of interleaved sampling, the sampling clock must be provided by a $2^m - 1$ or a $2^m + 1$ divider (refer to Figure 3.35) if Golay-sequences of length 2^n are applied.

Annex C: Electromagnetic Field

C.1
Time Domain Reciprocity Relation

We will ask for the mutual relation between two electromagnetic fields $[\mathbf{E}_1(\mathbf{r}),\ \mathbf{H}_1(\mathbf{r})]$ and $[\mathbf{E}_2(\mathbf{r}),\ \mathbf{H}_2(\mathbf{r})]$ caused from the source distributions $\mathbf{J}_1(\mathbf{r}_1)$ or

$\mathbf{J}_2(\mathbf{r}_2)$. For that purpose, we relate the fields created by source 1 at the position of source 2, i.e. $[\mathbf{E}_1(\mathbf{r}_2),\ \mathbf{H}_1(\mathbf{r}_2)]$, to the fields created by source 2 at position of source 1, i.e. $[\mathbf{E}_2(\mathbf{r}_1),\ \mathbf{H}_2(\mathbf{r}_1)]$. Figure 5.2 illustrates the considered scenario for two point sources.

The so-called Lorentz reciprocity gives us the wanted relation between the two fields. In the literature, it is usually written in frequency domain notation by an integral or differential form:

$$\begin{aligned}\oint_S \left(\underline{\mathbf{E}}_1(\mathbf{r}_2)\times\underline{\mathbf{H}}_2(\mathbf{r}_1)-\underline{\mathbf{E}}_2(\mathbf{r}_1)\times\underline{\mathbf{H}}_1(\mathbf{r}_2)\right)\cdot \mathrm{d}\mathbf{A} &= \int_V \left(\underline{\mathbf{J}}_1(\mathbf{r}_1)\cdot\underline{\mathbf{E}}_2(\mathbf{r}_1)-\underline{\mathbf{E}}_1(\mathbf{r}_2)\cdot\underline{\mathbf{J}}_2(\mathbf{r}_2)\right)\mathrm{d}V\\ \nabla\cdot\left(\underline{\mathbf{E}}_1(\mathbf{r}_2)\times\underline{\mathbf{H}}_2(\mathbf{r}_1)-\underline{\mathbf{E}}_2(\mathbf{r}_1)\times\underline{\mathbf{H}}_1(\mathbf{r}_2)\right) &= \underline{\mathbf{J}}_1(\mathbf{r}_1)\cdot\underline{\mathbf{E}}_2(\mathbf{r}_1)-\underline{\mathbf{E}}_1(\mathbf{r}_2)\cdot\underline{\mathbf{J}}_2(\mathbf{r}_2)\end{aligned} \tag{C.1}$$

One can show[7] that the surface integral vanishes in the case of localised sources (i.e. sources of final dimension) and if the integration volume covers all sources. Thus, (C.1) simplifies to the form which is important for our purposes:

$$\int_V \left(\underline{\mathbf{J}}_1(\mathbf{r}_1)\cdot\underline{\mathbf{E}}_2(\mathbf{r}_1)-\underline{\mathbf{E}}_1(\mathbf{r}_2)\cdot\underline{\mathbf{J}}_2(\mathbf{r}_2)\right)\mathrm{d}V = 0. \tag{C.2}$$

A reciprocity theorem for arbitrary time-dependent fields has been given in [10], [11], [12] [8] by:

$$\begin{aligned}&\int\int_V \left(\mathbf{J}_1(t,\mathbf{r}_1)\cdot\mathbf{E}_2(t-\tau,\mathbf{r}_1)-\mathbf{E}_1(t+\tau,\mathbf{r}_2)\cdot\mathbf{J}_2(t,\mathbf{r}_2)\right)\mathrm{d}V\mathrm{d}t = 0\\ &\text{with}\quad \tau=\frac{|\mathbf{r}_1-\mathbf{r}_2|}{c}\end{aligned} \tag{C.3}$$

Note that (C.3) represents a correlation as function of τ. The relation supposes however retarded ($\mathbf{E}_2$) as well as “advanced” fields ($\mathbf{E}_1$). Advanced field contradicts the causality. This may be acceptable for field simulations and theoretical purposes but for understanding the behaviour of measurement scenario causality should be respected. Furthermore, the reciprocity relation (C.3) is restricted to lossless field propagation as we will see later (refer also to [10], [12]).

To be generic, we will introduce another form of the Lorentz reciprocity which we can simply gain from the conversion of the well accepted frequency domain relation (C.2) into the time domain by Fourier transform:

$$\begin{gathered}\int_V \left(\underline{\mathbf{J}}_1(f,\mathbf{r}_1)\cdot\underline{\mathbf{E}}_2(f,\mathbf{r}_1)-\underline{\mathbf{E}}_1(f,\mathbf{r}_2)\cdot\underline{\mathbf{J}}_2(f,\mathbf{r}_2)\right)\mathrm{d}V = 0\\ \downarrow\ FT\\ \int_V \left(\mathbf{J}_1(t,\mathbf{r}_1)*\mathbf{E}_2(t,\mathbf{r}_1)-\mathbf{E}_1(t,\mathbf{r}_2)*\mathbf{J}_2(t,\mathbf{r}_2)\right)\mathrm{d}V = 0\end{gathered} \tag{C.4}$$

7) The simplest way to show this is by supposing small propagation losses. In this case, the fields decay exponentially with distance and the surface integral vanishes for a large diameter of S.

8) Relation (C.3) actually refers to the last of the three references which uses the difference of the two integral expressions. In the two other references, the sum of both terms is used which is less common to the more familiar Lorentz reciprocity relation (C.2).

Proof:

From Maxwell's equations we get the following:

We suppose that a source $\mathbf{J}_1$ located at position $\mathbf{r}_1$ causes fields at position $\mathbf{r}_2$ according to:

$$\nabla \times \mathbf{H}_1(t, \mathbf{r}_2) = \frac{\partial \mathbf{D}_1(t, \mathbf{r}_2)}{\partial t} + \mathbf{J}_1(t, \mathbf{r}_1) \tag{C.5}$$

$$\nabla \times \mathbf{E}_1(t, \mathbf{r}_2) = -\frac{\partial \mathbf{B}_1(t, \mathbf{r}_2)}{\partial t}. \tag{C.6}$$

Correspondingly, we can assume that a source $\mathbf{J}_2$ located at position $\mathbf{r}_2$ causes the fields in the point $\mathbf{r}_1$ related to:

$$\nabla \times \mathbf{H}_2(t, \mathbf{r}_1) = \frac{\partial \mathbf{D}_2(t, \mathbf{r}_1)}{\partial t} + \mathbf{J}_2(t, \mathbf{r}_2) \tag{C.7}$$

$$\nabla \times \mathbf{E}_2(t, \mathbf{r}_1) = -\frac{\partial \mathbf{B}_2(t, \mathbf{r}_1)}{\partial t} \tag{C.8}$$

Introducing the exterior-convolution operator $\boxtimes$ as

$$\mathbf{A}(t) \boxtimes \mathbf{B}(t) = \int \mathbf{A}(\tau) \times \mathbf{B}(t-\tau) \mathrm{d}\tau,$$

the dot-convolution operator $*$ as

$$\mathbf{A}(t) * \mathbf{B}(t) = \int \mathbf{A}(\tau) \cdot \mathbf{B}(t-\tau) \mathrm{d}\tau,$$

using the vector identity $-\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{A})$ and convolving (C.5) to (C.8) with $\mathbf{E}_2,\ \mathbf{H}_2,\ \mathbf{E}_1$ respectively $\mathbf{H}_1$ lead to:

$$\mathbf{E}_1 * \nabla \times \mathbf{H}_2 - \mathbf{H}_2 * \nabla \times \mathbf{E}_1 = -\nabla \cdot (\mathbf{E}_1 \boxtimes \mathbf{H}_2) = \mathbf{E}_1 * \mathbf{J}_2 + \mathbf{E}_1 * \frac{\partial \mathbf{D}_2}{\partial t} + \mathbf{H}_2 * \frac{\partial \mathbf{B}_1}{\partial t}$$

$$\mathbf{E}_2 * \nabla \times \mathbf{H}_1 - \mathbf{H}_1 * \nabla \times \mathbf{E}_2 = -\nabla \cdot (\mathbf{E}_2 \boxtimes \mathbf{H}_1) = \mathbf{E}_2 * \mathbf{J}_1 + \mathbf{E}_2 * \frac{\partial \mathbf{D}_1}{\partial t} + \mathbf{H}_1 * \frac{\partial \mathbf{B}_2}{\partial t}$$

For shortness, we have omitted the arguments $\mathbf{r}$ and t in above expressions. After subtraction of both equations we have:

$$\begin{aligned} &-\nabla \cdot (\mathbf{E}_1 \boxtimes \mathbf{H}_2 - \mathbf{E}_2 \boxtimes \mathbf{H}_1) \\ &\quad = (\mathbf{E}_1 * \mathbf{J}_2 - \mathbf{E}_2 * \mathbf{J}_1) + \left(\mathbf{E}_1 * \frac{\partial \mathbf{D}_2}{\partial t} - \mathbf{E}_2 * \frac{\partial \mathbf{D}_1}{\partial t} \right) + \left(\mathbf{H}_2 * \frac{\partial \mathbf{B}_1}{\partial t} - \mathbf{H}_1 * \frac{\partial \mathbf{B}_2}{\partial t} \right) \end{aligned} \tag{C.9}$$

The two bracket terms on the right of (C.9) still have to be considered in detail. Firstly, we can write for two time functions $a(t)$ and $b(t)$:

$$a(t) * \frac{\partial}{\partial t} b(t) = \frac{\partial}{\partial t} a(t) * b(t) = \frac{\partial}{\partial t} (a(t) * b(t))$$

$$a(t) * (u_1(t) * b(t)) = (u_1(t) * a(t)) * b(t) = u_1(t) * (a(t) * b(t))'$$

which immediately results from doublet notation or the basic rules of Fourier transform and the commutative law of convolution (see also Table B-4). Secondly, the relation

$$\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}=\mathbf{C}\cdot\mathbf{B}\cdot\mathbf{A}$$

holds between the two vectors **A** and **C** if the matrix $\mathbf{B}=\mathbf{B}^T$ is a symmetric one.

Hence, under the conditions (which are the prerequisites of reciprocity) that

- the tensors of material parameters are symmetric (which is always the case for isotropic materials), i.e. $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^T;\ \boldsymbol{\mu}=\boldsymbol{\mu}^T$,
- the material parameters are independent from the strength of the fields (i.e. all substances behave linearly), i.e. $\boldsymbol{\varepsilon}\neq\boldsymbol{\varepsilon}(\mathbf{E},\mathbf{H},\mathbf{J});\ \boldsymbol{\mu}\neq\boldsymbol{\mu}(\mathbf{E},\mathbf{H},\mathbf{J})$
- the material is homogeneously distributed in space, i.e. $\boldsymbol{\varepsilon}(\mathbf{r})=\boldsymbol{\varepsilon};\ \boldsymbol{\mu}(\mathbf{r})=\boldsymbol{\mu}$ (later, this condition will be dropped), and
- the properties of the considered space are time independent[9] (only with respect to observation time), i.e. $\boldsymbol{\varepsilon}(t,T_R)=\boldsymbol{\varepsilon}(t);\ \boldsymbol{\mu}(t,T_R)=\boldsymbol{\mu}(t)$,

we can write:

$$\left(\mathbf{E}_1*\frac{\partial\mathbf{D}_2}{\partial t}-\mathbf{E}_2*\frac{\partial\mathbf{D}_1}{\partial t}\right)=\frac{\partial}{\partial t}(\mathbf{E}_1*\boldsymbol{\varepsilon}*\mathbf{E}_2-\mathbf{E}_2*\boldsymbol{\varepsilon}*\mathbf{E}_1)=\frac{\partial}{\partial t}\left(\mathbf{E}_1*\boldsymbol{\varepsilon}*\mathbf{E}_2-\mathbf{E}_1*\boldsymbol{\varepsilon}^T*\mathbf{E}_2\right)=\mathbf{0}$$

$$\left(\mathbf{H}_2*\frac{\partial\mathbf{B}_1}{\partial t}-\mathbf{H}_1*\frac{\partial\mathbf{B}_2}{\partial t}\right)=\frac{\partial}{\partial t}(\mathbf{H}_2*\boldsymbol{\mu}*\mathbf{H}_1-\mathbf{H}_1*\boldsymbol{\mu}*\mathbf{H}_2)=\frac{\partial}{\partial t}\left(\mathbf{H}_2*\boldsymbol{\mu}*\mathbf{H}_1-\mathbf{H}_2*\boldsymbol{\mu}^T*\mathbf{H}_1\right)=\mathbf{0}$$

If we switch to the integral representation of (C.9) (applying Gauss' divergence theorem) and supposing that all sources are covered by the considered integration volume, we end up in:

$$-\oint_S(\mathbf{E}_1\boxtimes\mathbf{H}_2-\mathbf{E}_2\boxtimes\mathbf{H}_1)\cdot\mathrm{d}\mathbf{A}=\int_V\left(\mathbf{E}_1*\mathbf{J}_2-\mathbf{E}_2*\mathbf{J}_1\right)\mathrm{d}V=0 \qquad \text{(C.10)}$$

what we already expected from (C.4). Here, we made again use of the fact that the surface integral tends to zero if we consider a sufficiently large region (see also remarks concerning (C.1) and (C.2))

However with respect to (C.4), we did not require a homogenous propagation medium as we did above under (C.10). But actually, we can drop this requirement because we can divide an inhomogeneous propagation scenario in several homogenous parts on which a modified consideration may be stepwise extended which also respects boundaries. Such an exercise is to be found e.g. in [13]. Further readings on reciprocity are given e.g. in [14].

9) Time independence does not exclude relaxation effects etc. It only refers to "long time effects", i.e. the substances exposed by the field must not change their properties during the recording time of any signals or fields.

Finally, we will repeat the above exercise via a second procedure in which we convolve (C.5) to (C.8) with time reversed versions of $\mathbf{E}_2,\ \mathbf{H}_2,\ \mathbf{E}_1$respectively $\mathbf{H}_1$ by following schematic, i.e. we are performing correlation:

$$
\begin{aligned}
\nabla \times \mathbf{H}_1(t, \mathbf{r}_2) &= \frac{\partial \mathbf{D}_1(t, \mathbf{r}_2)}{\partial t} + \mathbf{J}_1(t, \mathbf{r}_1) \quad \Bigg| * \mathbf{E}_2(-t, \mathbf{r}_1) \\
\nabla \times \mathbf{E}_1(t, \mathbf{r}_2) &= -\frac{\partial \mathbf{B}_1(t, \mathbf{r}_2)}{\partial t} \quad \Bigg| * \mathbf{H}_2(-t, \mathbf{r}_1) \\
\nabla \times \mathbf{H}_2(-t, \mathbf{r}_1) &= \frac{\partial \mathbf{D}_2(-t, \mathbf{r}_1)}{\partial t} + \mathbf{J}_2(-t, \mathbf{r}_2) \quad \Bigg| * \mathbf{E}_1(t, \mathbf{r}_2) \\
\nabla \times \mathbf{E}_2(-t, \mathbf{r}_1) &= -\frac{\partial \mathbf{B}_2(-t, \mathbf{r}_1)}{\partial t} \quad \Bigg| * \mathbf{H}_1(t, \mathbf{r}_2)
\end{aligned}
$$

An equivalent calculation as before, leads us to:

$$
\begin{aligned}
&-\nabla \cdot (\mathbf{E}_1(t, \mathbf{r}_2) \boxtimes \mathbf{H}_2(-t, \mathbf{r}_1) - \mathbf{E}_2(-t, \mathbf{r}_1) \boxtimes \mathbf{H}_1(t, \mathbf{r}_2)) \\
&= \mathbf{E}_1(t, \mathbf{r}_2) * \mathbf{J}_2(-t, \mathbf{r}_2) - \mathbf{E}_2(-t, \mathbf{r}_1) * \mathbf{J}_1(t, \mathbf{r}_1) + \frac{\partial}{\partial t}(x(t) + y(t))
\end{aligned}
$$

where at the scalar functions $x(t)$ and $y(t)$ are:

$$
\begin{aligned}
x(t) &= \mathbf{E}_1(t, \mathbf{r}_2) * \mathbf{D}_2(-t, \mathbf{r}_1) - \mathbf{E}_2(-t, \mathbf{r}_1) * \mathbf{D}_1(t, \mathbf{r}_2) \\
&= \mathbf{E}_1(t, \mathbf{r}_2) * \boldsymbol{\varepsilon}(-t, \mathbf{r}_1) * \mathbf{E}_2(-t, \mathbf{r}_1) - \mathbf{E}_2(-t, \mathbf{r}_1) * \boldsymbol{\varepsilon}(t, \mathbf{r}_2) * \mathbf{E}_1(t, \mathbf{r}_2) \\
&= \mathbf{E}_1(t, \mathbf{r}_2) * \boldsymbol{\varepsilon}(-t, \mathbf{r}_1) * \mathbf{E}_2(-t, \mathbf{r}_1) - \mathbf{E}_1(t, \mathbf{r}_2) * \boldsymbol{\varepsilon}^T(t, \mathbf{r}_2) * \mathbf{E}_2(-t, \mathbf{r}_1)
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \mathbf{H}_2(-t, \mathbf{r}_1) * \mathbf{B}_1(t, \mathbf{r}_2) - \mathbf{H}_1(t, \mathbf{r}_2) * \mathbf{B}_2(-t, \mathbf{r}_1) \\
&= \mathbf{H}_2(-t, \mathbf{r}_1) * \boldsymbol{\mu}(t, \mathbf{r}_2) * \mathbf{H}_1(t, \mathbf{r}_2) - \mathbf{H}_1(t, \mathbf{r}_2) * \boldsymbol{\mu}(-t, \mathbf{r}_1) * \mathbf{H}_2(-t, \mathbf{r}_1) \\
&= \mathbf{H}_2(-t, \mathbf{r}_1) * \boldsymbol{\mu}(t, \mathbf{r}_2) * \mathbf{H}_1(t, \mathbf{r}_2) - \mathbf{H}_2(-t, \mathbf{r}_1) * \boldsymbol{\mu}^T(-t, \mathbf{r}_1) * \mathbf{H}_1(t, \mathbf{r}_2)
\end{aligned}
$$

Both expressions equal zero if the following holds:

$$
\begin{aligned}
\boldsymbol{\varepsilon}(t, \mathbf{r}_1) = \boldsymbol{\varepsilon}^T(-t, \mathbf{r}_2) \quad &\underset{IFT}{\overset{FT}{\rightleftarrows}} \quad \underline{\boldsymbol{\varepsilon}}(f, \mathbf{r}_1) = \underline{\boldsymbol{\varepsilon}}^H(f, \mathbf{r}_2) \\
\boldsymbol{\mu}(t, \mathbf{r}_2) = \boldsymbol{\mu}^T(-t, \mathbf{r}_1) \quad &\underset{IFT}{\overset{FT}{\rightleftarrows}} \quad \underline{\boldsymbol{\mu}}(f, \mathbf{r}_2) = \underline{\boldsymbol{\mu}}^H(f, \mathbf{r}_1)
\end{aligned}
$$

That means, that beside the already mentioned conditions, the frequency domain material parameters must be additionally purely real (i.e. lossless) which furthermore involves that material parameters must also be frequency independent in order to meet causality (i.e. $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ have to respect (2.150) – Kramers-Kronig relation – which leads to a constant real part if the imaginary part must be zero). Hence, the resulting reciprocity relation

$$
\begin{aligned}
&-\oint_S (\mathbf{E}_1(t, \mathbf{r}_2) \boxtimes \mathbf{H}_2(-t, \mathbf{r}_1) - \mathbf{E}_2(-t, \mathbf{r}_1) \boxtimes \mathbf{H}_1(t, \mathbf{r}_2)) \cdot \mathrm{d}\mathbf{A} \\
&\qquad = \int_V (\mathbf{E}_1(t, \mathbf{r}_2) * \mathbf{J}_2(-t, \mathbf{r}_2) - \mathbf{E}_2(-t, \mathbf{r}_1) * \mathbf{J}_1(t, \mathbf{r}_1))\, \mathrm{d}V = 0
\end{aligned}
\tag{C.11}
$$

is restricted to lossless propagation which exactly holds only for vacuum. Note that above relation is nothing but the generalised case of (C.3) for arbitrary time lag τ. The frequency domain counterpart of (C.11) follows from Fourier transform to:

$$
-\oint_S \left(\underline{\mathbf{E}}_1 \times \underline{\mathbf{H}}_2^* - \underline{\mathbf{E}}_2^* \times \underline{\mathbf{H}}_1\right) \cdot \mathrm{d}\mathbf{A} = \int_V \left(\underline{\mathbf{J}}_2^* \cdot \underline{\mathbf{E}}_1 - \underline{\mathbf{E}}_2^*(\mathbf{r}_2) \cdot \underline{\mathbf{J}}_1\right) \mathrm{d}V = 0 \tag{C.12}
$$

For completeness, we should also mention that the derivation of (C.10) indeed includes propagation losses due to relaxation phenomena (expressed by dependency from propagation time $\boldsymbol{\varepsilon}(t)$) but we did not yet explicitly include losses by conductivity $\boldsymbol{\sigma}$. It is a usual praxis for time harmonic fields to join permittivity and conductivity by $\underline{\boldsymbol{\varepsilon}}_\sigma = \underline{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}/j2\pi f$ in which $\underline{\boldsymbol{\varepsilon}}$ only respects relaxation effects. For the time domain, this results in the operation $\boldsymbol{\varepsilon}_\sigma(t) * \cdots = (\boldsymbol{\varepsilon}(t) + u_{-1}(t) * \boldsymbol{\sigma}(t)) * \cdots$ if we apply doublet notation. Insertion in (C.9), let us come to comparable conclusions for $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ with respect to the reciprocity relation (C.10) as above.

C.2
Scattering of Plane Waves at a Planar Interface

We consider the scenario as illustrated in Figure C.1 which symbolises the scattering of a pulse shaped plane wave at a planar interface between two dispersion less dielectric media having the intrinsic impedance $Z_n = \sqrt{\mu_0/\varepsilon_n}$ and the propagation speed $c_n = 1/\sqrt{\mu_0 \varepsilon_n}$.

We write the incident wave in the normalised form:

$$\mathbf{W}_i(t, \mathbf{r}) = A_i w_0 \left(t - \frac{\mathbf{e}_i \cdot \mathbf{r}}{c_1} \right) * \mathbf{q}_i(t) \tag{C.13}$$

in which $\mathbf{e}_i$ and $\mathbf{q}_i$ are the propagation direction and polarisation respectively of the incident wave. $w_0(t)$ relates to the time shape with unity amplitude and A_i is the amplitude of the incident wave. In the case of non-dispersive material, we can assume that the time shape of the involved waves rest the same so that reflected

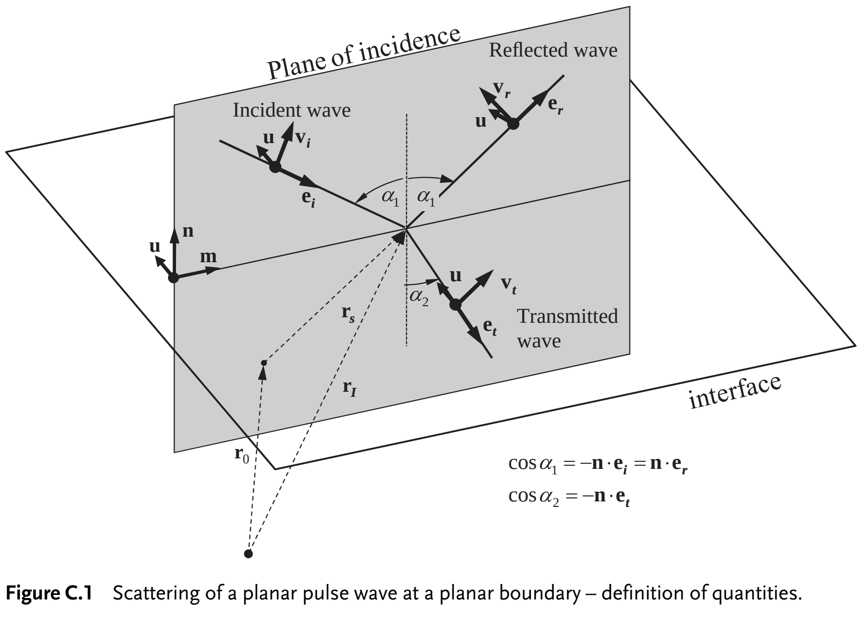


Figure C.1 Scattering of a planar pulse wave at a planar boundary – definition of quantities.

and refracted wave may be written as:

$$\mathbf{W}_r(t,\mathbf{r}) = A_r w_0\left(t - \frac{\mathbf{e}_r\cdot\mathbf{r}}{c_1}\right) * \mathbf{q}_r(t) \tag{C.14}$$

$$\mathbf{W}_t(t,\mathbf{r}) = A_t w_0\left(t - \frac{\mathbf{e}_t\cdot\mathbf{r}}{c_2}\right) * \mathbf{q}_t(t) \tag{C.15}$$

If the incident wave bounces the interface the boundary conditions (5.26) must be satisfied. This usually requires the creation of two new waves – one is penetrating and a second is reflected. In what follows, we like to determine these two waves. The boundary condition (5.26) relates to the tangential components of the electric and magnetic field respectively. In order to determine them, we first summarise some vector relations describing the geometry of the scenario. Refer to Figure 5.15 and Figure C.1 for the definition of the vectors.

Any point whose position vector $\mathbf{r}_I$ obeys the condition

$$\mathbf{n}\cdot(\mathbf{r}_I - \mathbf{r}_0) = \mathbf{n}\cdot\mathbf{r}_s = 0 \tag{C.16}$$

is located in the scattering plane. The plane of incidence is defined by its unity normal vector $\mathbf{u}$:

$$\mathbf{u} = \frac{\mathbf{n}\times\mathbf{e}_i}{|\mathbf{n}\times\mathbf{e}_i|} \tag{C.17}$$

The intersection line of the plane of incidence with the scattering interface is given by the unity vector $\mathbf{m}$:

$$\mathbf{m} = \mathbf{u}\times\mathbf{n} \tag{C.18}$$

Finally, we still define the unit vector $\mathbf{v}_n$ lying in the plan of incidence and perpendicular to the propagation of the incident field:

$$\mathbf{v}_n = \mathbf{e}_n \times \mathbf{u}. \tag{C.19}$$

The vectors $[\mathbf{e},\mathbf{u},\mathbf{v}_n]$ are forming a right hand system. The index n is n = i for incident field, n = r for the reflected and n = t for the refracted wave.

The boundary conditions (5.26) have to be satisfied within the whole boundary at any time. Hence, we have to require:

$$\begin{aligned}
&\mathbf{n}\times(\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{n}\times((\mathbf{E}_i + \mathbf{E}_r) - \mathbf{E}_t) = \mathbf{0} \quad \Rightarrow \\
&\sqrt{Z_1}\mathbf{n}\times\left(A_i\, w_0\left(t - \frac{\mathbf{e}_i\cdot(\mathbf{r}_I - \mathbf{r}_0)}{c_1}\right)\mathbf{q}_i + A_r w_0\left(t - \frac{\mathbf{e}_r\cdot(\mathbf{r}_I - \mathbf{r}_0)}{c_1}\right)\mathbf{q}_r\right) \\
&\qquad = \sqrt{Z_2}A_t w_0\left(t - \frac{\mathbf{e}_t\cdot(\mathbf{r}_I - \mathbf{r}_0)}{c_2}\right)\mathbf{n}\times\mathbf{q}_\mathbf{t}
\end{aligned} \tag{C.20}$$

$$\begin{aligned}
&\mathbf{n}\times(\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{n}\times((\mathbf{H}_i + \mathbf{H}_r) - \mathbf{H}_t) = \mathbf{0} \quad \Rightarrow \\
&\frac{1}{\sqrt{Z_1}}\mathbf{n}\times\left(A_i\, w_0\left(t - \frac{\mathbf{e}_i\cdot(\mathbf{r}_I - \mathbf{r}_0)}{c_1}\right)\mathbf{e}_i\times\mathbf{q}_i + A_r w_0\left(t - \frac{\mathbf{e}_r\cdot(\mathbf{r}_I - \mathbf{r}_0)}{c_1}\right)\mathbf{e}_r\times\mathbf{q}_r\right) \\
&\qquad = \frac{1}{\sqrt{Z_2}}A_t w_0\left(t - \frac{\mathbf{e}_t\cdot(\mathbf{r}_I - \mathbf{r}_0)}{c_2}\right)\mathbf{n}\times\mathbf{e}_t\times\mathbf{q}_t
\end{aligned} \tag{C.21}$$

Propagation term: These conditions are only met at any time if

$$\frac{\mathbf{e}_i \cdot (\mathbf{r}_I - \mathbf{r}_0)}{c_1} = \frac{\mathbf{e}_r \cdot (\mathbf{r}_I - \mathbf{r}_0)}{c_1} = \frac{\mathbf{e}_t \cdot (\mathbf{r}_I - \mathbf{r}_0)}{c_2}. \tag{C.22}$$

With $\mathbf{r}_s = \mathbf{r}_I - \mathbf{r}_0$, we may split (C.22) into

$$(\mathbf{e}_i - \mathbf{e}_r) \cdot \mathbf{r}_s = 0 \tag{C.23}$$

and

$$\left(\frac{\mathbf{e}_i}{c_1} - \frac{\mathbf{e}_t}{c_2}\right) \cdot \mathbf{r}_s = 0 \tag{C.24}$$

Both conditions are only valid if

$$\mathbf{e}_i - \mathbf{e}_r = \lambda\, \mathbf{n} \tag{C.25}$$

and

$$\mathbf{e}_i - \eta_{21}\, \mathbf{e}_t = \mu\, \mathbf{n} \tag{C.26}$$

where at λ, μ represents two scalar values and

$$\eta_{21} = \frac{c_1}{c_2} = \frac{Z_1}{Z_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon 1}} \tag{C.27}$$

is the refraction index. Since from (C.25) follows that $\mathbf{e}_i \cdot \mathbf{m} = \mathbf{e}_r \cdot \mathbf{m}$, we result in the reflection law:

$$\mathbf{e}_r = \mathbf{e}_i - 2(\mathbf{n} \cdot \mathbf{e}_i) \cdot \mathbf{n} \tag{C.28}$$

which is better known under

$$\alpha_i = \alpha_r = \alpha_1 \tag{C.29}$$

Equation (C.28) may also be expressed in matrix notation:

$$\mathbf{e}_r = T_H^{(\mathbf{n})}\, \mathbf{e}_i \tag{C.30}$$

$\mathbf{T}_H^{(\mathbf{n})}$ - Householder matrix – see (A.163)
Corresponding we get from (C.26)

$$\mathbf{e}_i \cdot \mathbf{m} = \eta_{21}\, \mathbf{e}_t \cdot \mathbf{m} \tag{C.31}$$

which is better know under

$$\sin \alpha_i = \sin \alpha_1 = \eta_{21} \sin \alpha_t = \eta_{21} \sin \alpha_2 \tag{C.32}$$

representing Snell's law. Hence the vector $\mathbf{e}_t$ gets:

$$\begin{aligned} \eta_{21}\, \mathbf{e}_t &= \mathbf{e}_i - \left(\sqrt{\eta_{21}^2 - (\mathbf{e}_i \cdot \mathbf{m})^2} + \mathbf{e}_i \cdot \mathbf{n}\right)\mathbf{n} \\ &= \mathbf{e}_i - \left(\sqrt{\eta_{21}^2 - \sin^2 \alpha_1} - \cos \alpha_1\right)\mathbf{n} \end{aligned} \tag{C.33}$$

Amplitude relations: In order to determine the amplitude of the reflected and refracted wave, we decompose the fields in $\mathbf{u}$ and $\mathbf{v}_n$ components. The $\mathbf{u}$-polarised field components[10] are parallel to the interface plane and the $\mathbf{v}_n$-polarised field components[11] are parallel to the plane of incidence. Thus we can write for the involved waves:

$$\mathbf{W}_n = (\mathbf{u} \cdot \mathbf{W}_n)\mathbf{u} + (\mathbf{v}_n \cdot \mathbf{W}_n)\mathbf{v}_n = U_n\mathbf{u} + V_n\mathbf{v}_n; \quad n \in [i, r, t] \tag{C.34}$$

where at holds $A_n^2 = U_n^2 + V_n^2$.

$\mathbf{u}$-*polarised wave*: Applying (C.27) and insertion of the $\mathbf{u}$-polarised components of the waves into (C.20) and (C.21) gives:

$$\begin{aligned} \sqrt{\eta_{21}}\,(U_i + U_r) &= U_t \\ \mathbf{n} \times (U_i\mathbf{e}_i \times \mathbf{u} + U_r\mathbf{e}_r \times \mathbf{u}) &= \sqrt{\eta_{21}}\mathbf{n} \times \mathbf{e}_t \times \mathbf{u} \end{aligned} \tag{C.35}$$

Using the vector identity (A.52) and respecting $\mathbf{u} \cdot \mathbf{n} = 0$, we get $\mathbf{n} \times (\mathbf{e} \times \mathbf{u}) = -\mathbf{u}(\mathbf{n} \cdot \mathbf{e})$ so that the reflection coefficient $R_u = U_r/U_i$ and transmission coefficient $T_u = U_t/U_i$ become:

$$\Lambda_{uu} = \frac{U_r}{U_i} = \frac{\eta_{21}\mathbf{n} \cdot \mathbf{e}_t - \mathbf{n} \cdot \mathbf{e}_i}{\mathbf{n} \cdot \mathbf{e}_r - \eta_{21}\mathbf{n} \cdot \mathbf{e}_t} = \frac{Z_2 \cos\alpha_1 - Z_1 \cos\alpha_2}{Z_2 \cos\alpha_1 + Z_1 \cos\alpha_2} \tag{C.36}$$

$$T_{uu} = \frac{U_t}{U_i} = \frac{2\sqrt{\eta_{21}}\mathbf{n} \cdot \mathbf{e}_r}{\mathbf{n} \cdot \mathbf{e}_r - \eta_{21}\mathbf{n} \cdot \mathbf{e}_t} = \frac{2\sqrt{Z_1 Z_2}\cos\alpha_1}{Z_2 \cos\alpha_1 + Z_1 \cos\alpha_2} \tag{C.37}$$

$\mathbf{v}$-*polarised waves*: The $\mathbf{v}$-polarised fields inserted in (C.20) and (C.21) gives:

$$\begin{aligned} \sqrt{\eta_{21}}\,\mathbf{n} \times (V_i\,\mathbf{v}_i + V_r\,\mathbf{v}_r) &= V_t\,\mathbf{n} \times \mathbf{v}_t \\ \mathbf{n} \times (V_i\,\mathbf{e}_i \times \mathbf{v}_i + V_r\,\mathbf{e}_r \times \mathbf{v}_r) &= \sqrt{\eta_{21}}V_t\,\mathbf{n} \times (\mathbf{e}_t \times \mathbf{v}_t) \end{aligned} \tag{C.38}$$

Insertion of (C.19) and using the identities $\mathbf{n} \times \mathbf{v} = \mathbf{n} \times (\mathbf{e} \times \mathbf{u}) = -\mathbf{u}(\mathbf{n} \cdot \mathbf{e})$ as well as $\mathbf{n} \times (\mathbf{e} \times \mathbf{v}) = \mathbf{n} \times (\mathbf{e} \times (\mathbf{e} \times \mathbf{u})) = -\mathbf{n} \times \mathbf{u}$ leads to

$$\Lambda_{vv} = \frac{V_r}{V_i} = \frac{\eta_{21}\mathbf{n} \cdot \mathbf{e}_i - \mathbf{n} \cdot \mathbf{e}_t}{\mathbf{n} \cdot \mathbf{e}_t - \eta_{21}\mathbf{n} \cdot \mathbf{e}_r} = \frac{Z_1 \cos\alpha_1 - Z_2 \cos\alpha_2}{Z_1 \cos\alpha_1 + Z_2 \cos\alpha_2} \tag{C.39}$$

$$T_{vv} = \frac{V_t}{V_i} = \frac{2\sqrt{\eta_{21}}\mathbf{n} \cdot \mathbf{e}_r}{\eta_{21}\mathbf{n} \cdot \mathbf{e}_r - \mathbf{n} \cdot \mathbf{e}_t} = \frac{2\sqrt{Z_1 Z_2}\cos\alpha_1}{Z_1 \cos\alpha_1 + Z_2 \cos\alpha_2} \tag{C.40}$$

The relations (C.36), (C.37), (C.39) and (C.40) are denoted as Fresnel equations (for normalised waves). They are valid for time and frequency domain notations as long as the material parameters are frequency independent. Otherwise, they are only valid for frequency domain notation.

C.3
Scattering of a Plane Wave at a Sphere

We refer to the scattering geometry depicted in Figure C.2 and a perfect conducting sphere of radius a placed in the origin of the coordinate system. A planar time

10) They are often referred as horizontal polarised waves.
11) They are often referred as vertical polarised waves.

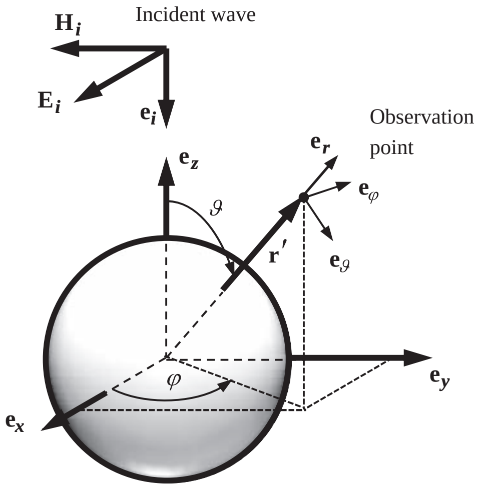


Figure C.2 Planar wave scattering at a sphere.

harmonic electromagnetic wave propagates along the negative z-axis. It should be polarised in x-direction so that electric and the magnetic field as well as the normalised wave are expressed by:

$$\mathbf{E}_i = E_0 \; e^{jkz} \; \mathbf{e}_x$$

$$\mathbf{H}_i = -\frac{E_0}{Z_s} \; e^{jkz} \; \mathbf{e}_y$$

$$\mathbf{W}_i = \frac{E_0}{\sqrt{Z_s}} \; e^{jkz} \; \mathbf{e}_x$$

The scattered far field in the observation point $\mathbf{r}' = [r', \vartheta, \varphi]$ and for $k\,a = 2\,\pi\frac{a}{\lambda} = \frac{2\,\pi f a}{c}$ is determined by [10]:

Scattered field $\mathbf{e}_\vartheta$ - polarized:

$$W_{S,\vartheta}(r', \vartheta, \varphi) = \frac{E_\vartheta(r', \vartheta, \varphi)}{\sqrt{Z_s}} = \sqrt{Z_s} H_\varphi(r', \vartheta, \varphi) = \underline{\Lambda}_\vartheta(k\,a, \vartheta, \varphi)\, \frac{e^{-jkr'}}{r'}\, \frac{E_0}{\sqrt{Z_s}} \tag{C.41}$$

$$\underline{\Lambda}_\vartheta(k\,a, \vartheta, \varphi) = \frac{j\cos\varphi}{k} \begin{cases} -\sum\limits_{n=1}^{\infty}(-1)^n\left(n+\frac{1}{2}\right)(b_n - a_n); & \vartheta = 0 \\ \sum\limits_{n=1}^{\infty}(-1)^n \frac{2n+1}{n(n+1)}\left(b_n \frac{\partial P_n^1(\cos\vartheta)}{\partial\vartheta} - a_n \frac{P_n^1(\cos\vartheta)}{\sin\vartheta}\right); & \vartheta \in (0, \pi) \\ -\sum\limits_{n=1}^{\infty}\left(n+\frac{1}{2}\right)(b_n + a_n); & \vartheta = \pi \end{cases}$$

Scattered field $\mathbf{e}_\varphi$ - polarized:

$$W_{S,\varphi}(r',\vartheta,\varphi) = \frac{E_\varphi(r',\vartheta,\varphi)}{\sqrt{Z_s}} = -\sqrt{Z_s}H_\vartheta(r',\vartheta,\varphi) = \underline{\Lambda}_\varphi(k\,a,\vartheta,\varphi)\,\frac{e^{-jkr'}}{r'}\,\frac{E_0}{\sqrt{Z_s}} \tag{C.42}$$

$$\underline{\Lambda}_\varphi(k\,a,\vartheta,\varphi) = \frac{j\sin\varphi}{k}\begin{cases} \sum_{n=1}^{\infty}(-1)^n\left(n+\frac{1}{2}\right)(b_n - a_n) & \vartheta = 0 \\ -\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}\left(b_n\frac{P_n^1(\cos\vartheta)}{\sin\vartheta} - a_n\frac{\partial P_n^1(\cos\vartheta)}{\partial\vartheta}\right); & \vartheta \in (0,\pi) \\ -\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)(b_n + a_n) & \vartheta = \pi \end{cases}$$

Whereat:

$$a_n = \frac{S_n(ka)}{\zeta_n^{(1)}(ka)}; \quad b_n = \frac{\partial S_n(ka)/\partial ka}{\partial\zeta_n^{(1)}(ka)/\partial ka}$$

$S_n(x) = xj_n(x)$; $\zeta_n^{(1)}(x) = xh_n^{(1)}(x)$ – Riccati-Bessel functions

$j_n(x),\ h_n^{(1)}(x)$ – spherical Bessel/Hankel function

$P_n^1(x)$ – associated Legendre function of first order

a – radius of sphere

$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$ – wave number

The bi-static scattering cross section normalised to the geometric cross section of the sphere is calculated by:

$$\frac{\sigma(\vartheta,\varphi)}{\pi\,a^2} = \frac{4}{a^2}\left(|\underline{\Lambda}_\vartheta|^2 + |\underline{\Lambda}_\varphi|^2\right) \tag{C.43}$$

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