

## 1 Atomic Kinematics

In this chapter, we introduce a model for a two-level system, the so-called two-level atom. We consider a collection of many such systems, and introduce different atomic states (Dicke, coherent, and squeezed states). We also discuss a generalization to systems with an arbitrary structure of energy levels.

### 1.1 Kinematics of an Atom with Two Energy Levels

The state vector for a two-level system can be written as a linear superposition

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \quad (1.1)$$

where vectors  $|0\rangle$  and  $|1\rangle$  represent respectively the ground and excited states of the system and form a basis in an abstract two-dimensional Hilbert space. The coefficients  $c_k$  satisfy the normalization condition  $|c_0|^2 + |c_1|^2 = 1$ , and  $|c_k|^2$  represents the probability of finding the system in the state “ $k$ ”. In a representation where

$$|0\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |1\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

an arbitrary pure state  $|\psi\rangle$  is described by a complex two-dimensional vector

$$|\psi\rangle \rightarrow \mathbf{c} = \begin{bmatrix} c_1 \\ c_0 \end{bmatrix}$$

Up to a common phase, the amplitudes  $c_k$  can be conveniently parameterized as follows:

$$c_1 = \cos \frac{\vartheta}{2} e^{-i\varphi/2}, \quad c_0 = \sin \frac{\vartheta}{2} e^{i\varphi/2} \quad (1.2)$$

In the same representation, an arbitrary linear operator  $\hat{f}$ , acting on the two-level system, has a matrix realization of the form

$$\hat{f} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



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and can be expressed in terms of the Pauli matrices,

$$\begin{aligned}\sigma_x = \sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \sigma_y = \sigma_2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ \sigma_z = \sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, & I = \sigma_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}\quad (1.3)$$

as

$$\hat{f} = \sum_{j=0}^3 f_j \sigma_j$$

where the coefficient  $f_j$  is given by

$$f_j = \frac{1}{2} \text{Tr}(\hat{f} \sigma_j)$$

If the operator is Hermitian,  $\hat{f}^\dagger = \hat{f}$ , the coefficients  $f_j$  are real. Average values of the Pauli matrices in an arbitrary state  $|\psi\rangle$  in the parameterization (Equation 1.2) are

$$\langle \sigma_x \rangle = 2\text{Re}(c_1 c_0) = \sin \vartheta \cos \varphi$$

$$\langle \sigma_y \rangle = 2\text{Im}(c_1 c_0) = \sin \vartheta \sin \varphi$$

$$\langle \sigma_z \rangle = |c_1|^2 - |c_0|^2 = \cos \vartheta$$

From these, one can infer the average value for  $\hat{f}$ .

In the two-level atom, the physical interpretation of the operator  $\sigma_z$  can be understood from the Schrödinger equation for a free system:

$$i\hbar \frac{d\mathbf{c}}{dt} = H_0 \mathbf{c}, \quad H_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_0 \end{bmatrix}$$

If we choose the zero energy to lie exactly between  $E_0$  and  $E_1$ , so that  $E_0 = -\varepsilon$  and  $E_1 = \varepsilon$ , the free Hamiltonian of the system is then

$$H_0 = \frac{\omega_0 \hbar}{2} \sigma_z, \quad \omega_0 = \frac{E_1 - E_0}{\hbar} = \frac{2\varepsilon}{\hbar}$$

Clearly, the states  $|0\rangle$  and  $|1\rangle$  are eigenstates of the free Hamiltonian  $H_0$  (these states are sometimes called *bare states*).

The simplest physical realization of a two-level system is clearly a single electron spin in an external magnetic field. To find more applications for a two-level system, let us consider the interaction of a general multilevel atom with an external electromagnetic field. Let us suppose that the atom has only one electron and designate by  $|\Psi_n\rangle$ ,  $n = 0, 1, 2, \dots$  the eigenstates of the free Hamiltonian  $H_0$  of the atom

$$H_0 |\Psi_n\rangle = E_n |\Psi_n\rangle \quad (1.4)$$



We assume that the Hamiltonian describing the interaction with the electric field has the simplified form

$$H_{\text{int}} = -(\mathbf{d} \cdot \mathcal{E})$$

where  $\mathbf{d} = e\mathbf{r}$  is the electric dipole operator,  $e$  is the charge, and  $\mathbf{r}$  is the electron position vector. In the basis spanned by the  $|\Psi_n\rangle$ 's, we obtain

$$H_0 = \begin{bmatrix} \dots & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & E_0 \end{bmatrix}, \quad H_{\text{int}} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & H_{11} & H_{10} \\ \dots & H_{01} & H_{00} \end{bmatrix}$$

where

$$H_{kn} = \langle \Psi_k | H_{\text{int}} | \Psi_n \rangle = - \left( \mathcal{E} \cdot \int d\mathbf{r} \Psi_k^*(\mathbf{r}) e\mathbf{r} \Psi_n(\mathbf{r}) \right) \equiv -(\mathcal{E} \cdot \mathbf{d}_{kn})$$

Since the Hamiltonian is invariant under reflections, the wave functions  $|\Psi_n\rangle$  have definite parity. Thus, the diagonal matrix elements of the interaction Hamiltonian vanish:  $H_{nn} = -e\mathcal{E} \cdot \langle \Psi_n | \mathbf{r} | \Psi_n \rangle = 0$ . The off-diagonal matrix elements  $H_{kn}$  describe transitions between the states with energies  $E_k$  and  $E_n$ , respectively. These transitions have frequency  $\omega_a = (E_k - E_n)/\hbar$ . To stimulate these transitions, the frequency of the external field must be close to  $\omega_a$ . As the energy levels of an atom are usually not equidistant, the external field does not, in practice, induce transition between levels other than  $E_k$  and  $E_n$ . This means that, in the matrix for  $H_{\text{int}}$ , we can focus on a single  $2 \times 2$  block corresponding to the elements at the intersection of the  $k$ th and  $n$ th rows with the  $k$ th and  $n$ th columns. This leads us to the approximation of a two-level atom. We assign an index '0' to the lowest state and an index '1' to the highest state.

The Hamiltonian of the atom takes on the form

$$H = H_0 + H_{\text{int}}, \quad H_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_0 \end{bmatrix} \quad (1.5)$$

where

$$H_{\text{int}} = -(\mathcal{E} \cdot \mathbf{d}) = \begin{bmatrix} 0 & g \\ g^* & 0 \end{bmatrix}$$

and we have introduced the notation

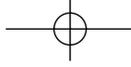
$$-e\langle 1 | (\mathcal{E} \cdot \mathbf{r}) | 0 \rangle \equiv g = g_r + ig_i$$

Here,  $g_r$  and  $g_i$  are the real and imaginary parts of the complex number  $g$ . We obtain

$$\begin{aligned} \langle H_{\text{int}} \rangle &= (c_1^* \langle 1 | + c_0^* \langle 0 |) H_{\text{int}} (c_1 | 1 \rangle + c_0 | 0 \rangle) \\ &= g_r (c_1^* c_0 + c_0^* c_1) + ig_i (c_0 c_1^* - c_0^* c_1) = g_r \langle \sigma_x \rangle - g_i \langle \sigma_y \rangle \end{aligned}$$

Thus, the dipole moment of a two-level atom can be expressed in terms of operators  $\sigma_x$  and  $\sigma_y$ , as follows:





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$$\mathbf{d} = (\text{Re } \mathbf{d}) \sigma_x - (\text{Im } \mathbf{d}) \sigma_y \quad (1.6)$$

where  $\mathbf{d} = e\langle 1|\mathbf{r}|0\rangle$ . The last equation provides a physical interpretation of the operators  $\sigma_x$  and  $\sigma_y$ .

Note that  $\mathcal{E}$  and  $\mathbf{d}$  are vectors in the real physical three-dimensional space. On the other hand, average values of the Pauli matrices  $\sigma_{x,y,z}$  form a pseudospin vector in the isotopic space ( $\sigma_{x,y,z}$  are sometimes called *energy spin operators* or *isotopic spin operators*). These two spaces should not be confused. (At this point, the difference from spin dynamics in the external magnetic field appears: in the last case the mean values of the spin operators  $\sigma$  form a vector in the real space).

In the case of a pure atomic state, the values  $\langle \sigma_k \rangle$  are the components of a unit vector in the isotopic space:

$$\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 = 1$$

This vector is called the *Bloch vector*. The tip of this vector is point located on the surface of a unit sphere, called the *Bloch sphere*. The orientation of the Bloch vector on the sphere is uniquely specified by two angles  $\vartheta, \varphi$ . To each pure state of a two-level system corresponds a point on the Bloch sphere. The north pole of the sphere corresponds to the excited state  $|1\rangle$  and the south pole to the ground state  $|0\rangle$ . At either pole, the phase  $\varphi$  is not defined.

If the system is in an arbitrary state (pure or mixed), described by a density matrix  $\rho$ , where

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{bmatrix}$$

the Bloch vector  $\mathbf{R} = \text{Tr}(\rho\sigma)$ , can be written in the following way:

$$R_x = 2\text{Re}\rho_{01}, \quad R_y = 2\text{Im}\rho_{01}, \quad R_z = \rho_{11} - \rho_{00}$$

Inversely, given a Bloch vector, one can recover the density matrix via

$$\rho = \frac{1}{2} (1 + \mathbf{R}\sigma)$$

It is easy to see that  $|\mathbf{R}|^2 = R_x^2 + R_y^2 + R_z^2 = 2\text{Tr}(\rho^2) - 1 \leq 1$ , with the equality satisfied only for pure states. This means that, while pure states are represented as points on the surface of the sphere, mixed states are represented as points inside the sphere. The completely mixed atomic state, which is represented by a diagonal matrix  $\rho = I/2$ , corresponds to the zero-length Bloch vector. In spherical coordinates  $(r, \vartheta, \varphi)$ , the density matrix elements take the form

$$\rho_{00} = \frac{1}{2} (1 - r \cos \vartheta), \quad \rho_{01} = \frac{1}{2} r \sin \vartheta e^{i\varphi}, \quad \rho_{11} = \frac{1}{2} (1 + r \cos \vartheta)$$

where  $r = 1$  corresponds to a pure state.

Note that the basis for linear operators acting in the Hilbert space of the states of a two-level system can be chosen in a different way:

$$|j\rangle\langle k|, \quad j, k = 0, 1 \quad (1.7)$$



In this basis, Pauli matrices can be written as follows:

$$\begin{aligned}\sigma_x &= |0\rangle\langle 1| + |1\rangle\langle 0|, & \sigma_y &= i(|0\rangle\langle 1| - |1\rangle\langle 0|), \\ \sigma_z &= |1\rangle\langle 1| - |0\rangle\langle 0|, & I &= |1\rangle\langle 1| + |0\rangle\langle 0|\end{aligned}$$

## 1.2 Dicke States

Let us consider a collection of  $A$  two-level atoms with the same transition frequency  $\omega_0$ . We assume that the atoms are distinguishable and do not interact with each other. We designate the  $n$ th state of the  $j$ th atom as  $|j, n\rangle$ ,  $n = 0, 1$ . Let  $k$  be the number of atoms in the excited state, so that  $A - k$  atoms are in the ground state. The possible outcomes of measuring the energy of the system are

$$E = \hbar\omega_0 \left( k - \frac{A}{2} \right), \quad k = 0, 1, \dots, N$$

Thus, the energy levels are equally spaced and the spectrum is bounded. One example of a vector that describes a state with  $k$  excited atoms has the form

$$|1, 1\rangle \dots |k, 1\rangle |k+1, 0\rangle \dots |A, 0\rangle \quad (1.8)$$

This energy level has a degeneracy of  $C_k^A = (A! / k!(A-k)!)$ , since any state differing from the state Equation 1.8 only by permutation of the atoms,

$$|j_1, 1\rangle \dots |j_k, 1\rangle |j_{k+1}, 0\rangle \dots |j_A, 0\rangle$$

has the same energy. (Here,  $A!$  is the total number of atomic permutations and  $k!(A-k)!$  is the number of permutations that do not interchange excited and nonexcited atoms.) Now suppose that the states that have the same energy are indistinguishable and consider a normalized symmetric linear combination of all the functions that correspond to the same energy

$$|k, A\rangle = \sqrt{\frac{k!(A-k)!}{A!}} \sum_p |j_1, 1\rangle \dots |j_k, 1\rangle |j_{k+1}, 0\rangle \dots |j_A, 0\rangle \quad (1.9)$$

where the sum is made over all the possible permutations that interchange excited and nonexcited atoms. For example, for two atoms these symmetric states are

$$\begin{aligned}|0, 2\rangle &= |1, 0\rangle |2, 0\rangle \\ |1, 2\rangle &= (|1, 1\rangle |2, 0\rangle + |1, 0\rangle |2, 1\rangle) / \sqrt{2} \\ |2, 2\rangle &= |1, 1\rangle |2, 1\rangle\end{aligned}$$

Let us define the *collective operators* for a system of  $A$  two-level atoms in the following way:

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$$S_{x,y,z} = \frac{1}{2} \sum_{j=1}^A \sigma_{x,y,z}^{(j)} \quad (1.10)$$

where  $\sigma_{x,y,z}^{(j)}$  are individual operators for each of the atoms. It is obvious that in the collective basis (1.9),

$$S_z |k, A\rangle = \left(k - \frac{A}{2}\right) |k, A\rangle$$

Thus, the operator  $S_z$  defines the inversion in the atomic system, that is, the difference between the number of excited and nonexcited atoms. The operators

$$\sigma_{\pm}^{(j)} = \frac{1}{2} \left( \sigma_x^{(j)} \pm i\sigma_y^{(j)} \right)$$

satisfy the commutation relations

$$[\sigma_+^{(j)}, \sigma_-^{(j)}] = \sigma_z^{(j)}, \quad [\sigma_z^{(j)}, \sigma_{\pm}^{(j)}] = \pm 2\sigma_{\pm}^{(j)}$$

and act on the basis elements of each atom as

$$\sigma_+^{(j)} |j, 0\rangle = |j, 1\rangle, \quad \sigma_+^{(j)} |j, 1\rangle = 0, \quad \sigma_-^{(j)} |j, 1\rangle = |j, 0\rangle, \quad \sigma_-^{(j)} |j, 0\rangle = 0$$

Thus, the collective operators

$$S_{\pm} = \sum_{j=1}^A \sigma_{\pm}^{(j)} = S_x \pm iS_y \quad (1.11)$$

act on symmetric states (Equation 1.9) as follows:

$$\begin{aligned} S_- |k, A\rangle &= \sqrt{\frac{k!(A-k)!}{A!}} (A-k+1) \sum_p |j_1, 1\rangle \dots |j_{k-1}, 1\rangle |j_k, 0\rangle \dots |j_A, 0\rangle \\ &= \sqrt{k(A-k+1)} |k-1, A\rangle \end{aligned}$$

Similarly, one obtains the action of the operator  $S_+$ . Finally, the action of the collective operators on the basis of the subspace of symmetric states (1.9) is given as

$$\begin{aligned} S_+ |k, A\rangle &= \sqrt{(k+1)(A-k)} |k+1, A\rangle \\ S_- |k, A\rangle &= \sqrt{k(A-k+1)} |k-1, A\rangle \\ S_z |k, A\rangle &= \left(\frac{k-A}{2}\right) |k, A\rangle \end{aligned} \quad (1.12)$$

The collective operators  $S_{x,y,z}$  satisfy the familiar  $su(2)$  algebra commutation relations:

$$[S_x, S_y] = iS_z, \quad [S_y, S_z] = iS_x, \quad [S_z, S_x] = iS_y \quad (1.13)$$

or, in terms of the operators  $S_{\pm}$

$$[S_z, S_{\pm}] = \pm S_{\pm}, \quad [S_+, S_-] = 2S_z \quad (1.14)$$

Since  $S_z^2 |k, A\rangle = \left(\frac{k-A}{2}\right)^2 |k, A\rangle$ , and

$$(S_x^2 + S_y^2) |k, A\rangle = \frac{1}{2} (S_+ S_- + S_- S_+) |k, A\rangle = \left(Ak - k^2 + \frac{A}{2}\right) |k, A\rangle$$

the state  $|k, A\rangle$  is the eigenstate of the *Casimir operator*

$$S^2 = S_z^2 + S_x^2 + S_y^2 = S_z^2 + \frac{1}{2} (S_+ S_- + S_- S_+) \quad (1.15)$$

The corresponding eigenvalue does not depend on  $k$ :

$$S^2 |k, A\rangle = \frac{A}{2} \left(\frac{A}{2} + 1\right) |k, A\rangle$$

Moreover, it is easy to prove that  $[S_{x,y,z}, S^2] = 0$ . This means that the symmetric states  $|k, A\rangle$  form a basis of the  $(A+1)$ -dimensional *irreducible representation* of the  $su(2)$  algebra. The vectors  $|k, A\rangle$  are called *Dicke states* [13].

### 1.3

#### Atomic Coherent States

Let us again consider a system of  $A$  (distinguishable) two-level atoms that do not interact with each other. Now we suppose that all the atoms are in the same quantum state, so that the state of the whole system is simply the product of the states of individual atoms. In the polar parameterization (Equation 1.2), we obtain

$$|\vartheta, \varphi\rangle = \prod_{j=1}^A \left( \cos \frac{\vartheta}{2} e^{-i\varphi/2} |j, 1\rangle + \sin \frac{\vartheta}{2} e^{i(\varphi/2)} |j, 0\rangle \right) \quad (1.16)$$

It is easy to see that this product transforms into

$$|\vartheta, \varphi\rangle = \sum_{k=0}^A \cos^k \frac{\vartheta}{2} \sin^{A-k} \frac{\vartheta}{2} e^{i((A-2k)\varphi/2)} \sum_p |j_1, 1\rangle \dots |j_k, 1\rangle |j_{k+1}, 0\rangle \dots |j_A, 0\rangle$$

where, on the right hand side, one can recognize the Dicke state  $|k, A\rangle$

$$\begin{aligned} |\vartheta, \varphi\rangle &= e^{i\frac{A}{2}\varphi} \sum_{k=0}^A \sqrt{\frac{A!}{k!(A-k)!}} e^{-ik\varphi} \cos^k \frac{\vartheta}{2} \sin^{A-k} \frac{\vartheta}{2} |k, A\rangle \\ &= e^{-i\varphi S_z} \sum_{k=0}^A \sqrt{\frac{A!}{k!(A-k)!}} \cos^k \frac{\vartheta}{2} \sin^{A-k} \frac{\vartheta}{2} |k, A\rangle \end{aligned} \quad (1.17)$$

The state  $|\vartheta, \varphi\rangle$  is, in fact, a coherent state for the  $su(2)$  group [14] (see Appendix 11.2). Let us replace  $\vartheta = \theta + \pi$  in Equation 1.17 and introduce the state [15, 16]

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$$|\xi\rangle = e^{-i\varphi A/2} |\vartheta = \theta + \pi, \varphi\rangle, \quad \xi = -\tan\left(\frac{\theta}{2}\right) e^{-i\varphi}$$

The state  $|\xi\rangle$  can be generated from vector  $|0, A\rangle$ , describing a state with no atom excited, by the displacement operator  $D(z)$ :

$$D(z) = \exp(zS_+ - z^*S_-), \quad z = -\frac{\theta}{2} e^{-i\varphi} \quad (1.18)$$

$$D^\dagger(z) = D^{-1}(z) = D(-z)$$

The displacement operator can be rewritten in different ways:

$$\begin{aligned} D(z) &= \exp(-i\varphi S_z) \exp(-i\theta S_y) \exp(i\varphi S_z) \\ &= \exp(\xi S_+) \exp[\ln(1 + |\xi|^2) S_z] \exp(-\xi^* S_-) \end{aligned}$$

Thus, the ket  $|\xi\rangle$  takes the form

$$|\xi\rangle = D(z)|0, A\rangle = (1 + |\xi|^2)^{-A/2} \exp[\xi S_+] |0, A\rangle \quad (1.19)$$

It is easy to write the expansion of the coherent state  $|\xi\rangle$  in the basis of the Dicke states:

$$|\xi\rangle = \frac{1}{(1 + |\xi|^2)^{A/2}} \sum_{k=0}^A \xi^k \sqrt{\frac{A!}{k!(A-k)!}} |k, A\rangle \quad (1.20)$$

The coherent state (1.17) can also be obtained by applying the operator  $\exp(-i\varphi S_z) \exp(-i\vartheta S_y)$  to the fully excited state  $|A, A\rangle$ :

$$|\vartheta, \varphi\rangle = \exp(-i\varphi S_z) \exp(-i\vartheta S_y) |A, A\rangle$$

To understand the physical properties of the atomic coherent states, we calculate the expectation values of the operators  $S_{x,y,z}$ . It follows directly from Equation 1.20 that

$$\langle \xi | S_+ | \xi \rangle = \frac{1}{(1 + |\xi|^2)^A} \sum_{k,n=0}^A \frac{\xi^{*k} \xi^n}{k!n!} \langle A, 0 | S_-^k S_+ S_+^n | 0, A \rangle$$

and, using the relation

$$\langle A, 0 | S_-^k S_+^{n+1} | 0, A \rangle = \delta_{k,n+1} \frac{k!A!}{(A-k)!}$$

we immediately obtain

$$\langle \xi | S_+ | \xi \rangle = \frac{\xi^*}{(1 + |\xi|^2)^A} \sum_{k=1}^A \frac{|\xi|^{2(k-1)} A!}{(k-1)!(A-k)!} = \frac{A \xi^*}{1 + |\xi|^2}$$

or, in the angular representation

$$\langle \xi | S_+ | \xi \rangle = -\frac{A}{2} \sin \theta e^{i\varphi} = \langle \xi | S_- | \xi \rangle^*$$

If we now observe that

$$S_+ | \xi \rangle = \frac{1}{\xi} \left( S_z + \frac{A}{2} \right) | \xi \rangle, \quad S_- | \xi \rangle = \xi \left( \frac{A}{2} - S_z \right) | \xi \rangle \quad (1.21)$$

we can easily evaluate  $\langle \xi | S_z | \xi \rangle = \xi \langle \xi | S_+ | \xi \rangle - A/2$ , and finally, we have

$$\langle \xi | S_x | \xi \rangle = -\frac{A}{2} \sin \theta \cos \varphi, \quad \langle \xi | S_y | \xi \rangle = -\frac{A}{2} \sin \theta \sin \varphi, \quad \langle \xi | S_z | \xi \rangle = -\frac{A}{2} \cos \theta$$

or for the coherent states (Equations 1.16 and 1.17),

$$\langle \vartheta, \varphi | \mathbf{S} | \vartheta, \varphi \rangle = \frac{A}{2} \mathbf{n}, \quad \mathbf{n} = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \quad (1.22)$$

where  $\mathbf{S} = (S_x, S_y, S_z)$  and  $\mathbf{n}$  is the unit vector that determines the direction of the classical angular momentum. The averages of the operators  $S_j$  then form a vector in the direction of the classical momentum (collective Bloch vector):

$$\mathbf{R} = \langle \vartheta, \varphi | \mathbf{S} | \vartheta, \varphi \rangle = \frac{A}{2} \mathbf{n}$$

These average values are identical to those obtained for the single atom case, except for the change in the value of the isotopic spin:  $1/2 \rightarrow A/2$ . This suggests that a coherent state  $|\vartheta, \varphi\rangle$  can be interpreted as a symmetric generalization of the one-atom state. In fact, Equation 1.22 follows from the observation that the spin coherent states are eigenstates of the operator  $(\mathbf{S} \cdot \mathbf{n})$ , as shown in Appendix 11.2.

Fluctuations of the pseudospin projection  $S_x$  in the state  $|\vartheta, \varphi\rangle$

$$\langle \Delta S_x^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2, \quad \langle \dots \rangle \equiv \langle \vartheta, \varphi | \dots | \vartheta, \varphi \rangle$$

can be easily found from Equation 1.17. However, it is more interesting to use the definition (Equation 1.16) directly. We have

$$\langle S_x^2 \rangle = \frac{1}{4} \left\langle \left( \sum_{j=1}^A \sigma_x^{(j)} \right)^2 \right\rangle = \frac{1}{4} \left\langle \sum_{j=1}^A \left( \sigma_x^{(j)} \right)^2 + \sum_{i \neq j} \sigma_x^{(i)} \sigma_x^{(j)} \right\rangle$$

In the first term on the far right,  $(\sigma_x^{(j)})^2 = 1$  while the additional term  $\langle \sigma_x^{(i)} \sigma_x^{(j)} \rangle = \langle \sigma_x^{(i)} \rangle \langle \sigma_x^{(j)} \rangle = \sin^2 \vartheta \cos^2 \varphi$ , due to the factorized form of the state (1.16). This leads to

$$\langle S_x^2 \rangle = \frac{A}{4} + \frac{A(A-1)}{4} \sin^2 \vartheta \cos^2 \varphi$$

Using Equation 1.22, we obtain

$$\langle \Delta S_x^2 \rangle = \frac{A}{4} (1 - \sin^2 \vartheta \cos^2 \varphi)$$

Relative fluctuations are

$$\delta_x = \frac{\sqrt{\langle \Delta S_x^2 \rangle}}{\langle S_x \rangle} = \frac{1}{\sqrt{A}} \frac{\sqrt{1 - \sin^2 \vartheta \cos^2 \varphi}}{\sin \vartheta \cos \varphi}$$

For the states with  $\varphi = \pm \pi/2$  as well as for the states  $|0\rangle$  and  $|1\rangle$  corresponding to the values  $\vartheta = 0$  and  $\vartheta = \pi$  (north and south poles of the Bloch sphere),  $\langle S_x \rangle = 0$  and relative fluctuations  $\delta_x \rightarrow \infty$ . For all other coherent states, relative fluctuations  $\delta_x$  tend toward zero in the limit of a large number of atoms,  $A \rightarrow \infty$ . States with the lowest fluctuations of  $S_x$  correspond to the points on the equator of the Bloch sphere,  $\vartheta = \pi/2$ . In this case, half of the atoms are excited. Fluctuations  $\langle \Delta S_x^2 \rangle$  completely disappear when  $\vartheta = \pi/2$  and  $\varphi = 0$ , or  $\varphi = \pi$ : these two coherent states are eigenstates of the operator  $S_x$ . It is worth noting that the directions of the axes of the coordinate system can always be chosen in such a way that fluctuations of a given operator ( $S_x$ ,  $S_y$  or  $S_z$ ) depend on one angle only. For example, in the case of operator  $S_x$ , this angle is between the  $x$  axis and the direction of  $\mathbf{n}$ .

Finally, fluctuations for the operators  $S_{x,y,z}$  in the atomic coherent states  $|\vartheta, \varphi\rangle$  are

$$\langle \Delta S_x \rangle^2 = \frac{A}{4} (1 - \sin^2 \vartheta \cos^2 \varphi) \quad (1.23)$$

$$\langle \Delta S_y \rangle^2 = \frac{A}{4} (1 - \sin^2 \vartheta \sin^2 \varphi)$$

$$\langle \Delta S_z \rangle^2 = \frac{A}{4} (1 - \cos^2 \vartheta)$$

One can observe that the square fluctuations in the coherent states are  $A$  times larger than in the case of one atom. Further note that

$$\langle \Delta S_x \rangle^2 + \langle \Delta S_y \rangle^2 + \langle \Delta S_z \rangle^2 = \langle S^2 \rangle - \langle \mathbf{S} \rangle^2 = \frac{A}{2}$$

It is also easy to calculate the fluctuations of the component of the vector operator  $\mathbf{S}$  in the direction perpendicular to the direction  $\mathbf{n}$  of the average spin  $\mathbf{R} = \langle \vartheta, \varphi | \mathbf{S} | \vartheta, \varphi \rangle$ . For this goal one can write

$$\mathbf{S} = \mathbf{n}(\mathbf{S} \cdot \mathbf{n}) + \mathbf{S}_\perp$$

where  $\mathbf{S}_\perp$  is a two-dimensional vector that belongs to the plane tangent to the sphere at point  $(\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ :

$$\mathbf{S}_\perp = \mathbf{S}_\perp^{(1)} + \mathbf{S}_\perp^{(2)}, \quad \mathbf{S}_\perp^{(1)} \perp \mathbf{S}_\perp^{(2)}$$

Using Equation 1.22, we have

$$\langle \vartheta, \varphi | \mathbf{S}_\perp | \vartheta, \varphi \rangle = 0 \quad (1.24)$$

As stated earlier,  $|\vartheta, \varphi\rangle$  is the eigenvector of the operator  $(\mathbf{S} \cdot \mathbf{n})$ :

$$(\mathbf{S} \cdot \mathbf{n})|\vartheta, \varphi\rangle = \frac{A}{2} |\vartheta, \varphi\rangle$$

thus,

$$\langle (\Delta \mathbf{S}_\perp)^2 \rangle = \langle S^2 \rangle - \langle (\mathbf{S} \cdot \mathbf{n}) \rangle^2 = \frac{A}{2} \left( \frac{A}{2} + 1 \right) - \left( \frac{A}{2} \right)^2 = \frac{A}{2}$$

where the reader is reminded that  $\langle \dots \rangle$  is taken over the coherent state  $|\vartheta, \varphi\rangle$ .

From Equation 1.19, we conclude that, for the coherent states, the fluctuations of the components of vector  $\mathbf{S}_\perp$  are equal. This is obvious for the coherent state  $|0, A\rangle$ , i.e. the coherent state located at  $\theta = 0$  (or, equivalently, at  $\vartheta = \pi$ ). As physical properties of the coherent states do not depend on rotations, these properties are not changed by the action of the operator  $D(\theta, \varphi)$ . Thus, we obtain

$$\langle \vartheta, \varphi | (\Delta \mathbf{S}_\perp^{(1)})^2 | \vartheta, \varphi \rangle = \langle \vartheta, \varphi | (\Delta \mathbf{S}_\perp^{(2)})^2 | \vartheta, \varphi \rangle = \frac{A}{4} \quad (1.25)$$

This equation, taken in conjunction with Equation 1.22, allows us to visualize a coherent atomic state as a symmetric segment of radius  $\sqrt{A}/2$  on a Bloch sphere of radius  $A/2$ , centered at the point determined by the vector  $\mathbf{n}$  of Equation 1.22. In the classical limit, where  $A \rightarrow \infty$ , the radius of this segment divided by the Bloch sphere radius is proportional to  $1/\sqrt{A} \rightarrow 0$ . This means that the relative fluctuations in the directions perpendicular to the Bloch vector disappear.

It is important to note that, for a system described by a coherent state, the number of excitations is not well defined, except at the poles, where  $\vartheta = 0$  or  $\pi$ . From Equation 1.17, one can obtain the probability of finding  $k$  excitations in the system

$$p_k = |\langle \vartheta, \varphi | k, A \rangle|^2 = \frac{A!}{k!(A-k)!} x^k (1-x)^{A-k}$$

where  $x = \sin^2 \vartheta/2$ . This expression corresponds to the binomial distribution, which tends to the Poisson distribution when  $A \rightarrow \infty$  (and  $A \sin^2 \vartheta/2$  is a finite number). Assuming that each excitation has the energy  $\hbar\omega_0/2$  and each nonexcited state has the energy  $-\hbar\omega_0/2$ , we find the average energy in a coherent state:

$$E = \frac{\hbar\omega_0}{2} \sum_k (2k - A) p_k = \hbar\omega_0 \frac{A}{2} \cos \vartheta$$

This result may be obtained by calculating the mean energy as the average of the free Hamiltonian over the coherent state and using Equation 1.22.

$$E = \langle \vartheta, \varphi | H_0 | \vartheta, \varphi \rangle, \quad H_0 = \hbar\omega_0 S_z$$

It can easily be seen that at the poles,  $\vartheta = 0$  and  $\vartheta = \pi$ , the energy takes its maximum and minimum values, while for points on the equator,  $\vartheta = \pi/2$ , the mean energy equals to zero.

The free evolution of the coherent state  $|\vartheta, \varphi\rangle$

$$|\Psi(t)\rangle = U(t)|\vartheta, \varphi\rangle, \quad U(t) = \exp\left(-it \frac{H_0}{\hbar}\right)$$

is obtained from Equation 1.17 replacing  $\varphi \rightarrow \varphi + \omega_0 t$ , since

$$H_0 |k, A\rangle = E_k |k, A\rangle, \quad E_k = \hbar\omega_0 \left(k - \frac{A}{2}\right)$$

## 1.4

## Squeezed Atomic States

As we have seen, if an atomic system is prepared in a coherent state, the square fluctuations of the components of the vector  $\mathbf{S}$  in any direction perpendicular to  $\langle \mathbf{S} \rangle$  are equal to  $A/4$ . Thus fluctuations are uniformly distributed, that is, fluctuations of any component lying in the plane tangent to the sphere and perpendicular to the average spin vector are the same. Physically, this is clear since in a coherent state, individual atoms are all in the same state (1.16) and, thus, the square of the fluctuation of the component of  $\mathbf{S}$  is simply the sum of the fluctuations corresponding to different atoms. One may say that the atoms are *not correlated*, in the sense that the mean value of any observable can be calculated by averaging over individual atomic states.

In principle, it is possible to form a *correlated* superposition of atomic states in such a way that fluctuations are no longer distributed uniformly in the tangent plane and, in some directions, the fluctuations would be smaller than  $A/4$ , at the expense of the fluctuations in the other perpendicular directions, which will be larger than  $A/4$ .

To correlate the atoms with each other, a transformation generated by a nonlinear combination of generators of the  $su(2)$  algebra representation is required, since linear transformations only rotate individual isotopic spins and would not produce between the atoms a correlation of the type described above. One way to introduce a correlation into a collective atomic system consists in inserting a nontrivial phase in Equation 1.17, that depends on a set of parameters  $\mu$

$$|\vartheta, \varphi, \mu\rangle = e^{iA/2\varphi} \sum_{k=0}^A \sqrt{\frac{A!}{k!(A-k)!}} e^{-ik\varphi - if(k, \mu)} \cos^k \frac{\vartheta}{2} \sin^{A-k} \frac{\vartheta}{2} |k, A\rangle$$

Here  $f(k, \mu)$  is a nonlinear function of  $k$ , which is equivalent to the application to the coherent state (1.17) of a unitary transformation, which is nonlinear in the collective inversion operator  $S_z$ :  $|\vartheta, \varphi, \mu\rangle = U_s(\mu)|\vartheta, \varphi\rangle$ . A simple model for such a transformation was discussed in [17]

$$U_s(\mu) = \exp[-i\mu S_z^2] \quad (1.26)$$

This model is analogous to the field-mode evolution in the presence of the Kerr medium; however, it has a finite number of states and we call it the *finite Kerr medium*. Upon applying the transformation (1.26) to the atomic operators  $S_{\pm}$ , nonlinear phases are generated:

$$S_+(\mu) = U_s^\dagger S_+ U_s = S_+ \exp\left[2i\mu \left(S_z + \frac{1}{2}\right)\right]$$

$$S_-(\mu) = U_s^\dagger S_- U_s = \exp\left[-2i\mu \left(S_z + \frac{1}{2}\right)\right] S_-$$

which leads to a redistribution of quantum fluctuations in the plane tangent to the Bloch sphere. Note that the operators  $S_z^2$  and  $\{S_+, S_-\} \equiv S_+ S_- + S_- S_+$  commute

with  $S_z$ , and are thus unchanged by the transformation of (Equation 1.26). The value of fluctuations in the tangent plane are obviously invariant with respect to rotations about the  $z$  axis. In other words, the distribution of fluctuations in the tangent plane does not depend on the phase  $\varphi$  of the Bloch vector; so, we assume, without loss of generality, that  $\varphi = 0$ . The direction in the plane tangent to the Bloch sphere at the point with coordinates  $(\vartheta, \varphi = 0)$  is determined by a unit vector

$$\mathbf{n}_\perp = (-\cos \vartheta \sin \delta, \cos \delta, \sin \vartheta \sin \delta), \quad \mathbf{n} \perp \mathbf{n}_\perp$$

where the unit vector  $\mathbf{n} = (\sin \vartheta, 0, \cos \vartheta)$  defines the direction of the collective Bloch vector (Equation 1.22) and  $\delta$  is the angle between the vector  $\mathbf{n}_\perp$  and the positive direction of the  $y$  axis. Fluctuations of the projection of the vector  $\mathbf{S}$  in the direction  $\mathbf{n}_\perp$ ,

$$s_\delta = (\mathbf{S} \cdot \mathbf{n}_\perp) = -S_x \cos \vartheta \sin \delta + S_y \cos \delta + S_z \sin \delta \sin \vartheta \quad (1.27)$$

depend on the angle  $\vartheta$  that determines the position on the Bloch sphere. For a given value of the angle  $\vartheta$ , we can find the direction of maximum squeezing by changing the angle  $\delta$ . It is easy to see that

$$[s_\delta, s_{\delta+\pi/2}] = \mathbf{S} \cdot \mathbf{n}$$

Formally, the atomic squeezing can be quantified by the ratio

$$\zeta = \frac{\min \sigma_{s_\delta}}{\sqrt{A/4}} \quad (1.28)$$

where  $\min \sigma_{s_\delta}^2 = \min [ \langle s_\delta^2 \rangle - \langle s_\delta \rangle^2 ]$  is the smallest variation of the isotopic spin component normal to the mean spin  $\langle \mathbf{S} \rangle$ . Clearly, in the coherent state  $\zeta = 1$ , and the state is squeezed if  $\zeta < 1$ .

Let us find, for example, the amount of squeezing generated in a set of  $A$  atoms by the transformation (1.26) applied to a coherent atomic state  $|\pi/2, 0\rangle$ , located at the equator of the Bloch sphere (i.e.  $\vartheta = \pi/2$ ):

$$|\psi_c\rangle = U_s(\mu)|\pi/2, 0\rangle \quad (1.29)$$

From Equation 1.27, we obtain

$$\begin{aligned} \langle \psi_c | s_\delta | \psi_c \rangle &= \left\langle S_z \sin \delta + \frac{\cos \delta}{2i} (S_+(\mu) - S_-(\mu)) \right\rangle_0 \\ \langle \psi_c | s_\delta^2 | \psi_c \rangle &= \langle S_z^2 \rangle_0 \sin^2 \delta - \langle S_+(\mu) + S_-(\mu) \rangle_0 \frac{\cos^2 \delta}{4} \\ &\quad + \langle \{S_z, S_+(\mu)\} - \{S_z, S_-(\mu)\} \rangle_0 \frac{\sin 2\delta}{4i} \end{aligned}$$

where  $\langle \dots \rangle_0$  means the average over the coherent state  $|\pi/2, 0\rangle$ . Following Equation 1.17, we have

$$\langle S_z \rangle_0 = 0, \quad \langle S_z^2 \rangle_0 = \frac{A}{4}, \quad \langle \{S_+, S_-\} \rangle_0 = \frac{A(A-1)}{2}, \quad \langle \exp [2i\mu S_z] \rangle_0 = \cos^A \mu$$

It is easy to find, using the relations (1.21), that

$$\begin{aligned}\langle S_+(\mu) \rangle_0 &= \frac{A}{2} \cos^{A-1} \mu \\ \langle [S_+(\mu)]^2 \rangle_0 &= \frac{A(A-1)}{4} \cos^{A-2} 2\mu \\ \langle \{S_z, S_+(\mu)\} \rangle_0 &= i \frac{A(A-1)}{2} \cos^{A-2} \mu \sin \mu\end{aligned}\quad (1.30)$$

We obtain the following expression for the fluctuations of the spin projection (1.27) in the state (1.29):

$$\begin{aligned}(\Delta s_\delta(\mu))^2 &= \langle \psi_c | s_\delta^2 | \psi_c \rangle - (\langle \psi_c | s_\delta | \psi_c \rangle)^2 \\ &= \frac{A}{4} \left[ 1 + \frac{A-1}{4} \left( a + \sqrt{a^2 + b^2} \cos(2\delta - 2\nu) \right) \right]\end{aligned}\quad (1.31)$$

where

$$a = 1 - \cos^{A-2} 2\mu, \quad b = 4 \sin \mu \cos^{A-2} \mu, \quad \tan 2\nu = \frac{b}{a}$$

From Equation 1.31, we can see that the fluctuations are minimized in the direction determined by the angle

$$\delta = \frac{1}{2} \arctan \frac{b}{a} + \frac{\pi}{2}\quad (1.32)$$

and reach the value

$$(\Delta s(\mu))_{\min}^2 = \frac{A}{4} \left[ 1 - \frac{A-1}{4} \left( \sqrt{a^2 + b^2} - a \right) \right] \leq \frac{A}{4}\quad (1.33)$$

Obviously, fluctuations in the direction perpendicular to Equation 1.32 are maximized. It is clear that no squeezing can occur in the case of one atom (since this atom would not have another to correlate to). In the particular case of two atoms, the minimum value of the fluctuations (Equation 1.31) may be zero. It may also happen that both the mean spin and  $\min \sigma_{s_\delta}^2$  takes zero values at some state (for instance, in the two-atom Dicke state  $|1, A=2\rangle$ ), so that the parameter (Equation 1.28) becomes undefined.

It is instructive to visualize the state (1.29) on the two-dimensional sphere in terms of the so-called  $Q$  function (see, e.g. Chapter 10)

$$Q_\mu(\vartheta, \varphi) = \left| \langle \vartheta, \varphi | e^{-i\mu S_z^2} | \pi/2, 0 \rangle \right|^2\quad (1.34)$$

where  $|\vartheta, \varphi\rangle$  is the coherent state (1.17), in the limit of large number of atoms,  $A \gg 1$ . The matrix element appearing in Equation 1.34 takes the form

$$\langle \vartheta, \varphi | e^{-i\mu S_z^2} | \pi/2, 0 \rangle = \frac{1}{(1 + |\xi|^2)^{A/2}} \frac{1}{2^{A/2}} \sum_{k=0}^A \xi^{*k} C_k^A e^{-i\mu(k-(A/2))^2}\quad (1.35)$$

where  $\xi = \cot(\vartheta/2) e^{-i\varphi}$  and  $C_k^A$  are the binomial coefficients. In the limit  $A \gg 1$ , the binomial coefficients can be approximated by a Gaussian

$$C_k^A \approx \frac{2^{2S}}{\sqrt{\pi S}} \exp\left[-\frac{(k-S)^2}{S}\right] \quad (1.36)$$

where  $S = A/2$ . Substituting Equation 1.36 in Equation 1.35 and changing the summation index  $k - S = n$ , we get

$$\langle \vartheta, \varphi | e^{-i\mu S_z} | \pi/2, 0 \rangle = \frac{1}{(1 + |\xi|^2)^S} \frac{\xi^S}{2^S} \sum_{n=-S}^S \exp\left[-\frac{n^2}{S} - i\mu n^2 + n \ln |\xi| + in\varphi\right]$$

Using the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dx f(x) e^{2\pi i m x}$$

we obtain (extending the summation to infinity),

$$\sum_{n=-S}^S \exp\left[-\frac{n^2}{S} - i\mu n^2 + n \ln |\xi| + in\varphi\right] \approx \sqrt{\frac{\pi}{a}} \sum_{m=-\infty}^{\infty} e^{(b_m^2/4a)}$$

where

$$a = \frac{1}{S} + i\mu, \quad b_m = \ln |\xi| + i(\varphi + 2\pi m)$$

Finally, the  $Q$  function takes the form of cyclic Gaussian (describing, in particular, the formation of the Schrödinger cats on the sphere)

$$Q_\mu(\vartheta, \varphi) = \frac{2^{2S}}{\sqrt{1 + S^2\mu^2}} \left(\frac{|\xi|}{1 + |\xi|^2}\right)^{2S} \left| \sum_{m=-\infty}^{\infty} e^{(b_m^2/4a)} \right|^2 \quad (1.37)$$

Let us note that  $|a| \ll 1$  for typical values of the parameter  $\mu$ . With this assumption, the maximally squeezed state corresponds to a situation where each Gaussian in Equation 1.37 is very narrow. In this case, the dominant contribution to the Equation 1.37 comes from the  $m = 0$  term. In this case, the approximate expression for the  $Q$  function takes the form

$$Q_\mu(\vartheta, \varphi) = \frac{2^A}{\sqrt{1 + A^2\mu^2/4}} \left(\frac{|\xi|}{1 + |\xi|^2}\right)^A \exp\left[\frac{A((\ln |\xi|)^2 - \varphi^2) + A^2\mu\varphi \ln |\xi|}{4(1 + A^2\mu^2/4)}\right] \quad (1.38)$$

In particular, the coherent states correspond to the value 0 of the parameter  $\mu$

$$Q_0(\vartheta, \varphi) = 2^A \left(\frac{|\xi|}{1 + |\xi|^2}\right)^A \exp\left[-\frac{A}{4} (\varphi^2 - (\ln |\xi|)^2)\right] \quad (1.39)$$

Recalling that in the limit  $A \gg 1$ , relative fluctuations in the direction perpendicular to  $\mathbf{n} = (1, 0, 0)$  are of order  $\sim A^{-1/2}$ , we can expand  $\xi$  in series of  $\zeta$ ,  $|\zeta| \ll 1$ , where  $\vartheta \sim \pi/2 - \zeta$ , thereby obtaining an expression for the projection of the  $Q$  function onto the tangent plane  $(\varphi, \zeta)$ :

$$Q_0(\varphi, \zeta) = \exp \left[ -\frac{A}{4} (\varphi^2 + \zeta^2) \right] \quad (1.40)$$

with the plane centered on the  $x$  axis.

The corresponding level curves for  $Q_\mu(\vartheta, \varphi)$  take the form of ellipses for values of  $\mu$  that are not very large, i.e. the initial coherent state becomes squeezed in a certain direction. For larger values of  $\mu$ , these ellipses become twisted and deformed due to quantum self-interference effect (appearance of Schrödinger cat states on the sphere [18]). This deformation and deviation from the elliptical shape put limits to the maximum possible squeezing that can be achieved in the system with a given number of atoms subjected by the transformation (1.26).

An approximate expression for the  $Q$  function on the tangent  $(\varphi, \zeta)$  plane can be obtained by expanding  $|\xi\rangle$  in series of  $\zeta$ , giving

$$Q_\mu(\zeta, \varphi) = \frac{1}{\sqrt{1 + A^2\mu^2/4}} \exp \left[ -A \frac{\zeta^2 (2 + A^2\mu^2) + 2A\mu\varphi\zeta + 2\varphi^2}{8(1 + A^2\mu^2/4)} \right] \quad (1.41)$$

The major axis of the ellipse makes an angle

$$\tan 2\phi = \frac{2}{A\mu}$$

with the  $\varphi$  axes. In the rotated frame,

$$\varphi' = \varphi \cos \phi + \zeta \sin \phi$$

$$\zeta' = \zeta \cos \phi - \varphi \sin \phi$$

the  $Q$  function takes the form

$$Q_\mu(\zeta', \varphi') = \frac{1}{\sqrt{A^2\mu^2/4 + 1}} \exp \left[ -\frac{A}{8} \left( \frac{\varphi'^2}{a^2} + \frac{\zeta'^2}{b^2} \right) \right]$$

where

$$a^2 = \left( 2 - \frac{A\mu}{\sqrt{A^2\mu^2/4 + 1}} \right)^{-1}, \quad b^2 = \left( 2 + \frac{A\mu}{\sqrt{A^2\mu^2/4 + 1}} \right)^{-1}$$

For large values of  $\mu$ ,  $a^2 \rightarrow A^2\mu^2/4$ ,  $b^2 \rightarrow 1/4$ , exhibiting growing squeezing – beyond the maximum value (Equation 1.33). The above equations show that the tangent plane description of the atomic squeezing fails for large values of  $\mu$ .

Another transformation that generates squeezing when applied to atomic coherent states is of the form

$$V_s(\mu) = \exp(\mu S_+^2 - \mu^* S_-^2)$$

Unfortunately, calculations for this transformation are not as simple as for the transformation (1.26). However, in the limit of small  $\mu$ , we can use the Taylor expansion and find the transformed collective atomic operators up to the second order in  $\mu$  with a consecutive calculation of fluctuations in the tangent plane.

## 1.5 Atoms with $n > 2$ Energy Levels

This section is devoted to the description of atomic systems with more than two energy levels [19]. First, we introduce a general formalism for a collection of  $A$  atoms, each atom containing  $n$  energy levels. We assume that the atoms are distinguishable, so the state vector describing the system is symmetric under permutation of indices labeling the individual atoms. We then specialize the discussion to systems with three energy levels, and then we consider a particular case of systems with three energy levels.

### 1.5.1 Systems with $n$ Energy Levels

Let us consider a single atom with  $n$  energy levels. Vectors  $|j\rangle$ ,  $j = 1, 2, \dots, n$ , describing states with energies  $E_j$  (bare states) form an orthonormal basis,  $\langle k|j\rangle = \delta_{jk}$ ,  $\sum_{j=1}^n |j\rangle\langle j| = I$ , in the  $n$ -dimensional Hilbert space for this single atom. We represent the states  $|j\rangle$  as column vectors in the  $n$ -dimensional linear space

$$|1\rangle \rightarrow \begin{bmatrix} 0 \\ \dots \\ 0 \\ \dots \\ 1 \end{bmatrix}, \quad |j\rangle \rightarrow \begin{bmatrix} 0 \\ \dots \\ 1_j \\ \dots \\ 0 \end{bmatrix}, \quad |n\rangle \rightarrow \begin{bmatrix} 1 \\ \dots \\ 0 \\ \dots \\ 0 \end{bmatrix} \quad (1.42)$$

It is easy to see that a parameterization, similar to Equation 1.2, of an arbitrary state  $|\psi\rangle$  has the form

$$|\psi\rangle \rightarrow \begin{bmatrix} e^{i\varphi_1} \cos \frac{\vartheta_1}{2} \\ e^{i\varphi_2} \sin \frac{\vartheta_1}{2} \cos \frac{\vartheta_2}{2} \\ \dots \\ \dots \\ e^{i\varphi_{n-1}} \sin \frac{\vartheta_1}{2} \dots \sin \frac{\vartheta_{n-2}}{2} \cos \frac{\vartheta_{n-1}}{2} \\ \sin \frac{\vartheta_1}{2} \dots \sin \frac{\vartheta_{n-1}}{2} \end{bmatrix} \quad (1.43)$$

where the phase of the ground state is chosen to be zero for simplicity. Clearly, the transformations preserving the norm of the state vector (1.43) are elements of the group  $U(n)$ .

The free Hamiltonian of the system has the form

$$H_0 = \sum_{j=1}^n E_j |j\rangle\langle j| \quad (1.44)$$

The configuration of the energy levels may be arbitrary, but it is convenient to choose the energies in such a way that  $E_j \leq E_{j+1}$ . Operators that generate

transitions between levels are  $s^{ij} = |j\rangle\langle i|$ , ( $j \neq i$ ),  $i, j = 1, \dots, n$  and satisfy the following commutation relations:

$$[s^{ij}, s^{kl}] = \delta_{ik}s^{lj} - \delta_{jl}s^{ik} \quad (1.45)$$

which are those for the generators of the  $U(n)$  algebra (see Appendix 11.1).

Let us recall that any element of the  $U(n)$  group can be represented as the product of an element of the  $su(n)$  group by an element of the  $U(1)$  group. Obviously the element of the  $U(1)$  group is related to a global phase of the system state. To exclude this phase, we use the traceless combinations

$$s_z^{jj+1} = \frac{1}{2} (|j+1\rangle\langle j+1| - |j\rangle\langle j|), \quad j = 1, \dots, n-1$$

These operators are interpreted as atomic inversions between levels  $j+1$  and  $j$ . This interpretation is convenient considering the ordering of our energy levels. Obviously, there are  $n-1$  independent operators  $s_z^{jj+1}$ . The projectors  $|j\rangle\langle j|$  are expressed in terms of  $s_z^{jj+1}$  as

$$s^{jj} = |j\rangle\langle j| = \frac{1}{n} + 2 \sum_{k=1}^{j-1} s_z^{kk+1} - \frac{2}{n} \sum_{k=1}^{n-1} (n-k) s_z^{kk+1} \quad (1.46)$$

Thus, the free Hamiltonian (1.44) in terms of the operators  $s_z^{jj+1}$  has the form

$$H_0 = 2 \sum_{k=1}^{n-1} s_z^{kk+1} \left[ Ek - \sum_{j=1}^k E_j \right] + E \quad (1.47)$$

where  $E$  is the “energy of the center of the spectrum”, which is defined as  $E = \frac{1}{n} \sum_{j=1}^n E_j$ . The operators  $s_z^{jj+1}$ ,  $s^{ij} = |i\rangle\langle j|$ ,  $j \neq i$ ,  $i, j = 1, \dots, n$  form the  $n$ -dimensional representation of the  $su(n)$  algebra.

In the case of  $A$  identical atoms with  $n$  energy levels, the collective operators

$$S^{ij} = \sum_{a=1}^A |a, i\rangle\langle a, j|, \quad i, j = 1, \dots, n \quad (1.48)$$

are introduced where  $|a, i\rangle$  is a state of the  $a$ th atom. The operators  $S^{ij}$  obviously satisfy commutation relations identical to those of the individual atomic operators of Equation 1.45. Thus the problem is reduced to the construction of the irreducible representations for the  $su(n)$  group.

If the atoms are *identical* (the Hamiltonian is invariant under permutations of atoms), it is sufficient to consider only the symmetric representation. A useful method for obtaining the matrix elements of operators acting in the space of this representation of  $su(n)$  is the so-called Schwinger construction. Let us introduce a set of auxiliary creation and annihilation operators satisfying the usual bosonic commutation relations:

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0$$

We may think of  $b_i$  and  $b_i^\dagger$  as the creation and annihilation operators for the excitation at the  $i$ th atomic level ( $i = 1, 2, \dots, n$ ). In terms of  $b_i$  and  $b_i^\dagger$ , the collective atomic operators have the form

$$S^{ij} = b_j^\dagger b_i, \quad i, j = 1, 2, \dots, n \quad (1.49)$$

It is directly checked that these operators satisfy the commutation relations of the  $u(n)$  algebra (1.45). The collective operators (1.49) act on the Hilbert space of  $n$  harmonic oscillators with the basis

$$|k_1, k_2, \dots, k_n\rangle = |k_1\rangle \otimes |k_2\rangle \otimes \dots \otimes |k_n\rangle$$

such that

$$b_i |k_1, \dots, k_i, \dots, k_n\rangle = \sqrt{k_i} |k_1, \dots, k_i - 1, \dots, k_n\rangle$$

It can easily be seen that the following operator commutes with all the operators of the algebra:

$$\hat{A} = \sum_{j=1}^n S^{jj}, \quad [\hat{A}, S^{ij}] = 0$$

The eigenvalue of this operator  $\hat{A}$  is just  $k_1 + k_2 + \dots + k_n = A$ . Here,  $k_i$  represents the number of atoms at the  $i$ th energy level and  $A$  is the total number of atoms. The free atomic Hamiltonian takes on the form

$$H_0 = \sum_{i=1}^n E_i S^{ii} = \sum_{i=1}^n E_i b_i^\dagger b_i$$

The atomic inversion operators are now introduced as

$$S_z^{ij} = \frac{1}{2} (S^{jj} - S^{ii}) = \frac{1}{2} (b_j^\dagger b_j - b_i^\dagger b_i)$$

and the transition operators  $S_{\pm}^{ij}$ , ( $j > i$ )

$$S_+^{ij} = S^{ij} = b_j^\dagger b_i, \quad S_-^{ij} = (S_+^{ij})^\dagger = S^{ji} = b_i^\dagger b_j$$

It is easy to find the matrix elements of the operators  $S_{\pm}^{ij}$ , Using Equation 1.49

$$\begin{aligned} S_+^{ij} |k_1, \dots, k_i, \dots, k_j, \dots, k_n\rangle &= \sqrt{k_i(k_j + 1)} |k_1, \dots, k_i - 1, \dots, k_j + 1, \dots, k_n\rangle, \\ S_-^{ij} |k_1, \dots, k_i, \dots, k_j, \dots, k_n\rangle &= \sqrt{k_j(k_i + 1)} |k_1, \dots, k_i + 1, \dots, k_j - 1, \dots, k_n\rangle, \\ S_z^{ij} |k_1, \dots, k_i, \dots, k_j, \dots, k_n\rangle &= \frac{1}{2} (k_j - k_i) |k_1, \dots, k_i, \dots, k_j, \dots, k_n\rangle \end{aligned} \quad (1.50)$$

The operators  $\{S_z^{ij}, S_+^{ij}, S_-^{ij}\}_{i \neq j}$  span a representation of the  $su(n)$  algebra. Note that the operators  $S_z^{ij}$  and  $S_{\pm}^{ij}$  form  $su(2)$  subalgebras:

$$[S_z^{ij}, S_{\pm}^{ij}] = \pm S_{\pm}^{ij} \quad (1.51)$$

## 1.5.2

**Systems with Three Energy Levels**

Let us consider a particular case of systems with three energy levels. A pure state is a superposition of bare states  $|j\rangle, j = 1, 2, 3$ ,

$$|\psi\rangle = \cos \vartheta_1 |1\rangle + e^{i\varphi_1} \sin \vartheta_1 \cos \vartheta_2 |2\rangle + e^{i\varphi_2} \sin \vartheta_2 \sin \vartheta_1 |3\rangle$$

where  $0 \leq \varphi_{1,2} \leq 2\pi$ ,  $0 \leq \vartheta_{1,2} \leq \pi/2$ . In terms of the diagonal projectors  $|j\rangle\langle j|$ , the free Hamiltonian for a single atom has the form

$$H_0 = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2| + E_3 |3\rangle\langle 3|$$

so that  $|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = I$ .

We suppose that the transitions in atomic systems only take place in the dipole approximation, that is, there are no transitions between levels with the same parity. Thus, there are only three possible configurations for systems with three energy levels: cascade ( $\Xi$ ), lambda ( $\Lambda$ ), and V configurations. Enumerating energy levels in such a way that  $E_1 \leq E_2 \leq E_3$ , we see that the transitions allowed in the  $\Xi$  system are  $1 \leftrightarrow 2 \leftrightarrow 3$ ; in the  $\Lambda$  system,  $1 \leftrightarrow 3 \leftrightarrow 2$ ; and in the V system  $3 \leftrightarrow 1 \leftrightarrow 2$ . Inversion operators relevant to each physically different configuration can be constructed without difficulty:

1. For the  $\Xi$ -type system the inversion operators are defined as

$$s_z^{12} = \frac{1}{2} (|2\rangle\langle 2| - |1\rangle\langle 1|), \quad s_z^{23} = \frac{1}{2} (|3\rangle\langle 3| - |2\rangle\langle 2|)$$

and the free Hamiltonian takes on the form

$$H_0 = \frac{2}{3} (E_2 + E_3 - 2E_1) s_z^{12} + \frac{2}{3} (2E_3 - E_1 - E_2) s_z^{23} + E \quad (1.52)$$

Here, the energy of the center of a spectrum  $E$  is equal to

$$E = \frac{1}{3} (E_1 + E_2 + E_3)$$

2. For the  $\Lambda$ -type system, the inversion operators are defined as

$$s_z^{13} = \frac{1}{2} (|3\rangle\langle 3| - |1\rangle\langle 1|), \quad s_z^{23} = \frac{1}{2} (|3\rangle\langle 3| - |2\rangle\langle 2|)$$

and the free Hamiltonian takes on the form

$$H_0 = \frac{2}{3} (E_2 + E_3 - 2E_1) s_z^{13} + \frac{2}{3} (E_1 + E_3 - 2E_2) s_z^{23} + E \quad (1.53)$$

3. For the V-type system the inversion operators are defined as

$$s_z^{12} = \frac{1}{2} (|2\rangle\langle 2| - |1\rangle\langle 1|), \quad s_z^{13} = \frac{1}{2} (|3\rangle\langle 3| - |1\rangle\langle 1|)$$

and the free Hamiltonian takes on the form

$$H_0 = \frac{2}{3} (2E_2 - E_1 - E_3) s_z^{12} + \frac{2}{3} (2E_3 - E_1 - E_2) s_z^{13} + E \quad (1.54)$$

The transition operators in the system are given by Equations 1.7 and 1.45. Some of the transitions are absent in the  $\Xi$ ,  $\Lambda$ , or  $V$  configurations. Explicitly, the transition operators for these cases are

$$\Xi : |2\rangle\langle 1|, |3\rangle\langle 2|, \text{ h.c.}$$

$$\Lambda : |3\rangle\langle 1|, |3\rangle\langle 2|, \text{ h.c.}$$

$$V : |2\rangle\langle 1|, |3\rangle\langle 1|, \text{ h.c.}$$

We can immediately write, using Equations 1.48, the operators describing a collection of  $A$  indistinguishable atoms with three energy levels. There are still three configurations, and appropriate operators are simply obtained by substituting  $s_z^{jk} \leftrightarrow S_z^{jk}$  and  $E \rightarrow AE$ . Then the atomic transition operators are

$$\Xi : S_{\pm}^{12}, S_{\pm}^{23}$$

$$\Lambda : S_{\pm}^{13}, S_{\pm}^{23}$$

$$V : S_{\pm}^{12}, S_{\pm}^{13}$$

The matrix realization of each  $S_z^{jk}, S_{\pm}^{jk}$  for each configuration form a symmetric irreducible representation of the  $su(3)$  algebra of the dimension  $(A+1)(A+2)/2$ . The operators  $\{S_z^{jk}, S_{+}^{jk}, S_{-}^{jk}\}$  form the  $su(2)$  subalgebras (1.51) and all the  $S_z^{jk}$  operators commute. All the other commutation relations and the matrix elements of the collective operators in symmetric representations are easily obtained using the Schwinger representation for the collective atomic operators.

## 1.6 Problems

- 1.1** Show that the Von Neumann entropy of a two-level system, defined as  $S = -\text{Tr}(\rho \log \rho)$  is a monotonous function of the purity of the atomic state,  $\mathcal{P} = \text{Tr}(\rho^2)$  in the interval  $1/2 \leq \mathcal{P} \leq 1$ . Hint: prove that  $S = -\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2$ , where  $\lambda_{1,2} = (1 \pm \sqrt{2\mathcal{P} - 1})/2$ .
- 1.2** For squeezed states in the case of two atoms,  $A = 2$ , find the state  $|\psi_0\rangle = U_s(\mu_0)|\pi/2, 0\rangle$  and direction  $\delta_0$  for which the fluctuation of the projection of vector  $\mathbf{S}$  is zero (i.e.  $|\psi_0\rangle$  is the eigenstate of the operator  $s(\delta_0)$ ). *Answer:*

$$|\psi_0\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ -i\sqrt{2} \\ 1 \end{bmatrix}, \quad \delta_0 = \frac{3\pi}{4}$$

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- 1.3 Prove that for squeezed states in the case of many atoms,  $A \gg 1$  and  $|\mu| \ll 1$ , the following approximation can be made for  $\langle \Delta s^2(\mu) \rangle$  in Equation 1.27,

$$\langle \Delta s^2(\mu) \rangle \approx \frac{A}{4} \left( \frac{1}{4\alpha^2} + \frac{2}{3}\beta^2 \right)$$

where  $\alpha = (A/2)\mu$  and  $\beta = (A/2)\mu^2$ , and its minimum value,

$$\langle \Delta s^2(\mu) \rangle_{\min} \approx \frac{1}{2} \left( \frac{A}{6} \right)^{1/3} \quad (1.55)$$

is achieved when

$$\mu_0 = \frac{24^{1/6}}{2} \left( \frac{A}{2} \right)^{-2/3} \quad (1.56)$$