1
Superconductivity: Basics and Formulation

1.1
Introduction

The Bardeen–Cooper–Schrieffer (BCS) theory captures the essential physics that gives rise to the condensation of the Cooper pairs into a coherent superconducting state. It specifically deals with a homogeneous and clean system in three dimensions and is essentially a self-consistent, mean field treatment. Following its success, subsequent theories are based on a mean field treatment on the BCS or related models. In order to account for inhomogeneous systems, such as those occurring in multilayer systems of superconductor- insulators, or normal metals, de Gennes, and independently Bogoliubov, derived a system of coupled equations between electrons and holes that yield the solutions for the fermionic quasiparticles above the superconducting condensate, which are separated from the condensate by the superconducting energy gap, Δ. Such a formalism is useful for computing the properties of ultrathin SC films and clean SC nanowires, as will be discussed in Chapter 2.

Prior to the BCS microscopic theory, Ginzburg and Landau sought to generalize the Landau theory of second order phase transitions to the superconductivity problem. The result is the celebrated Ginzburg–Landau (GL) theory of superconductivity. The free energy is written in terms of the superconducting order parameter, which in this case is a complex quantity. This theory is expected to be valid near T_c, and was derived from a microscopic BCS model by Gorkov.

Gorkov invented a powerful methodology by deducing the equations of motion for the Green's functions. An anomalous Green's function, F, accounting for pair-correlations was introduced in addition to the normal electron Green's function. The two Green's functions form a closed set of equations, the solutions of which yield all the results of the BCS theory, and moreover, can be readily extended to incorporate dirty systems with impurities, as well as deal with nonlinearities, dynamics, and so on. Thus, type-II superconductors can readily be described. From the perspective of this text, the central importance of the Gorkov equations is the ultimate deduction of the Usadel equation for related Green's functions (to the Gorkov GF's), in the limit of dirty systems. The derivation of the Usadel diffusion equation was based on the works of Eilenberger and of Larkin and Ovchinnikov,

One-Dimensional Superconductivity in Nanowires, First Edition. F. Altomare and A.M. Chang

who independently applied the quasiclassical approximation to the Gorkov equations, and identified an energy-integrated version of the Gorkov Green's functions. These approaches led to a simplification of the Gorkov equations into Boltzmann transport-like equations for these modified Green's functions.

The Usadel diffusion equation is much more tractable and amenable to numerical implementations, enabling realistic experimental geometries and situations to be analyzed. In particular, issues of quasiparticle injection at the normal-superconductor interface, nonequilibrium quasiparticle distribution, and so on, are readily computed. These methodologies based on the Usadel equation are naturally suited to analyzing systems with 1D SC nanowires. In fact, Dr. Pauli Virtanen provides downloadable C++ programs for such computations on his website: http://ltl.tkk.fi/~theory/usadel1/!

Beyond these standard methodologies, a powerful technique for analyzing complex superconducting phenomena has been developed over the past 20 years, which is particularly well-suited for understanding quantum phase transitions, such as those in dissipative Josephson junction arrays of 2D and 1D varieties, dissipative single Josephson junctions, vortex states and associated transitions, as well as transitions in the 1D superconducting nanowire system. This technique is based on the path integral formulation. Thus, introducing such a formulation is important in bridging the gap to enable the study of quantum-tunneling processes, such as the macroscopic quantum tunneling of the phase slip, which is central to the understanding of the unique behaviors in 1D superconducting nanowires.

1.2 BCS Theory

Conventional superconductivity has, as its key components, the binding of the Cooper pairs, and the Bose condensation of such Cooper pairs. The binding arises from an attractive interaction between electrons mediated by phonons or other excitations, for example, plasmons. The essential behaviors and outstanding characteristics are captured by a self-consistent mean field approximation. The celebrated BCS theory, proposed by Bardeen, Cooper, and Schrieffer in 1957 [1], provides both a microscopic model, which provides the underpinning of our current understanding of conventional superconductivity and a basic formalism, which is most conveniently cast in terms of the second-quantized formalism, that is, in terms of the creation and destruction operators of the single particle electronic states.

The Cooper instability [2] indicates that the Fermi sea is unstable to the formation of bound-pairs (Cooper pairs) of electrons, in the presence of an effective, attractive interaction. In the original formulation, and in most conventional superconductors to date, the attractive interaction is a time-retarded interaction mediated by lattice vibration phonons. A passing electron, whose velocity, given by the Fermi velocity, is much larger than the phonon propagation speed, polarizes the local ions: before the lattice can relax, a second electron arrives and feels the attraction from the still-polarized positive ions. This attraction produces a pairing of electrons which gives

rise to an entity which is bosonic in character. The two electrons thus form a Cooper pair whose size-scale is denoted by the superconducting coherence length ξ.

The Bose condensation of such Cooper pairs into a macroscopic, coherent quantum ground state is the source of the unusual physical properties associated with superconductivity. In a weak-coupling superconductor, where the electron–phonon coupling is much smaller than the Fermi energy, the size of the Cooper pair, or ξ, is typically much larger than the Fermi wavelength λ_F. In a homogeneous system, the attraction is maximal when the relative total momentum of the two electrons is zero, that is, when the center of mass is stationary. This gives the requirement of pairing between $+\boldsymbol{k}$ and $-\boldsymbol{k}$ states. At the same time, the s-wave channel has the largest attractive potential, leading to an s-wave superconductor, for which the energy gap created by the condensation is isotropic in k-space. As the spatial wavefunction is symmetric, in order to satisfy Pauli's exclusion principle, the spin must be in an antisymmetric, singlet state.

To simplify the calculations, the BCS model assumes that the attractive phonon-mediated potential is a delta-function in real space (a contact interaction). This is sensible since the interaction occurs on the length scale of the Fermi wavelength λ_F, and the length scale of the phonons associated with the Debye frequency, of order the lattice constant, whereas the Cooper pair has a much larger scale given by the superconducting coherence length, ξ. The coherence length has its minimum value at $T = 0$, with a value typically exceeding 10 nm.

The second quantization formulation is then equivalent to a formulation based on the occupation of single-particle states – in this case of a homogeneous, disorder-free system, the $\boldsymbol{k}$-states. The Pauli exclusion in the occupation dictating that a given state to be either singly occupied or unoccupied, with multi-occupation forbidden, is encoded in the anticommutation relation of the creation and annihilation operators. In the $\boldsymbol{k}$-representation, the model Hamiltonian is given by

$$H = \sum_{\boldsymbol{k}\sigma} \epsilon_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \sum_{\boldsymbol{k},\boldsymbol{k}':|\epsilon_{\boldsymbol{k}}|,|\epsilon_{\boldsymbol{k}'}|\leq\hbar\omega_D} V c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} \,. \tag{1.1}$$

Here, $c^{\dagger}_{\boldsymbol{k}\sigma}$ is the creation operator for an electron in momentum-state $\boldsymbol{k}$ and spin σ. $\epsilon_{\boldsymbol{k}} = \hbar^2\boldsymbol{k}^2/(2m) - E_F$ is the energy of the single-particle $\boldsymbol{k}$-state measured from the Fermi energy of the system, $V > 0$ for an attractive interaction, and the scattering is between $(\boldsymbol{k}, -\boldsymbol{k})$ – and $(\boldsymbol{k}', -\boldsymbol{k}')$-states. The pairing in real space is an s-wave, which is symmetric under exchange, while the spin is an antisymmetric singlet, in order to satisfy Pauli exclusion. Because phonons mediate the interaction, the scattering term is restricted to states within $\hbar\omega_D$ of the Fermi surface, where ω_D is the Debye frequency. This form of the phonon-mediated attractive interaction between electrons is equivalent to a contact potential interaction in real space, that is,

$$V(\boldsymbol{r}_i, \boldsymbol{r}_j) = -V\mathsf{V}\delta(\boldsymbol{r}_i - \boldsymbol{r}_j) = -g\delta(\boldsymbol{r}_i - \boldsymbol{r}_j) \,, \tag{1.2}$$

between electrons at positions $\boldsymbol{r}_i$, $\boldsymbol{r}_j$, where V is the interacting strength, V is the system volume, and g the volume-independent Gorkov coupling.

To find the ground state energy of the system, a variational wavefunction in the form is assumed

$$|\Psi_{\rm BCS}\rangle = \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} \right) |0\rangle \,, \tag{1.3}$$

where $|0\rangle$ is the vacuum, and $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ are the amplitude of electron-like Cooper pairs and hole-like Cooper pairs, respectively, chosen to be real, and subjected to the normalization condition $u^2_{\boldsymbol{k}} + v^2_{\boldsymbol{k}} = 1$. By varying the expectation value of the ground state $E_{\rm g}$ energy and minimizing $E_{\rm g} = \langle \Psi_{\rm BCS}|H|\Psi_{\rm BCS}\rangle$ with respect to $v_{\boldsymbol{k}}$, one finds the condition for

$$2\epsilon_{\boldsymbol{k}} u_{\boldsymbol{k}} v_{\boldsymbol{k}} = \sum_{\boldsymbol{k}':|\epsilon_{(\boldsymbol{k},\boldsymbol{k}')}|\leq \hbar\omega_{\rm D}} V \left(u^2_{\boldsymbol{k}} - v^2_{\boldsymbol{k}} \right) u_{\boldsymbol{k}'} v_{\boldsymbol{k}'} \tag{1.4}$$

for the ground state energy

$$\langle \Psi_{\rm BCS}|H|\Psi_{\rm BCS}\rangle = 2\sum_{\boldsymbol{k}'} v^2_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} - \sum_{\boldsymbol{k},\boldsymbol{k}':|\epsilon_{(\boldsymbol{k},\boldsymbol{k}')}|\leq \hbar\omega_{\rm D}} V u_{\boldsymbol{k}} v_{\boldsymbol{k}} u_{\boldsymbol{k}'} v_{\boldsymbol{k}'} \,. \tag{1.5}$$

Changing to the customary variables of $E_{\boldsymbol{k}}$ and $\Delta_{\boldsymbol{k}}$, where

$$u_{\boldsymbol{k}} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\epsilon_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}} \tag{1.6}$$

and

$$v_{\boldsymbol{k}} = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\epsilon_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}} \tag{1.7}$$

with

$$E_{\boldsymbol{k}} = \sqrt{\epsilon^2_{\boldsymbol{k}} + \Delta^2_{\boldsymbol{k}}} \,, \tag{1.8}$$

the minimization condition yields the self-consistency expression

$$\Delta_{\boldsymbol{k}} = \frac{1}{2} \sum_{\boldsymbol{k}':|\epsilon_{\boldsymbol{k}'}|\leq \hbar\omega_{\rm D}} \frac{V\Delta_{\boldsymbol{k}'}}{\sqrt{\epsilon^2_{\boldsymbol{k}'} + \Delta^2_{\boldsymbol{k}'}}} \,. \tag{1.9}$$

This is the BCS gap equation for the superconducting gap of a quasi-particle of momentum $\boldsymbol{k}$. The restriction in $\boldsymbol{k}'$ leads to a $\boldsymbol{k}$-independent gap

$$\Delta_{\boldsymbol{k}} = \Delta \,, \quad |\epsilon_{\boldsymbol{k}}| < \hbar\omega_{\rm D} \,, \tag{1.10}$$

and

$$\Delta_{\boldsymbol{k}} = 0 \,, \quad \text{otherwise} \,. \tag{1.11}$$

Converting to an integral using the normal electron density of states, and approximating the density of states with its value at the Fermi energy $N(0)$ gives

$$\frac{1}{N(0)g} = \int_0^{\hbar\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} \,. \tag{1.12}$$

The solution yields the celebrated BCS expression for the energy gap Δ

$$\Delta = 1.14\hbar\omega_D e^{-1/N(0)g} = 1.14\hbar\omega_D e^{-1/[N(0)VV]} \,, \tag{1.13}$$

with $g \equiv VV$ the Gorkov coupling constant. The nonanalytic dependence on g indicates that this result cannot be obtained by a perturbation calculation in the small parameter V/E_F!

At finite temperatures $T > 0$, the gap equation is modified by the Fermi–Dirac occupation of the fermionic excitations across the gap ($\beta = 1/(k_B T)$)

$$\frac{1}{N(0)g} = \int_0^{\hbar\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} \left[1 - 2\frac{1}{1 + \exp(\beta\sqrt{\epsilon^2 + \Delta^2})}\right] \,. \tag{1.14}$$

At T_c, the gap vanishes, giving the condition

$$\frac{1}{N(0)g} = \int_0^{\hbar\omega_D} \frac{d\epsilon}{\epsilon} \tanh\frac{\epsilon}{2k_B T} \tag{1.15}$$

and the relation

$$\Delta = 1.76 k_B T_c \,. \tag{1.16}$$

The BCS gap, $\Delta(T)$, occupies a central role in the phenomenon of conventional superconductivity. From the existence of the gap, many important properties can be derived, such as the quasiparticle excitation spectrum

$$E_k = \sqrt{\epsilon_k^2 + \Delta_k^2} \tag{1.17}$$

and the density of states

$$D_{qp}(E_k) = D(E_F)\frac{E_k}{\sqrt{E_k^2 - \Delta_k^2}} = N(0)\frac{E_k}{\sqrt{E_k^2 - \Delta_k^2}} \,; \quad D(E_F) \equiv N(0) \,, \tag{1.18}$$

the London equations [3] accounting for the Meissner effect of flux expulsion, flux quantization, specific heat, critical current, and so on. Our purpose here is to introduce the essential aspect of BCS superconductivity, and to wet the appetite of those readers unfamiliar with this subject matter. We refer to the excellent standard textbooks, for example, de Gennes [4], Tinkham [5], Ashcroft and Mermin [6], Fetter and Walecka [7], and so on, for a comprehensive treatment.

1.3 Bogoliubov–de Gennes Equations – Quasiparticle Excitations

The BCS theory in the previous section is for a 3D, homogeneous system without disorder. An essential feature of the theory is the pairing between time-reversed states. The role of time-reversed states was highlighted by Anderson's well-known theoretical analysis, demonstrating that nonmagnetic impurities do not suppress the superconducting gap [8]. To go beyond a translational invariant system, and account for boundaries, and for the possibility of tunnel junctions or SNS bridges, the theories of Bogoliubov [9, 10] and de Gennes [11] are required.

In the method of Bogoliubov and de Gennes, a solution is sought for the fermionic electron-like and hole-like quasiparticle excitations above the gap, which separates the condensed Cooper pairs from these unbound, quasiparticles. The theory was developed in part to address situations where there are boundaries or coupling to normal metals [11]. Because the presence of boundaries is properly accounted for, this formalism lends itself readily to deduce the behaviors of clean, 1D superconducting nanowires in which the boundaries are in the two lateral directions, perpendicular to the length of the nanowire, as was done in the work of Shanenko, Croitoru, Peeters, and collaborators [12–14] (please see Chapter 2).

The fermionic electron- and hole-like quasiparticles are each linear combinations of the electron and hole wavefunctions in the normal state. However, because of the spatial dependences introduced by the boundaries, the linear combinations are in general not as simple as in the homogeneous case. The point contact attractive interaction between electrons may be spatially dependent: $-V(\boldsymbol{r})\mathsf{V}\delta(\boldsymbol{r}) \equiv -g(\boldsymbol{r})\delta(\boldsymbol{r})$ (see (1.2)).

The starting point is the equation of motion for the field operator $\psi_\sigma^\dagger(\boldsymbol{r}, t)$, which creates an electron of spin σ at position $\boldsymbol{r}$. In the special case of a homogeneous, clean system, it is related to the creation operator $c_{\boldsymbol{k}\sigma}^\dagger$ for the $\boldsymbol{k}\sigma$ state by the (inverse-)Fourier transform

$$\psi^\dagger(\boldsymbol{r}, t) = \frac{1}{\sqrt{2\pi}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} c_{\boldsymbol{k}}^\dagger(t) \; . \tag{1.19}$$

The usual commutation with the BCS Hamiltonian in real space gives the equation of motion:

$$\begin{aligned} -i\hbar \frac{\partial \psi_\sigma^\dagger(\boldsymbol{r}, t)}{\partial t} = &\left[\frac{\boldsymbol{p}^2}{2m} + U_{\mathrm{o}}(\boldsymbol{r}) \right] \psi_\sigma^\dagger(\boldsymbol{r}, t) \\ &- g(\boldsymbol{r}) \sum_{\sigma'} \psi_\sigma^\dagger(\boldsymbol{r}, t) \psi_{\sigma'}^\dagger(\boldsymbol{r}, t) \psi_{\sigma'}(\boldsymbol{r}, t) \; ; \quad g(\boldsymbol{r}) = V(\boldsymbol{r})\mathsf{V} \; . \end{aligned} \tag{1.20}$$

Here, the one-electron potential $U_{\mathrm{o}}(\boldsymbol{r})$ includes the boundary and static impurity potentials. Using a Hartree–Fock mean field approximation, the term cubic in the field operators is replaced in the anomalous channel by

$$\left\langle \psi_\sigma^\dagger(\boldsymbol{r}, t) \psi_{\sigma'}^\dagger(\boldsymbol{r}, t) \right\rangle \psi_{\sigma'}(\boldsymbol{r}, t) \; , \tag{1.21}$$

and, in the normal channel by

$$\left\langle \psi^{\dagger}_{\sigma'}(\boldsymbol{r},t)\psi_{\sigma}(\boldsymbol{r},t)\right\rangle \psi^{\dagger}_{\sigma}(\boldsymbol{r},t)\ . \tag{1.22}$$

Here, the averaging denotes a thermal average. For the anomalous channel, in anticipation of a coupling similar to s-wave coupling in the homogeneous case, only the dominant terms, which come from the down- and up-spin correlators (and up- and down-), are kept, that is,

$$g(\boldsymbol{r})\left\langle \psi^{\dagger}_{\downarrow}\psi^{\dagger}_{\uparrow}\right\rangle = V(\boldsymbol{r})\mathsf{V}\left\langle \psi^{\dagger}_{\downarrow}\psi^{\dagger}_{\uparrow}\right\rangle = \Delta^{\dagger}(\boldsymbol{r})\ , \tag{1.23}$$

which defines the partial potential $\Delta^{\dagger}(\boldsymbol{r})$. For the normal channel, the self-consistent potential contributes a term

$$U_{\rm sc}(\boldsymbol{r}) = -g(\boldsymbol{r})\left\langle \psi^{\dagger}_{\sigma'}\psi_{\sigma'}\right\rangle \tag{1.24}$$

to the single-particle potential $U(\boldsymbol{r}) = U_{\rm o}(\boldsymbol{r}) + U_{\rm sc}(\boldsymbol{r})$, where $U_{\rm o}$ represents the contributions from static background ions, impurities, and boundaries. New fermionic operators $\gamma^{\dagger}_{n\tau}$, which carry the usual anticommutation relationships, are introduced, namely,

$$\begin{aligned}
\psi^{\dagger}_{\uparrow}(\boldsymbol{r},t) &= \sum_n \left[u^{\dagger}_n(\boldsymbol{r})e^{iE_nt/\hbar}\gamma^{\dagger}_{n\uparrow} + v_n(\boldsymbol{r})e^{-iE_nt/\hbar}\gamma_{n\downarrow}\right]\ ,\\
\psi^{\dagger}_{\downarrow}(\boldsymbol{r},t) &= \sum_n \left[u^{\dagger}_n(\boldsymbol{r})e^{iE_nt/\hbar}\gamma^{\dagger}_{n\downarrow} - v_n(\boldsymbol{r})e^{-iE_nt/\hbar}\gamma_{n\uparrow}\right]\ ,
\end{aligned} \tag{1.25}$$

where the energies E_n are positive. The eigenfunctions $u_n(\boldsymbol{r})$ and $v_n(\boldsymbol{r})$ satisfy the following coupled equations, namely,

$$\begin{aligned}
E_n u_n(\boldsymbol{r}) &= \left[\frac{\boldsymbol{p}^2}{2m} + U(\boldsymbol{r})\right]u_n(\boldsymbol{r}) + \Delta(\boldsymbol{r})v_n(\boldsymbol{r})\\
E_n v_n(\boldsymbol{r}) &= -\left[\frac{\boldsymbol{p}^2}{2m} + U(\boldsymbol{r})\right]v_n(\boldsymbol{r}) + \Delta(\boldsymbol{r})^* u_n(\boldsymbol{r})\ .
\end{aligned} \tag{1.26}$$

The pair potential satisfies the self-consistency condition

$$\Delta(\boldsymbol{r}) = g(\boldsymbol{r})\sum_n v^{\dagger}_n(\boldsymbol{r})u_n(\boldsymbol{r})\left\{1 - 2\frac{1}{1+\exp[E_n/(k_{\rm B}T)]}\right\}\ . \tag{1.27}$$

In addition,

$$\begin{aligned}
U_{\rm sc}(\boldsymbol{r}) = -g(\boldsymbol{r})\sum_n \Bigg\{&|u_n(\boldsymbol{r})|^2\frac{1}{1+\exp[E_n/(k_{\rm B}T)]}\\
&+|v^{\dagger}_n(\boldsymbol{r})|^2\left[1 - \frac{1}{1+\exp[E_n/(k_{\rm B}T)]}\right]\Bigg\}\ .
\end{aligned} \tag{1.28}$$

In the presence of an external magnetic field, these equations are supplemented, of course, by the London equations. As expected, in the homogeneous case in three dimensions, they reproduce the BCS results.

1.4 Ginzburg–Landau Theory

The Ginzburg–Landau (GL) [15] approach to the description of conventional superconductivity is based on the notion of an order parameter, and follows Landau's phenomenological theory of second order phase transitions. The free energy of the superconducting state is envisioned to differ from the normal state by a contribution from the condensate when the order parameter becomes nonzero below a transition temperature T_c. The thermodynamic state is given by the solution which minimizes the free energy with respect to variations in the order parameter. In the case of a charged superconductor, which couples to the electromagnetic scalar and vector field, minimization with respect to the potentials leads to the London equations. The GL equations therefore represent a mean field treatment and are valid below, but near T_c. Very close to T_c, additional fluctuations may arise. Gorkov showed that the GL equations (see below) can be derived from a microscopic theory starting from a BCS model Hamiltonian [16].

For a superconductor, the order parameter Ψ is a complex quantity, having a magnitude, $|\Psi|$, and a phase, φ. The physical interpretation of this order parameter is that its modulus squared $2|\Psi|^2$ yields the density fraction of superfluid component. Thus, Ψ may be thought of as the "wavefunction" of the Cooper pairs; these pairs are coherent below T_c and are described by a single "wavefunction." The GL theory was used by Little [17], and Langer and Ambegaokar to analyze the phase-slip process, to account for the generation of a finite voltage below T_c in thin superconducting whiskers, at current levels below the expected critical current [18]. Thus, in some sense, the GL approach has been essential in providing a physical picture of a phase-slip defect. A time-dependent version was used by McCumber and Halperin to more accurately estimate the prefactor in the rate for the thermal generation of phase slips for an activated process just below T_c [19]. In this approach, relaxation to equilibrium from an external perturbation is characterized by a relaxation time, and modeled by the addition of a relaxation term which is proportional to the variation of the free energy density with respect to the order parameter.

The phenomenological GL free energy, in cgs units, has the following form:

$$F = F_n + \int d^3x \left[\alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{2M} \left| \left(\frac{\hbar}{i} \nabla + \frac{2e}{c} \boldsymbol{A} \right) \Psi \right|^2 + \frac{|\boldsymbol{B}|^2}{8\pi} \right] . \tag{1.29}$$

Here, M refers to the mass of the Cooper pair, and its charge is $-2e$. The inclusion of the gradient term with the covariant derivative $(\hbar/i\nabla + 2e/c\boldsymbol{A})$ accounts for the kinetic energy term when the superfluid is in motion, such as when a current is flowing.

The coefficients are dependent on temperature. Near T_c,

$$\alpha = a(T - T_c) + O((T - T_c)^2) , \tag{1.30}$$

$$\beta = b + O(T - T_c) , \tag{1.31}$$

where a and b are positive constants, and

$$M = M_o + O(T - T_c) \,. \tag{1.32}$$

Note that α changes sign and is negative below T_c. This leads to a minimum energy state which has a nonzero value for the order parameter.

Minimization with respect to the order parameter Ψ^* yields the GL equations

$$\frac{\delta F}{\delta \Psi^*} = \alpha(T)\Psi + \beta(T)|\Psi|^2\Psi - \frac{\hbar^2}{2M}\left(\nabla + \frac{2ie}{\hbar c}A\right)^2 \Psi = 0 \,. \tag{1.33}$$

Near T_c, the solution for a uniform state is given by

$$|\Psi|^2 = \frac{a(T_c - T)}{b} \,. \tag{1.34}$$

Thus, the superfluid density, n_s, goes as

$$n_s = 2|\Psi|^2 \propto \left(1 - \frac{T}{T_c}\right) \,. \tag{1.35}$$

Variation with respect to the vector potential gives

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} \,, \tag{1.36}$$

with

$$\begin{aligned} \boldsymbol{J} &= \frac{-2e}{2M}\left[\Psi^*\left(\frac{\hbar}{i}\nabla + \frac{2e}{c}A\right)\Psi - \Psi\left(\frac{\hbar}{i}\nabla - \frac{2e}{c}A\right)\Psi^*\right] \\ &= -\frac{2e\hbar}{M}|\Psi|^2\left(\nabla\phi + \frac{2e}{\hbar c}\boldsymbol{A}\right) \,. \end{aligned} \tag{1.37}$$

These are the phenomenological London equations, which were put forth by London [3] to account for the complete expulsion of magnetic flux from the interior of a bulk SC sample, beyond the London penetration depth λ.

In the Ginzburg–Landau theory, the coherence length ξ characterizing the size of the Cooper pair is given by

$$\xi(T) = \sqrt{\frac{\hbar^2}{2M|\alpha(T)|}} \,, \quad \xi(0) = \sqrt{\frac{\hbar^2}{2M|\alpha(T=0)|}} = \sqrt{\frac{\hbar^2}{2MaT_c}} \,, \tag{1.38}$$

while the penetration depth

$$\begin{aligned} &\lambda(T) = \sqrt{\frac{Mc^2}{16\pi e^2|\Psi|^2}} = \sqrt{\frac{Mc^2\beta}{16\pi e^2|\alpha(T)|}} \,, \quad \lambda(0) = \sqrt{\frac{Mc^2 b}{16\pi e^2 a T_c}} \,; \\ &\lambda(T) = \sqrt{\frac{M}{4\mu_o e^2|\Psi|^2}} \quad \text{[SI units]} \,. \end{aligned} \tag{1.39}$$

1.4.1
Time-Dependent Ginzburg–Landau Theory

In equilibrium, the minimization of the free energy yields the lowest energy state, for which the free energy does not change, to first order, with any variation of the order parameter or the fields. If, however, the system is not in an equilibrium state, the system should relax toward the equilibrium state. To account for the time-dependence of this relaxation process, the time-dependent Ginzburg–Landau equation is often used to describe the temporal behavior. The relaxation rate is assumed proportional to the variation of the free energy density with the order parameter ψ^*. The TDGL equation is written as

$$\left(i\hbar\frac{\partial}{\partial t} - 2\mu\right)\Psi = -\frac{i}{\tau_{\mathrm{GL}}}\frac{\hbar}{|\alpha(T)|}\frac{\partial f}{\partial \Psi^*} = -\frac{i}{\tau_{\mathrm{GL}}}\frac{\hbar}{|\alpha(T)|}\left[\alpha(T) + \beta(T)|\Psi|^2 - \frac{\hbar^2}{2M}\left(\nabla + \frac{2ie}{\hbar c}\boldsymbol{A}\right)^2\right]\Psi\,. \tag{1.40}$$

Here, τ_{GL} is the Ginzburg–Landau relaxation time $\tau_{\mathrm{GL}} = (\pi/8)\hbar/(T_{\mathrm{c}} - T)$, μ the chemical potential, and f is the free energy density. The last expression, of course, equals zero in equilibrium, but is nonzero when displaced from equilibrium.

This form of the TDGL is somewhat controversial and is believed to be accurate when close to T_{c} [20, 21]. It was used by McCumber and Halperin to estimate the attempt frequency to surmount the free energy barrier due to a phase slip in the vicinity of T_{c}. Review papers on applying the TDGL approach to the phase-slip phenomena in narrow superconducting devices are readily available in the literature [22, 23]. In addition, the generalized TDGL approach of Kramer and coworkers [24, 25] has been widely used to describe the property of both 1D [26, 27] and 2D [28, 29] superconducting systems, for example.

1.5
Gorkov Green's Functions, Eilenberger–Larkin–Ovchinnikov Equations, and the Usadel Equation

Gorkov developed a powerful method for understanding superconductivity by introducing a set of coupled equations for the dynamics (time evolution) of the Green's functions [30]. The equations couple the normal Green's functions, G, and the anomalous Green's, F, functions relevant to Cooper pairing. The equations of motion follow from the time evolution of the fermion field operators (see (1.20)), in which the interaction terms are again treated in a mean field manner. From such equations, essentially all interesting physical quantities can be obtained.

However, these coupled equations are difficult to solve. Starting from these equations, Eilenberger [31], and separately Larkin and Ovchinnikov [32], developed transport-like equations for a set of Green's functions closely related to Gorkov's Green's functions; these Green's functions are related to Gorkov's via an integration over the energy variable. By exploiting the fact that most quantities of interest are derivable from such Green's functions integrated over energy, and exploiting the quasiclassical approximation (which amounts to the neglect of the second order spatial derivatives relative to terms involving k_F times the first order derivatives), after averaging over the position of impurities, the number of Green's functions was reduced from four to two, and a transport-like equation for these Green's functions integrated over energy emerged.

Going one step further, Usadel noted that in the dirty limit, the Eilenberger–Larkin–Ovchinnikov Green's functions are nearly isotropic in space [33]. By exploiting this condition, the Eilenberger equations are further simplified to the transport-like Usadel equations describing diffusive motion of the Cooper pairs and the normal electrons. These equations now form the corner stone of many analyses of the dynamics of superconducting nanowires and SNS (superconductor–normal metal–superconductor) bridges. The purpose of this section is to provide a brief summary and some intuitive understanding of the transport-like Usadel equations, starting from the Gorkov formulation. These equations will be essential to understanding 1D nanowires, when either parts of the nanowires are driven into the normal state in a nonequilibrium situation, or else are connected to normal leads at one or multiple points along the nanowire.

The BCS Hamiltonian, which contains a contact interaction for the attraction between electrons, is written in the form

$$H = \int \left(\sum_\sigma \hat{\psi}_\sigma^\dagger \left[\frac{\left(\frac{\hbar}{i}\nabla + \frac{eA}{c} \right)^2}{2m} + u(\boldsymbol{r}) \right] \hat{\psi}_\sigma - \frac{g}{2} \sum_{\sigma \neq \sigma'} \hat{\psi}_\sigma^\dagger \hat{\psi}_{\sigma'}^\dagger \hat{\psi}_{\sigma'} \hat{\psi}_\sigma \right) d^3 r \,, \tag{1.41}$$

where $-e$ is the charge of the electron, σ denotes the spin (up or down), $u(\boldsymbol{r})$ is a one-particle potential, which includes impurity and boundary effects, $g = V\mathbb{V}$ is the Gorkov electron phonon coupling, and as before, the interaction between electrons is nonzero only within a region of the Debye energy $\hbar\omega_D$ of the Fermi surface. The Heisenberg field operators

$$\psi_\sigma(\boldsymbol{r}, t) = e^{iHt/\hbar} \hat{\psi}_\sigma(\boldsymbol{r}) e^{-iHt/\hbar} \,, \tag{1.42}$$

and

$$\psi_\sigma^\dagger(\boldsymbol{r}, t) = e^{iHt/\hbar} \hat{\psi}_\sigma^\dagger(\boldsymbol{r}) e^{-iHt/\hbar} \,, \tag{1.43}$$

satisfy the following equations of motion:

$$
\begin{aligned}
-i\hbar\frac{\partial\psi_\sigma(\boldsymbol{r},t)}{\partial t} &= -\left[\frac{\left(\frac{\hbar}{i}\nabla+\frac{e\boldsymbol{A}}{c}\right)^2}{2m}+u(\boldsymbol{r})-\mu\right]\psi_\sigma(\boldsymbol{r},t)\\
&\quad + g\psi^\dagger_{\sigma'}(\boldsymbol{r},t)\psi_{\sigma'}\boldsymbol{r},t)\psi_\sigma(\boldsymbol{r},t)\\
-i\hbar\frac{\partial\psi^\dagger_\sigma(\boldsymbol{r},t)}{\partial t} &= +\left[\frac{\left(\frac{\hbar}{i}\nabla-\frac{e\boldsymbol{A}}{c}\right)^2}{2m}+u(\boldsymbol{r})-\mu\right]\psi^\dagger_\sigma(\boldsymbol{r},t)\\
&\quad - g\psi^\dagger_\sigma(\boldsymbol{r},t)\psi^\dagger_{\sigma'}(\boldsymbol{r},t)\psi_{\sigma'}(\boldsymbol{r},t)\,,
\end{aligned}
\tag{1.44}
$$

where $\sigma' \neq \sigma$ and the terms $-\mu$ have been added to measure the energy relative to the chemical potential μ. Defining the normal Green's function, G, and anomalous Green's function, F, which measures pair correlations

$$
\begin{aligned}
G_1 &\equiv G_{\uparrow\uparrow}(\boldsymbol{r},t;\boldsymbol{r}',t') = -i\left\langle T(\psi_\uparrow(\boldsymbol{r},t)\psi^\dagger_\uparrow(\boldsymbol{r}',t')\right\rangle\,,\\
G_2 &\equiv G^\dagger_{\downarrow\downarrow}(\boldsymbol{r},t;\boldsymbol{r}',t') = -i\left\langle T(\psi^\dagger_\downarrow(\boldsymbol{r},t)\psi_\downarrow(\boldsymbol{r}',t')\right\rangle\,,\\
F_1 &\equiv F_{\uparrow\downarrow}(\boldsymbol{r},t;\boldsymbol{r}',t') = -i\left\langle T(\psi_\uparrow(\boldsymbol{r},t)\psi_\downarrow(\boldsymbol{r}',t')\right\rangle\,,\\
F_2 &\equiv F^\dagger_{\downarrow\downarrow}(\boldsymbol{r},t;\boldsymbol{r}',t') = -i\left\langle T(\psi^\dagger_\downarrow(\boldsymbol{r},t)\psi^\dagger_\uparrow(\boldsymbol{r}',t')\right\rangle\,.
\end{aligned}
\tag{1.45}
$$

Here, the chemical potential μ is given by the relationship $2\mu = 2(\partial E/\partial N) = E_{N+2} - E_N$. Equation (1.45) represents the thermodynamic Green's functions, where the brackets around an operator O denote $\langle O\rangle \equiv \mathrm{Tr}(e^{-\beta H}O)$.

For these Green's functions, their dynamical equations of motion form a closed set of coupled equations. The coupled equations are derived from the equations of motion for ψ and $\psi^\dagger$, making use of the same mean field approximation employed in the BCS solution as well as the BdG formulations to account for pair correlations. Defining a Green's function matrix $\hat{G}$,

$$
\hat{G} \equiv \begin{pmatrix} G_1 & F_1 \\ -F_2 & G_2 \end{pmatrix}\,,
\tag{1.46}
$$

and going over to imaginary time with $t \to -i\tau$, the dynamical equations of motion for the Green's functions in matrix notation are found to be [30–32]

$$
\begin{aligned}
&\begin{pmatrix} \left[-\hbar\frac{\partial}{\partial\tau}-\frac{\left(\frac{\hbar}{i}\nabla+\frac{e\boldsymbol{A}}{c}\right)^2}{2m}-u(\boldsymbol{r})+\mu\right] & \Delta(\boldsymbol{r}) \\ -\Delta^*(\boldsymbol{r}) & \left[\hbar\frac{\partial}{\partial\tau}-\frac{\left(\frac{\hbar}{i}\nabla-\frac{e\boldsymbol{A}}{c}\right)^2}{2m}-u(\boldsymbol{r})+\mu\right] \end{pmatrix}\hat{G}\\
&= \delta(\boldsymbol{r}-\boldsymbol{r}')\delta(\tau-\tau')\hat{\mathbf{1}}\,.
\end{aligned}
\tag{1.47}
$$

Here,

$$\Delta^*(\boldsymbol{r},\boldsymbol{\tau}) = -gF^{\dagger}_{\downarrow\uparrow}(\boldsymbol{r},\tau;\boldsymbol{r},\tau) \tag{1.48}$$

is the gap function.

The potential $u(\boldsymbol{r})$ may include the random impurity potentials, as well as that from an applied electric potential ϕ. From these equations, many properties, including nonequilibrium properties, can be obtained. However, the equations are difficult to solve in realistic situations of relevance to experiment, except for special cases, such as thin films at weak perturbation of external drive.

Eilenberger, and independently Larkin and Ovchinnikov, applied a quasiclassical approximation to the Fourier transformed Green's functions, $\hat{G}(\boldsymbol{k};\boldsymbol{r},\tau,\tau')$, where the transform is taken in the relative coordinates $\boldsymbol{r}'' = \boldsymbol{r} - \boldsymbol{r}'$ [31, 32, 34]. These Green's functions, in matrix notation, are given by

$$\begin{aligned}\hat{G}(\boldsymbol{k};\boldsymbol{R},\tau,\tau') &= \int \hat{G}(\boldsymbol{r},\tau;\boldsymbol{r}',\tau')e^{-i\boldsymbol{k}\cdot\boldsymbol{r}''}d^3r'' \\ &= \int \hat{G}\left(\boldsymbol{R}+\frac{\boldsymbol{r}''}{2},\tau;\boldsymbol{R}-\frac{\boldsymbol{r}''}{2},\tau'\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}''}d^3r'' \,.\end{aligned} \tag{1.49}$$

Here, the choice of singling out the relative coordinate is predicated on the weak dependence of the Green's functions on the center-of-mass coordinate ($\boldsymbol{R} \equiv (\boldsymbol{r}+\boldsymbol{r}')/2$), once averaged over impurity positions. (Note that Eilenberger introduced extra gauge potential phase factors in his definition of the Green's functions, and thus these matrix equations are in a slightly different but equivalent form [34].) Following an average over the impurity positions, one arrives at a set of equations which are now first order in the covariant derivatives, rather than second order. In the absence of paramagnetic impurities that flip spins, the Gorkov equations become (from here on, we denote $\boldsymbol{R}$ by $\boldsymbol{r}$)

$$\begin{aligned}&\begin{pmatrix} -\hbar\frac{\partial}{\partial\tau} - \xi(\boldsymbol{k}) - \boldsymbol{v}(\boldsymbol{k}_{\rm F})\cdot\frac{\hbar}{i}\nabla & \Delta(\boldsymbol{r}) \\ -\frac{e\hbar}{mc}\boldsymbol{k}_{\rm F}\cdot\boldsymbol{A} & \\ -\Delta^*(\boldsymbol{r}) & \hbar\frac{\partial}{\partial\tau} - \xi(\boldsymbol{k}) - \boldsymbol{v}(\boldsymbol{k}_{\rm F})\cdot\frac{\hbar}{i}\nabla \\ & +\frac{e\hbar}{mc}\boldsymbol{k}_{\rm F}\cdot\boldsymbol{A} \end{pmatrix}\hat{G}(\boldsymbol{k};\boldsymbol{r},\tau,\tau') \\ &\quad - \int d\tau_1\hat{\Sigma}(\boldsymbol{k}_{\rm F};\boldsymbol{r},\tau,\tau_1)\hat{G}(\boldsymbol{k};\boldsymbol{r},\tau_1,\tau') = \delta(\tau-\tau')\hat{\mathbf{1}} \,,\end{aligned} \tag{1.50}$$

where

$$\xi(\boldsymbol{k}) = \frac{\hbar^2\boldsymbol{k}^2}{2m} - \mu \,, \tag{1.51}$$

and

$$\begin{aligned}\hat{\Sigma}(\boldsymbol{k}_{\rm F};\boldsymbol{r},\tau,\tau_1) &= \frac{1}{2}\int d^3q\, W(\boldsymbol{k}_{\rm F},\boldsymbol{q})\hat{G}(\boldsymbol{k};\boldsymbol{r},\tau_1,\tau') \\ &\approx \frac{1}{4i}\int_{S_{\rm F}} d^2q_{\rm F}\rho(\boldsymbol{q}_{\rm F})\,W(\boldsymbol{k}_{\rm F},\boldsymbol{q}_{\rm F})\frac{i}{\pi}\int d\xi'\,\hat{G}(\boldsymbol{q};\boldsymbol{r},\tau_1,\tau') \,.\end{aligned} \tag{1.52}$$

The evaluation of the integral over ξ' requires care to remove an unphysical divergence [31]. Here, $W(\boldsymbol{k}, \boldsymbol{q})$ is the probability of scattering from the $\boldsymbol{k}$-state to the $\boldsymbol{q}$-state. The approximation follows with the assumption that W varies slowly in the energy regions of relevance: $\xi(\boldsymbol{k})$ and $\xi'(\boldsymbol{q})$ (with $\xi'(\boldsymbol{q}) = \hbar^2 q^2/2m - \mu$). In the Born approximation

$$W(\boldsymbol{k}_\mathrm{F}, \boldsymbol{q}_\mathrm{F}) \approx 2\pi N(0) n_\mathrm{i} |u(\boldsymbol{k}_\mathrm{F} - \boldsymbol{q}_\mathrm{F})|^2 \,, \tag{1.53}$$

where n_i is the impurity concentration, and $u(\boldsymbol{k})$ is the Fourier transform of the impurity potential $u(\boldsymbol{r})$. The quasiclassical approximation amounts to neglecting the terms of second order in the covariant derivatives in comparison to the first order derivatives, that is, $(\nabla \pm eA/i\hbar)^2 \ll \boldsymbol{k}_\mathrm{F} \cdot (\nabla + e\boldsymbol{A}/i\hbar)$, a consequence of the fact that the length scales of relevance for superconductivity: λ, the penetration depth, and ξ, the coherence length, on which Δ and $\boldsymbol{A}$ vary, far exceeds the Fermi wavelength $\lambda_\mathrm{F} = 2\pi/k_\mathrm{F}$.

A key observation significantly simplifies the situation. As it turns out, physical quantities of interest, such as the self-consistent gap equation

$$\frac{\Delta(\boldsymbol{r})}{g} = \langle \psi_\downarrow(\boldsymbol{r}) \psi_\uparrow(\boldsymbol{r}) \rangle \,, \tag{1.54}$$

and the supercurrent density, and so on, can be written in terms of the energy integrated Green's functions

$$\hat{g}(\boldsymbol{k}_\mathrm{F}; \boldsymbol{r}, \tau, \tau') = \frac{i}{2\pi} \int d\xi \, \hat{G}(\boldsymbol{k}; \boldsymbol{r}, \tau_1, \tau') \,. \tag{1.55}$$

Here, the wave vector $\boldsymbol{k}$ is to be taken to indicate the direction on the Fermi surface, or in essence $\boldsymbol{k}_\mathrm{F}$, which points in the same direction as the vector $\boldsymbol{k}$, and the energy integration must be taken as the principal value for large energies ξ. The energy-variable integrated Green's functions satisfy the normalization condition

$$\int_0^{\hbar\beta} \hat{g}(\boldsymbol{k}_\mathrm{F}; \boldsymbol{r}, \tau, \tau_1) \hat{g}(\boldsymbol{k}_\mathrm{F}; \boldsymbol{r}, \tau_1, \tau') d\tau_1 = \delta(\tau - \tau') \hat{\mathbf{1}} \,, \tag{1.56}$$

with $\beta = 1/(k_\mathrm{B} T)$. In the simplest case of time-independent perturbations, written in terms of the individual components

$$\begin{aligned} g(\boldsymbol{k}_\mathrm{F}, \omega; \boldsymbol{r}) &= \hat{g}_{11} = \hat{g}_{22} \,, \\ f(\boldsymbol{k}_\mathrm{F}, \omega; \boldsymbol{r}) &= i\hat{g}_{12} \,, \\ f^\dagger(\boldsymbol{k}_\mathrm{F}, \omega; \boldsymbol{r}) &= i\hat{g}_{21} \,, \end{aligned} \tag{1.57}$$

where the normalization condition now reads

$$g(\boldsymbol{k}_\mathrm{F}, \omega; \boldsymbol{r}) = \left[1 - f(\boldsymbol{k}_\mathrm{F}, \omega; \boldsymbol{r}) f^\dagger(\boldsymbol{k}_\mathrm{F}, \omega; \boldsymbol{r})\right]^{1/2} \,, \tag{1.58}$$

the resultant equations of motion of these Green's functions that are integrated over the energy variable are given by

$$\begin{aligned}
&\left[-2i\hbar\frac{\partial}{\partial\tau}+\boldsymbol{v}(\boldsymbol{k}_\mathrm{F})\cdot\left(\nabla+i\frac{2e}{\hbar c}\boldsymbol{A}\right)\right]f(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})\\
&\quad=2\Delta(\boldsymbol{r})g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})+\int\limits_{S_\mathrm{F}}d^2q_\mathrm{F}\rho(\boldsymbol{q}_\mathrm{F})\,W(\boldsymbol{k}_\mathrm{F},\boldsymbol{q}_\mathrm{F})\\
&\qquad\times\left[g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})f(\boldsymbol{q}_\mathrm{F},\omega;\boldsymbol{r})-f(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})g(\boldsymbol{q}_\mathrm{F},\omega;\boldsymbol{r})\right]\\
&\left[2i\hbar\frac{\partial}{\partial\tau}-\boldsymbol{v}(\boldsymbol{k}_\mathrm{F})\cdot\left(\nabla-i\frac{2e}{\hbar c}\boldsymbol{A}\right)\right]f^\dagger(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})\\
&\quad=2\Delta^*(\boldsymbol{r})g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})+\int\limits_{S_\mathrm{F}}d^2q_\mathrm{F}\rho(\boldsymbol{q}_\mathrm{F})\,W^*(\boldsymbol{k}_\mathrm{F},\boldsymbol{q}_\mathrm{F})\\
&\qquad\times\left[g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})f^\dagger(\boldsymbol{q}_\mathrm{F},\omega;\boldsymbol{r})-f^\dagger(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})g(\boldsymbol{q}_\mathrm{F},\omega;\boldsymbol{r})\right]\,,
\end{aligned}\tag{1.59}$$

where S_F denotes the Fermi surface.

Defining the Fourier transform

$$\hat{g}(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})=\frac{1}{2}\int\limits_{-\hbar/(k_\mathrm{B}T)}^{\hbar/(k_\mathrm{B}T)}\hat{g}(\boldsymbol{k}_\mathrm{F};\boldsymbol{r},\tau,0)e^{i\omega\tau}d\tau\,,\tag{1.60}$$

and the inverse transform

$$\hat{g}(\boldsymbol{k}_\mathrm{F};\boldsymbol{r},\tau,0)=i\beta\sum_{n=-\infty}^{n=\infty}e^{-i\omega_n\tau}\hat{g}(\boldsymbol{k}_\mathrm{F},\omega_n;\boldsymbol{r})\,,\quad\beta=\frac{1}{k_\mathrm{B}T}\,,\tag{1.61}$$

the Eilenberger–Larkin–Ovchinnikov equations read

$$\begin{aligned}
&\left[2\omega+\boldsymbol{v}(\boldsymbol{k}_\mathrm{F})\cdot\left(\nabla+i\frac{2e}{\hbar c}\boldsymbol{A}\right)\right]f(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})\\
&\quad=2\frac{\Delta(\boldsymbol{r})}{\hbar}g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})+\int\limits_{S_\mathrm{F}}d^2q_\mathrm{F}\rho(\boldsymbol{q}_\mathrm{F})\,W(\boldsymbol{k}_\mathrm{F},\boldsymbol{q}_\mathrm{F})\\
&\qquad\times\left[g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})f(\boldsymbol{q}_\mathrm{F},\omega;\boldsymbol{r})-f(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})g(\boldsymbol{q}_\mathrm{F},\omega;\boldsymbol{r})\right]\\
&\left[2\omega-\boldsymbol{v}(\boldsymbol{k}_\mathrm{F})\cdot\left(\nabla-i\frac{2e}{\hbar c}\boldsymbol{A}\right)\right]f^\dagger(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})\\
&\quad=2\frac{\Delta^*(\boldsymbol{r})}{\hbar}g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})+\int\limits_{S_\mathrm{F}}d^2q_\mathrm{F}\rho(\boldsymbol{q}_\mathrm{F})\,W^*(\boldsymbol{k}_\mathrm{F},\boldsymbol{q}_\mathrm{F})\\
&\qquad\times\left[g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})f^\dagger(\boldsymbol{q}_\mathrm{F},\omega;\boldsymbol{r})-f^\dagger(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})g(\boldsymbol{q}_\mathrm{F},\omega;\boldsymbol{r})\right]\,.
\end{aligned}\tag{1.62}$$

These equations are reminiscent of Boltzmann transport equations, albeit for quantities which are complex, and thus can account for quantum-mechanical interference. The second term on the right-hand side has the appearance of a collision integral. An additional equation for $g(\boldsymbol{k}_\mathrm{F},\omega;\boldsymbol{r})$ is redundant.

These equations are supplemented by the self-consistent equations for the gap $\Delta(\boldsymbol{r})$ and for the relation between the magnetic field and the supercurrent, which now become

$$\begin{aligned}
&\Delta(\boldsymbol{r})N(0)\ln\frac{T}{T_c} + 2\pi N(0)k_B T \\
&\qquad\times\sum_{n=0}^{\infty}\left[\frac{\Delta(\boldsymbol{r})}{\omega_n} - \int_{S_F} d^2k_F\rho(\boldsymbol{k}_F)f(\boldsymbol{q}_F,\omega_n;\boldsymbol{r})\right] = 0 \\
&\boldsymbol{J}_s(\boldsymbol{r}) = \frac{1}{4\pi}\nabla\times i\left[\boldsymbol{B}(\boldsymbol{r}) - \boldsymbol{B}_e(\boldsymbol{r})\right] \\
&\qquad = i\frac{2e}{\hbar c}2\pi N(0)k_B T\sum_{n=0}^{\infty}\int_{S_F} d^2k_F\rho(\boldsymbol{k}_F)\boldsymbol{v}(\boldsymbol{k}_F)g(\boldsymbol{q}_F,\omega_n;\boldsymbol{r})\ ,
\end{aligned} \tag{1.63}$$

with $\boldsymbol{J}_s$ the supercurrent density and $\boldsymbol{B}_e$ the magnetic field generated by normal electrons. Using (1.62) and (1.63), all the important physical quantities can be derived. Written in terms of the f and $f^\dagger$ only using the relationship $g = (1-|f|^2)^{1/2}$, these equation are nonlinear, and can therefore go beyond the linear approximation used by de Gennes [11]. For instance, they readily reproduce the higher order corrections cubic in $|\Delta|$ calculated by Maki for dirty superconductors [35–37].

Although these equations are substantially more convenient than the Gorkov equations, and are more amenable to numerical implementation for computing the physical quantities under realistic experimental conditions, one further development rendered this entire theoretical machinery far more tractable. Usadel, following the earlier work by Lüders [38–41], recognized that a crucial further simplification can be obtained in the case of a very dirty superconductor. Starting with the Eilenberger–Larkin–Ovchinnikov equations in the form of (1.62), Usadel noted that the large amount of scattering in a dirty system renders the Green's functions nearly isotropic in space, and they can thus be written as the sum of a dominant isotropic part, plus a smaller part dependent on the direction on the Fermi surface. Based on this idea, and keeping the leading terms, Usadel transformed the equations into a diffusion equation valid in this dirty limit.

Separating to the dominant isotropic term and the Fermi velocity-dependent term

$$\begin{aligned}
f(\boldsymbol{k}_F,\omega;\boldsymbol{r}) &= F(\omega;\boldsymbol{r}) + \boldsymbol{k}_F\cdot\boldsymbol{F}(\omega;\boldsymbol{r}) \\
g(\boldsymbol{k}_F,\omega;\boldsymbol{r}) &= G(\omega;\boldsymbol{r}) + \boldsymbol{k}_F\cdot\boldsymbol{G}(\omega;\boldsymbol{r})\ ,
\end{aligned} \tag{1.64}$$

the quantities for g are expressible in terms of those for f via the normalization condition, which now, to leading order, reads

$$G(\omega;\boldsymbol{r}) = \left[1-|F(\omega;\boldsymbol{r})|^2\right]^{1/2}\ , \tag{1.65}$$

yielding, in addition,

$$\boldsymbol{G}(\omega;\boldsymbol{r}) = \frac{1}{2}\frac{F(\omega;\boldsymbol{r})\boldsymbol{F}^*(\omega;\boldsymbol{r}) - F^*(\omega;\boldsymbol{r})\boldsymbol{F}(\omega;\boldsymbol{r})}{G(\omega;\boldsymbol{r})}\ . \tag{1.66}$$

Under the conditions

$$G \gg 2\tau_{\rm tr}\omega\ , \quad F \gg 2\tau_{\rm tr}\Delta\ , \tag{1.67}$$

and of course $|F| \gg |\hat{\boldsymbol{k}}_{\rm F}\cdot\boldsymbol{F}|$, the resultant diffusion equation is given by

$$\begin{aligned} &2\omega F(\omega;\boldsymbol{r}) - D\left(\nabla + \frac{2ie}{\hbar c}\boldsymbol{A}\right) \\ &\quad\cdot\left[G(\omega;\boldsymbol{r})\left(\nabla + \frac{2ie}{\hbar c}\boldsymbol{A}\right)F(\omega;\boldsymbol{r}) - F(\omega;\boldsymbol{r})\nabla G(\omega;\boldsymbol{r})\right] \\ &= 2\frac{\Delta(\boldsymbol{r})}{\hbar}G(\omega;\boldsymbol{r}) \end{aligned} \tag{1.68}$$

with the diffusion constant $D = (1/3)\tau_{\rm tr}v_{\rm F}^2$, where $v_{\rm F}$ is the Fermi velocity. The transport time is given by the average of the scattering rate over the Fermi surface $S_{\rm F}$

$$\frac{1}{\tau_{\rm tr}} = \int_{S_{\rm F}} q_{\rm F}\rho(\boldsymbol{q}_{\rm F})\,W(\boldsymbol{k}_{\rm F},\boldsymbol{q}_{\rm F})\ , \quad W(\boldsymbol{k}_{\rm F},\boldsymbol{q}_{\rm F}) \approx 2\pi N(0)n_{\rm i}|u(\boldsymbol{k}_{\rm F}-\boldsymbol{q}_{\rm F})|^2\ , \tag{1.69}$$

where the approximation denotes the first Born approximation. Note that the frequencies ω are to be taken as the Matsubara frequency and are positive: $\omega = \omega_n = (2n+1)\pi k_{\rm B}T/\hbar > 0$.

1.6 Path Integral Formulation

The techniques described thus far are powerful and have been extremely successful in describing most of the properties of conventional superconductors and the phenomena associated with them as well. However, in order to formulate a fully quantum theory that captures the physics of 1D superconducting nanowires in the dirty limit, in which the mean free path $l_{\rm mfp} \ll \xi_0$, where ξ_0 is the clean limit coherence length, a path integral formulation has proven to be a convenient starting point. This formulation enables to go beyond their description close to $T_{\rm c}$, and is also naturally suited to describe issues pertaining to quantum phase transitions, such as those occurring in Josephson junctions coupled to a dissipative environment [42–44], such as those occurring in arrays of Josephson junctions in 2D and 1D, or in vortex matter [45].

In the path integral formulation, the action plays a central role. Theoretical developments in the 1970s [42] enabled one to make a connection from the computation of quantum evolution in time, to the partition function, with the introduction of the imaginary time. The nonequilibrium case, such as under current or voltage bias, is also readily incorporated through the generalization of the Keldysh formulation, with the ordered imaginary-time integrals in the expansion of the path integral [42, 46–49]. More to the point, the computation of the low temperature

behaviors of a 1D superconducting nanowire has required the use of path integral techniques based on instantons [50–52]. Instantons are saddle point solutions of the action and describe the quantum tunneling processes. Thus, such calculations yield the quantum tunneling rates for the phase slip.

Here, we summarize a formulation of conventional superconductors in a path integral formulation put forth by Otterlo, Golubev, Zaikin, and Blatter [49], which has developed out of earlier works [42, 46–48]. Such an approach not only reproduces the static results of the BCS theory, but enables the treatment of dynamical responses, including relaxation, collective modes, particularly the Carlson-Goldman [53] and Mooij–Schön modes [54], the latter being of direct relevance to 1D superconducting nanowires. Topological defects, such as the motion and the tunneling of vortices, as well as the quantum tunneling of phase slips in one dimension, are also included within this formalism. The Mooij–Schön mode will emerge to play a central role in the dynamics of phase slips and will be discussed in Chapter 3. The standard starting point to compute the partition function in the BCS model is written in terms of the path integral in the imaginary-time formulation [42, 49]

$$\begin{aligned} Z &= \int \mathcal{D}\psi \mathcal{D}\psi^* \mathcal{D}V \mathcal{D}^3 \boldsymbol{A} \exp\left(-\frac{S}{\hbar}\right) \\ \frac{S}{\hbar} &= \frac{1}{\hbar}\int d\tau d^3x \left\{ \psi_\uparrow^\dagger \left[\hbar\partial_\tau + ieV + \xi\left(\nabla + \frac{ie}{\hbar c}\boldsymbol{A}\right)\right]\psi_\uparrow \right. \\ &\quad + \psi_\downarrow^\dagger \left[\hbar\partial_\tau + ieV + \xi\left(\nabla + \frac{ie}{\hbar c}\boldsymbol{A}\right)\right]\psi_\downarrow - g\psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \\ &\quad \left. - ien_{\mathrm{i}}V + \frac{\boldsymbol{E}^2 + \boldsymbol{B}^2}{8\pi}\right\}, \end{aligned} \tag{1.70}$$

where $\xi(\nabla) \equiv -\nabla^2/2m - \mu$. The action includes the action of the electromagnetic field, and the coupling between the electrons and electromagnetic field is contained in the covariant derivative: $\partial/\partial_r = \nabla + (ie/\hbar c)\boldsymbol{A}$. The background ion charge density is n_i.

In order to perform the integral over the fermion Grassman fields ψ_σ and $\psi_\sigma^\dagger$, it is useful to decouple quartic attractive interaction term into terms bilinear in the ψs by the introduction of the superconducting gap $\Delta = |\Delta|e^{i\varphi}$ via the Hubbard–Stratonovich transformation, that is,

$$\begin{aligned} \exp\left(\frac{g}{\hbar}\int d\tau d^3x \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow\right) &= \left[\int \mathcal{D}^2\Delta \exp\left(-\frac{1}{\hbar g}\int d\tau d^3x |\Delta|^2\right)\right]^{-1} \\ &\times \int \mathcal{D}^2\Delta \exp\left[-\frac{1}{\hbar}\int d\tau d^3x \left(g^{-1}|\Delta|^2 + \Delta\psi_\uparrow^\dagger \psi_\downarrow^\dagger + \text{h.c.}\right)\right]. \end{aligned} \tag{1.71}$$

After integrating out the fermion degrees of freedom and discarding the normalization factor, which is not important for the dynamics we seek, the partition function becomes, with certain restrictions on the choice of gauge, that is, the Coulomb

gauge with $\nabla \cdot \boldsymbol{A} = 0$,

$$Z = \int \mathcal{D}^2 \Delta \mathcal{D} V \mathcal{D}^3 \boldsymbol{A} \exp\left[-\frac{(S_0 + S_1)}{\hbar} \right] , \tag{1.72}$$

where

$$\frac{S_0(V, \boldsymbol{A}, \Delta)}{\hbar} = \int d\tau d^3x \left(\frac{\boldsymbol{E}^2 + \boldsymbol{B}^2}{8\pi} - ien_{\mathrm{i}} V + \frac{|\Delta|^2}{g} \right) , \tag{1.73}$$

and

$$\begin{aligned} \frac{S_1}{\hbar} &= -\mathrm{Tr} \ln \hat{G}^{-1} \hat{G}_o[0] \, ; \\ \hat{G}_o[0] \equiv \hat{G}_o[\Delta = 0] &= \begin{pmatrix} \left[\frac{\hbar \partial}{\partial \tau} + \xi(\nabla) \right]^{-1} & 0 \\ 0 & \left[\frac{\hbar \partial}{\partial \tau} - \xi(\nabla) \right]^{-1} \end{pmatrix} . \end{aligned} \tag{1.74}$$

$\hat{G}$ denotes the Green's function matrix in Nambu space:

$$\hat{G} = \begin{pmatrix} G & F \\ \bar{F} & \bar{G} \end{pmatrix} , \tag{1.75}$$

while the inverse matrix

$$\begin{aligned} \hat{G}^{-1} &= \begin{pmatrix} \frac{\hbar \partial}{\partial \tau} + ieV + \xi(\nabla + \frac{ie}{\hbar c} \boldsymbol{A}) & \Delta \\ \Delta^* & \frac{\hbar \partial}{\partial \tau} - ieV - \xi(\nabla - \frac{ie}{\hbar c} \boldsymbol{A}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\hbar \partial}{\partial \tau} + \xi(\nabla) + ie\Phi + \frac{m v_{\mathrm{s}}^2}{2} & \Delta \\ -\frac{i\hbar}{2} \{\nabla, \boldsymbol{v}_{\mathrm{s}}\} & \\ \Delta^* & \frac{\hbar \partial}{\partial \tau} - \xi(\nabla) - ie\Phi - \frac{m v_{\mathrm{s}}^2}{2} \\ & -\frac{i\hbar}{2} \{\nabla, \boldsymbol{v}_{\mathrm{s}}\} \end{pmatrix} . \end{aligned} \tag{1.76}$$

The trace Tr is taken over the matrix in Nambu space and also over internal coordinates of frequency and momenta, and the curly brackets $\{\ldots, \ldots\}$ denote an anticommutator. In addition, the gauge invariant linear combinations of the electromagnetic fields and the order parameter phase φ are introduced:

$$\Phi = V + \frac{\hbar}{2e} \dot{\varphi} , \quad \boldsymbol{v}_{\mathrm{s}} = \frac{1}{2m} \left(\hbar \nabla \varphi + \frac{2e}{c} \boldsymbol{A} \right) . \tag{1.77}$$

The action may be expanded in terms of these gauge invariant quantities about a saddle point delineated below.

The Green's functions in space and real time are defined as

$$\begin{aligned} G(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2) &\equiv -i \langle | T\{\psi_\uparrow(\boldsymbol{r}_1, t_1) \psi_\uparrow^\dagger(\boldsymbol{r}_2, t_2)\} \rangle \\ \bar{G}(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2) &\equiv -i \langle | T\{\psi_\downarrow^\dagger(\boldsymbol{r}_1, t_1) \psi_\downarrow(\boldsymbol{r}_2, t_2)\} \rangle \\ F(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2) &\equiv -i \langle | T\{\psi_\uparrow(\boldsymbol{r}_1, t_1) \psi_\downarrow(\boldsymbol{r}_2, t_2)\} \rangle \\ \bar{F}(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2) &\equiv -i \langle | T\{\psi_\downarrow^\dagger(\boldsymbol{r}_1, t_1) \psi_\uparrow^\dagger(\boldsymbol{r}_2, t_2)\} \rangle , \end{aligned} \tag{1.78}$$

where $\psi_\sigma^\dagger$, and ψ_σ are the electron creation and annihilation field operators for spin $\sigma = \{\uparrow, \downarrow\}$, respectively. The imaginary-time operators are obtained by replacing t by $-i\tau$ and t' by $-i\tau'$.

The variation of (1.72) with respect to V, $\boldsymbol{A}$, and Δ, respectively, yields the equations describing the Thomas–Fermi screening, London screening, and the BCS-gap equations.

At this point, to further progress, the term with $\mathrm{Tr}\ln \hat{G}^{-1}$ is expanded up to second order, retaining Gaussian fluctuation terms around the saddle point solution: $\Delta = \Delta_0$, $\Phi = 0$, and $\boldsymbol{A} = 0$, and $V = V_\Delta$ a constant which will absorbed into the chemical potential μ. The gap is written as a uniform term plus a fluctuating term: $\Delta = \Delta_0 + \Delta_1$, with Δ_1 being the fluctuation, and Δ_0 is taken to be real. In addition, the inverse of the Green function $\hat{G}$ in (1.74)–(1.76) is split into an unperturbed part $\hat{G}_0^{-1}$ and $\hat{G}_1^{-1}$ with

$$\hat{G}_0 = \begin{pmatrix} G_0 & F_0 \\ \bar{F}_0 & \bar{G}_0 \end{pmatrix} \; ; \quad \bar{F}_0 = F_0 = \Delta_0 \tag{1.79}$$

for the saddle point solution, such that

$$\hat{G}_0^{-1} = \begin{pmatrix} \frac{\hbar\partial}{\partial\tau} + \xi(\nabla) & \Delta_0 \\ \Delta_0 & \frac{\hbar\partial}{\partial\tau} - \xi(\nabla) \end{pmatrix} , \tag{1.80}$$

and

$$\begin{aligned} \hat{G}_1^{-1} = & \\ & \begin{pmatrix} \left[\frac{m}{2}\left(\frac{e}{mc}\right)^2 \boldsymbol{A}^2 + ieV\right] - \frac{i}{2}\frac{\hbar e}{mc}\{\nabla, \boldsymbol{A}\} & \Delta_1 \\ \Delta_1^* & -\left[\frac{m}{2}\left(\frac{e}{mc}\right)^2 \boldsymbol{A}^2 + ieV\right] - \frac{i}{2}\frac{\hbar e}{mc}\{\nabla, \boldsymbol{A}\} \end{pmatrix} \\ & = \begin{pmatrix} +ie\Phi + \frac{mv_s^2}{2} - \frac{i\hbar}{2}\{\nabla, \boldsymbol{v}_s\} & \Delta_1 \\ \Delta_1^* & -ie\Phi - \frac{mv_s^2}{2} - \frac{i\hbar}{2}\{\nabla, \boldsymbol{v}_s\} \end{pmatrix} . \end{aligned} \tag{1.81}$$

The trace of the natural log of the inverse Green's function can thus be expanded as

$$\mathrm{Tr}\ln \hat{G}^{-1}\hat{G}_0[0] = \mathrm{Tr}\ln \hat{G}_0^{-1}\hat{G}_0[0] - \mathrm{Tr}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\hat{G}_0\hat{G}_1^{-1}\right)^n . \tag{1.82}$$

Note that since the gap Δ_0 in the unperturbed matrix G_0 is chosen to be real, we have $\bar{F}_0 = F_0$, $F_0(x_1, x_2) = F_0(x_2, x_1)$, and $G_0(x_1, x_2) = -G_0(x_2, x_1)$, where $x_i \equiv (\boldsymbol{r}_i, t_i)$.

From here on, various treatments diverge based on the assumption of Galilean invariance [46–48], gauge invariance, and so on [49]. In the formulation due to Otterlo *et al.*, by using Ward identities [49, 55] due to gauge invariance and charge and particles number conservation, they recast the effective action into four contributions:

$$S_{\mathrm{eff}} = S_{\mathrm{sc}}[\Delta_{\mathrm{L}}, \Phi, \boldsymbol{v}_{\mathrm{s}}] + S_{\mathrm{n}}[\boldsymbol{E}, \boldsymbol{B}] + S_{\mathrm{p\text{–}h}}[\Delta_{\mathrm{L}}, \Phi, \boldsymbol{v}_{\mathrm{s}}] + S_{\mathrm{em}}[\boldsymbol{E}, \boldsymbol{B}] , \tag{1.83}$$

where S_{sc} denotes the superconducting contribution, S_n the normal metallic contribution, $S_{p\text{–}h}$ the contribution from particle–hole symmetry breaking terms, and S_{em} the contribution of the free electromagnetic field. In addition, the gap fluctuation Δ_1 with $\Delta_1 = \Delta_L + i\Delta_T$, where Δ_L and Δ_T are both real, is written in terms of a longitudinal component Δ_L, and a phase-like transverse component Δ_T [49]. Please see (1.94) below regarding longitudinal and transverse projections. In this expansion, all terms linear in the expansion parameters vanish. Only quadratic or bilinear terms survive. Higher order terms are discarded.

To express these action terms explicitly, we make use of the gauge invariant quantities

$$\boldsymbol{v}_s = \frac{1}{2m}\left[\hbar\nabla\varphi + \frac{2e}{c}\boldsymbol{A}\right] \approx \frac{1}{2m}\left[\hbar\nabla\frac{\Delta_T}{\Delta_0} + \frac{2e}{c}\boldsymbol{A}\right], \tag{1.84}$$

$$\Phi = V + \frac{\hbar}{2e}\dot{\varphi} \approx V + \frac{\hbar}{2e}\frac{\dot{\Delta}_T}{\Delta_0}, \tag{1.85}$$

where the approximation holds for small fluctuations, that is, $|\Delta_1| \ll \Delta_0$. The second order expansion leads to to terms of the form $\mathrm{tr}[G\,O\,G'\,O]$. Here, G and G' symbolize any of the Green functions G, $\bar{G}$, F, or $\bar{F}$, and O is an operator. To evaluate the traces over the internal coordinates, $\mathrm{tr}[G\,O\,G'\,O']$, we define

$$q \equiv (\boldsymbol{q}, \omega_\mu). \tag{1.86}$$

$$\{B\}_{GG'}(q) = \frac{k_B T}{\hbar}\sum_{\omega_\mu}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,B\,G_0(p+q)\,G_0'(p), \tag{1.87}$$

where B denotes a function of frequency-momentum p and q. For example,

$$g_0 = \{1\}_{GG}(q), \quad g_1 = \left\{\frac{Q}{\boldsymbol{q}^2}\right\}_{GG}(q),$$
$$g_2 = \left\{\frac{Q^2}{\boldsymbol{q}^4}\right\}_{GG}(q), \quad g_3 = \left\{\frac{(\boldsymbol{p}\times\boldsymbol{q})^2}{2\boldsymbol{q}^4}\right\}_{GG}(q), \tag{1.88}$$

with $Q = \boldsymbol{q}\cdot(\boldsymbol{p}+\boldsymbol{q}/2)$, and likewise for f_i, h_i, and k_i denoting $\{\ldots\}_{FF}$, $\{\ldots\}_{GG^\dagger}$, $\{\ldots\}_{GF}$, respectively. Specializing to the various forms of the operators encountered, we have

$$\mathrm{tr}[G\,O\,G'\,O'] = \frac{k_B T}{\hbar}\sum_{\omega_\mu}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\,O(q)\,O'(-q)\{1\}_{GG'},$$

$$\mathrm{tr}[G\,O\,G'\{\nabla_a, \boldsymbol{O}_a'\}] = 2i\frac{k_B T}{\hbar}\sum_{\omega_\mu}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\,O(q)\,\boldsymbol{O}_a'(-q)\left\{\left(p+\frac{q}{2}\right)_a\right\}_{GG'},$$

$$\mathrm{tr}[G\{\nabla_a, \boldsymbol{O}_a\}G'\{\nabla_b, \boldsymbol{O}_b'\}] =$$
$$-4\frac{k_B T}{\hbar}\sum_{\omega_\mu}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\,\boldsymbol{O}_a(q)\,\boldsymbol{O}_b'(-q)\left\{\left(p+\frac{q}{2}\right)_a\left(p+\frac{q}{2}\right)_b\right\}_{GG'}. \tag{1.89}$$

The four contributions to the effective action can now be written as

$$\frac{S_{\rm sc}[\Delta_{\rm L},\boldsymbol{\Phi},\boldsymbol{v}_{\rm s}]}{\hbar} = -{\rm Tr}\ln\left(G_0^{-1}[\Delta_0]\hat{G}_0[0]\right) + \frac{1}{k_{\rm B}T}{\rm V}\frac{\Delta_0^2}{g}$$
$$+ \frac{k_{\rm B}T}{2}\sum_{\omega_\mu}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\left\{\left[\frac{2}{g}+h_0(q)+h_0(-q)+2f_0(q)\right]\Delta_{\rm L}(\boldsymbol{q})\Delta_{\rm L}(-\boldsymbol{q})\right.$$
$$\left.+\left(\frac{8e^2\Delta_0 k_0(q)}{i\omega_\mu}\right)\Phi(\boldsymbol{q})\Phi(-\boldsymbol{q}) - 8m\Delta_0 k_1(q)\boldsymbol{v}_{\rm s}(\boldsymbol{q})\boldsymbol{v}_{\rm s}(-\boldsymbol{q})\right\}, \tag{1.90}$$

$$\frac{S_{\rm n}[\boldsymbol{E},\boldsymbol{B}]}{\hbar} = +\frac{k_{\rm B}T}{2}\sum_{\omega_\mu}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3} + \left(\frac{2e^2 g_1(q)}{mi\omega_{mu}}\right)\boldsymbol{E}(\boldsymbol{q})\boldsymbol{E}(-\boldsymbol{q})$$
$$-\left(\frac{2e^2}{m^2c^2}\right)[g_2(q)+f_2(q)-g_3(q)-f_3(q)]\boldsymbol{B}(\boldsymbol{q})\boldsymbol{B}(-\boldsymbol{q}) \tag{1.91}$$

$$\frac{S_{\rm p\text{-}h}[\Delta_{\rm L},\boldsymbol{\Phi},\boldsymbol{v}_{\rm s}]}{\hbar} = k_{\rm B}T\sum_{\omega_\mu}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\left(\frac{1}{2}[h_0(q)-h_0(-a)]\Delta_{\rm L}(q)\Delta_{\rm L}(-q)\right.$$
$$+2ie[k_0(q)+k_0(-q)]\Phi(q)\Delta_{\rm L}(-q)$$
$$\left.+2[k_q(q)-k_q(-q)]\boldsymbol{q}\cdot\boldsymbol{v}_{\rm s}(q)\Delta_{\rm L}(-q)\right), \tag{1.92}$$

and

$$\frac{S_{\rm em}[\boldsymbol{E},\boldsymbol{B}]}{\hbar} = k_{\rm B}T\sum_{\omega_\mu}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\left(\frac{\boldsymbol{E}(\boldsymbol{q})\boldsymbol{E}(-\boldsymbol{q})+\boldsymbol{B}(\boldsymbol{q})\boldsymbol{B}(-\boldsymbol{q})}{8\pi}\right). \tag{1.93}$$

Here, V is the volume of the system. The subscripts "L" and "T" denote longitudinal and traverse projected components, respectively, based on the project operator associated with a momentum $\boldsymbol{q}$:

$$P_{\rm L}^{\alpha\beta} \equiv \frac{q^\alpha q^\beta}{\boldsymbol{q}^2}, \quad P_{\rm T}^{\alpha\beta} = \delta^{\alpha\beta} - P_{\rm L}^{\alpha\beta}, \tag{1.94}$$

with

$$P_{\rm L}^2 = P_{\rm L}, \quad P_{\rm T}^2 = P_{\rm T}, \quad P_{\rm L}P_{\rm T} = P_{\rm T}P_{\rm L} = 0, \quad P_{\rm L}+P_{\rm T} = 1. \tag{1.95}$$

For example, electric field $\boldsymbol{E}$ can be projected into

$$\boldsymbol{E}_{\rm L} = P_{\rm L}\boldsymbol{E} = (\boldsymbol{E}\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{\boldsymbol{q}^2}, \quad \boldsymbol{E}_{\rm T} = P_{\rm T}\boldsymbol{E} = \boldsymbol{E}-\boldsymbol{E}_{\rm L}. \tag{1.96}$$

In addition, the electric and magnetic fields are expressible in terms of $\boldsymbol{v}_{\rm s}$ and $\boldsymbol{\Phi}$ as

$$|\boldsymbol{E}(\boldsymbol{q})|^2 = \boldsymbol{q}^2|\Phi(\boldsymbol{q})|^2 + \frac{m^2\omega^2}{e^2}|\boldsymbol{v}_{\rm s}(\boldsymbol{q})|^2$$
$$-\frac{m\omega}{e}\left[\Phi(\boldsymbol{q})\boldsymbol{q}\cdot\boldsymbol{v}_{\rm s}(-\boldsymbol{q})+\Phi(-\boldsymbol{q})\boldsymbol{q}\cdot\boldsymbol{v}_{\rm s}(\boldsymbol{q})\right], \tag{1.97}$$

and

$$|\boldsymbol{B}(\boldsymbol{q})|^2 = \frac{m^2c^2}{e^2}\boldsymbol{q}^2 P_{\mathrm{T}}^{\alpha\beta}\boldsymbol{v}_{\mathrm{s}}^{\alpha}(q)\boldsymbol{v}_{\mathrm{s}}^{\beta}(q)\,. \tag{1.98}$$

Note that the various contributions to the effective action are expressed in terms of the *equilibrium, nonperturbed* normal and anomalous Green functions G_0, $\bar{G}_0$, F_0, and $\bar{F}_0$, for $\Delta = \Delta_0$, $V = V_\Delta = \mu$, and $\boldsymbol{A} = 0$ (for the normal contribution, $\Delta_0 = 0$). These standard Green functions are readily computed. See the appendices in references [49, 55], for example.

In the above, by introducing the longitudinal and transverse components, for example, Δ_{L} and Δ_{T} for the gap fluctuation Δ_1 about the BCS equilibrium value Δ_0, the effective action is cast in a form advantageous for illustrating the gauge invariant properties. For the 1D superconducting nanowire system, it is not necessary to do so. By repeated use of the Ward identities associated with gauge-invariance and particle number conservation, the second order expansion can be recast into the following convenient form [50, 51, 55]:

$$S_{\mathrm{eff}} = S_{\mathrm{sc}}[\Delta, \boldsymbol{\Phi}, \boldsymbol{v}_{\mathrm{s}}] + S_{\mathrm{n}}[\Delta, V, \boldsymbol{A}] + S_{\mathrm{em}}[\boldsymbol{E}, \boldsymbol{B}]\,, \tag{1.99}$$

where the particle–hole asymmetry contribution has been left out, as it is usually small. The contribution S_{sc} is given by

$$\begin{aligned}\frac{S_{\mathrm{sc}}[\Delta, \boldsymbol{\Phi}, \boldsymbol{v}_{\mathrm{s}}]}{\hbar} &= -\mathrm{tr}\ln\left(G_0^{-1}[\Delta]\right) + \frac{1}{\hbar}\int_0^\beta\int d\tau d^3x \frac{\Delta^2}{g} + \frac{(2e)^2}{2} \\ &\quad \times \mathrm{tr}[F_0\boldsymbol{\Phi}\,F_0\boldsymbol{\Phi}] - \frac{\hbar^2}{2}\mathrm{tr}[F_0\{\nabla, \boldsymbol{v}_{\mathrm{s}}\}F_0\{\nabla, \boldsymbol{v}_{\mathrm{s}}\}]\,,\end{aligned} \tag{1.100}$$

and the contribution S_{n} by

$$\begin{aligned}\frac{S_{\mathrm{n}}[\boldsymbol{E}, \boldsymbol{B}]}{\hbar} &= \frac{1}{\hbar}\int_0^\beta\int d\tau d^3x \frac{m}{2}\boldsymbol{u}^2 n_{\mathrm{e}}[\Delta] \\ &\quad - \frac{\hbar^2}{4}\mathrm{tr}[G_0\{\nabla, \boldsymbol{u}\}G_0\{\nabla, \boldsymbol{u}\}] + \frac{\hbar^2}{4}\mathrm{tr}[F_0\{\nabla, \boldsymbol{u}\}F_0\{\nabla, \boldsymbol{u}\}] \\ &= \frac{1}{\hbar}\int_0^\beta\int d\tau d^3x \frac{m}{2}\boldsymbol{u}^2 n_{\mathrm{e}}[\Delta] \\ &\quad + \frac{(2e)^2}{4}\mathrm{tr}[G_0 V G_0 V] - \frac{(2e)^2}{4}\mathrm{tr}[F_0 V F_0 V]\,,\end{aligned} \tag{1.101}$$

where n_{e} is the electron density. The velocity $\boldsymbol{u}$ does not depend on the phase of the order parameter φ, and is defined as

$$\boldsymbol{u} = \frac{-e}{m}\left(\int_{-\infty}^{\tau} d\tau' \nabla V(\tau') - \frac{1}{c}\boldsymbol{A}\right)\,. \tag{1.102}$$

The last line for $S_{\rm n}$ neglects the vector potential contribution, as the magnetic response tends to be small in normal metals. The contribution $S_{\rm em}$ remains as before, and is, in real space representation,

$$\frac{S_{\rm em}[\boldsymbol{E},\boldsymbol{B}]}{\hbar} = \int_0^\beta \int d\tau d^3x \left(\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}\right) . \tag{1.103}$$

Going further, and expanding Δ about its BCS value Δ_0, with $\Delta_1(\boldsymbol{r},\tau) = \Delta(\boldsymbol{r},\tau) - \Delta_0$, we arrive at the final form for the second order effective action, that is,

$$S^{(2)}_{\rm eff} = S_\Delta + S_{\rm J} + S_{\rm L} + S_{\rm D} + S_{\rm em} . \tag{1.104}$$

The various terms are

$$\frac{S_\Delta}{\hbar} = \int_0^\beta d\tau \int d^3x \frac{\Delta_0^2+|\Delta_1|^2}{g} + {\rm tr}\left[F_0\Delta_1^* F_0\Delta_1^* + G_0\Delta_1 G_0\Delta_1^*\right] , \tag{1.105}$$

$$\frac{S_{\rm J}}{\hbar} = \frac{(2e)^2}{2}{\rm tr}[F_0\Phi F_0\Phi] , \tag{1.106}$$

$$\frac{S_{\rm L}}{\hbar} = -\frac{\hbar^2}{2}{\rm tr}[F_0\{\nabla,\boldsymbol{v}_{\rm s}\}F_0\{\nabla,\boldsymbol{v}_{\rm s}\}] , \tag{1.107}$$

and

$$\frac{S_{\rm D}}{\hbar} = -\frac{(2e)^2}{4}{\rm tr}[G_0 V G_0 V] - \frac{(2e)^2}{4}{\rm tr}[F_0 V F_0 V] . \tag{1.108}$$

The equilibrium BCS energy gap as a function of temperature $\Delta_0(T)$ is assumed to obey the BCS gap equation (with $g=\lambda$)

$$\frac{1}{N(0)g} = \int_0^{\omega_{\rm D}} d\epsilon_n \frac{\tanh\left(\sqrt{\epsilon_n^2+\Delta_0^2}/(k_{\rm B}T)\right)}{\sqrt{\epsilon_n^2+\Delta_0^2}} . \tag{1.109}$$

Physically, the term S_Δ comes from fluctuations in the gap magnitude, $S_{\rm J}$ from Josephson coupling via the gauge invariant potential Φ, $S_{\rm L}$ pertains to the London screening of the magnetic field penetrating the superconductor, and the Drude contribution $S_{\rm D}$ accounts for the Ohmic dissipation of the normal electrons.

Based on this action, and specializing into one dimension, Zaikin, Golubev, *et al*. derived an effective action for 1D superconductors in terms of the Mooij–Schön plasmon mode, which is linearly dispersing, plus a phase-slip core contribution. The phase slips then arise as saddle point solutions (or instantons) of the effective action. Please see Chapter 3.

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