1 Rocket Fundamentals

Many people have had and still have misconceptions about the basic principle of a rocket. Here is a comment of an unknown editorial writer of the renowned *New York Times* from January 13, 1920, about the pioneer of US astronautics, Robert Goddard, who at that time was carrying out the first experiments with liquid propulsion engines:

> Professor Goddard . . . does not know the relation of action to reaction, and of the need to have something better than a vacuum against which to react – to say that would be absurd. Of course he only seems to lack the knowledge ladled out daily in high schools.

The publisher's doubt whether rocket propulsion in the vacuum could work is based on our daily experience that you can only move forward by pushing backward against an object or medium. Rowing is based on the same principle. You use the blades of the oars to push against the water. But this example already shows that the medium you push against, which is water, does not have to be at rest, it may move backward. So basically it would suffice to fill a blade with water and push against it by very quickly guiding the water backward with the movement of the oars. Of course, the forward thrust of the boat gained hereby is much lower compared with rowing with the oars in the water, as the large displacement resistance in the water means that you push against a far bigger mass of water. But the principle is the same. Instead of pushing water backward with a blade, you could also use a pile of stones in the rear of your boat and hurl them backward as fast as possible. With this, you would push ahead against the accelerating stone. And this is the basis of the propulsion principle of a rocket: it pushes against the gases it ejects backward with full brunt. So, with the propellant, the rocket carries the mass, against which it pushes to move forward, and this is why it also works in vacuum.

This repulsion principle, which is called the "rocket principle" in astronautics, is based on the physical principle of conservation of momentum. It states that the total (linear) momentum of a system remains constant with time: if, at initial time t_0, the boat (rocket) with mass m_1 plus stone (propellant) with mass m_2 had velocity $\boldsymbol{v}_0$, implying that the initial total momentum was $\boldsymbol{p}(t_0) = (m_1 + m_2)\boldsymbol{v}_0$, this must remain

Astronautics: The Physics of Space Flight, Second Edition. Ulrich Walter.

the same at some time $t_+ > t_0$ when the stone is hurled away with velocity $\boldsymbol{v}_2$, the boat has velocity $\boldsymbol{v}_1$ (neglecting water friction) and the total momentum is $\boldsymbol{p}(t_+) = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2$. That is,

$$\boldsymbol{p}(t_0) = \boldsymbol{p}(t_+) \quad \textbf{principle of the conservation of (linear) momentum}$$

from which follows

$$(m_1 + m_2) \cdot \boldsymbol{v}_0 = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2$$

Note: *The principle of the conservation of momentum is valid only for the vectorial form of the momentum equation, which is quite often ignored. A bomb that is ignited generates a huge amount of momentum out of nothing, which apparently would invalidate an absolute value form of the momentum equation. But if you add up the vectorial momentums of the bomb's fragments, it becomes obvious that the vectorial momentum has been conserved.*

Given m_1, m_2, $\boldsymbol{v}_0$, and velocity $\boldsymbol{v}_2$ of the stone (propellant) expelled, one is able to calculate from this equation the increased boat (rocket) velocity $\boldsymbol{v}_1$. Doing so, this equation affirms our daily experience that hurling the stone backward increases the speed of the boat, while doing it forward decreases its speed.

1.1 Rocket Principles

1.1.1 Momentum Thrust

With a rocket, the situation is a bit more complicated, as it does not eject one stone after the other, but it emits a continuous stream of tiny mass particles (typically molecules). In order to describe the gain of rocket speed by the continuous mass ejection stream adequately in mathematical and physical terms, we have to consider the ejected mass and time steps as infinitesimally small and in an external rest frame, the so-called inertial (unaccelerated, see Section 13.1) reference frame. This is depicted in Figure 1.1 where in an inertial reference frame with its origin at the center of the Earth, a rocket with mass m in space experiences no external forces.

At a given time t, the rocket may have velocity $\boldsymbol{v}$ and momentum $\boldsymbol{p}(t) = m\boldsymbol{v}$. By ejecting the propellant mass $dm_p > 0$ with exhaust velocity $\boldsymbol{v}_e$ and hence with momentum $\boldsymbol{p}_p(t + dt) = (\boldsymbol{v} + \boldsymbol{v}_e) \cdot dm_p$, it will lose part of its mass $dm = -dm_p < 0$ and hence gain rocket speed $d\boldsymbol{v}$ by acquiring momentum $\boldsymbol{p}_r(t + dt) = (m + dm)(\boldsymbol{v} + d\boldsymbol{v})$.

Note: *In literature, $dm > 0$ often denotes the positive mass flow rate of the propellant, and m the mass of the rocket. This is inconsistent, and leads to an erroneous mathematical description of the relationships, because if m is the mass of the rocket, logically dm has to be the mass change of the rocket, and thus it has to be negative. This is*

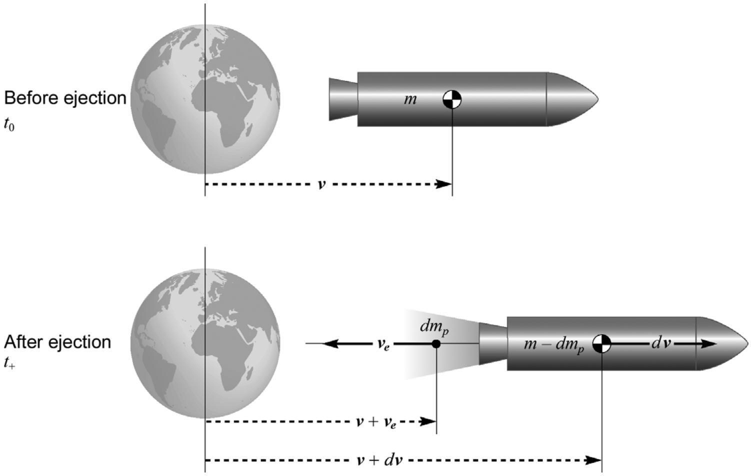

Figure 1.1 A rocket in force-free space before (above) and after (below) it ejected a mass dm_p with exhaust velocity $\boldsymbol{v}_e$, thereby gaining speed $d\boldsymbol{v}$. Velocities relative to the external inertial reference frame (Earth) are dashed and those with regard to the rocket are solid.

why in this book, we will always discriminate between rocket mass m and propulsion mass m_p using the consistent description $dm = -dm_p < 0$ implying $\dot{m} = -\dot{m}_p < 0$ for their flows.

For this line of events, we can apply the principle of conservation of momentum as follows:

$$\boldsymbol{p}(t) = \boldsymbol{p}(t+dt) = \boldsymbol{p}_p(t+dt) + \boldsymbol{p}_r(t+dt)$$

From this follows,

$$m\boldsymbol{v} = -dm(\boldsymbol{v}+\boldsymbol{v}_e) + (m+dm)(\boldsymbol{v}+d\boldsymbol{v}) = m\boldsymbol{v} - dm\cdot\boldsymbol{v}_e + m\cdot d\boldsymbol{v} + dm\cdot d\boldsymbol{v}$$

As the double differential $dm \cdot d\boldsymbol{v}$ mathematically vanishes with respect to the single differentials dm and $d\boldsymbol{v}$, we get with division by dt

$$m\dot{\boldsymbol{v}} = \dot{m}\boldsymbol{v}_e$$

According to Newton's second law (Eq. (7.1.12)), $\boldsymbol{F} = m\dot{\boldsymbol{v}}$, the term on the left side corresponds to a force, called *momentum thrust*, due to the repulsion of the propellant, which we correspondingly indicate by

$$\boldsymbol{F}_e = \dot{m}\boldsymbol{v}_e$$

Note: *This equation can alternatively be derived from the fact that the momentum of the expelled propellant mass is $d\boldsymbol{p}_p = dm_p\boldsymbol{v}_e$. The equivalent force according to Newton's second law (see Eq. (7.1.12)) is $\boldsymbol{F}_p = d\boldsymbol{p}_p/dt = \dot{m}_p\boldsymbol{v}_e$. This in turn causes a reaction force (Newton's third law Eq. (7.1.11)) on the rocket of $\boldsymbol{F}_e = -\boldsymbol{F}_p = -\dot{m}_p\boldsymbol{v}_e = \dot{m}\boldsymbol{v}_e$.*

This means that the thrust of a rocket is determined by the product of propellant mass flow rate and exhaust velocity. Observe that due to $\dot{m} = -\dot{m}_p < 0$, $\boldsymbol{F}_e$ is exactly in opposite direction to the exhaust velocity $\boldsymbol{v}_e$ (but depending on the steering angle of the engine, $\boldsymbol{v}_e$ and hence $\boldsymbol{F}_e$ does not necessarily have to be in line with the flight direction $\boldsymbol{v}$). Therefore, with regard to absolute values, we can write

$$F_e = -\dot{m}v_e = \dot{m}_p v_e \quad \textbf{momentum thrust} \tag{1.1.1}$$

The term *momentum thrust* is well chosen, because if the expression $\dot{m}_p v_e$ is integrated with regard to time, one obtains the momentum $m_p v_e$, which is merely the recoil momentum of the ejected propellant.

1.1.2
Effective Exhaust Velocity

In the above, we have considered a simple mechanism, namely, propellant mass expelled at rate $\dot{m}_p$ and at relative speed v_e, that causes momentum thrust. As we will see in the following sections, there exist other mechanisms that in conjunction with the propellant mass flow – or from a relativistic point of view, that employs the expenditure of energy – causes an additional thrust F_+. We can take this into account by writing quite formally $F_+ = \dot{m}_p v_+$. On the other hand, there are often side effects that decrease the momentum thrust such as jet straying that leads to divergence losses (see Section 4.4.2). All such losses are generally accounted for by a total loss factor η. We can hence write the total thrust as

$$F_* = F_e + F_+ = \dot{m}_p(\eta v_e + v_+) =: \dot{m}_p v_*$$

We therefore can understand the **total thrust** F_* as caused by an effective exhaust velocity v_*. From this point of view,

> The **effective exhaust velocity** v_* is an effective conversion factor of all physical effects that convert the employed propellant flow $\dot{m}_p$ into total thrust F_*.

If we would only have the above mechanism of expelling mass with some stray losses, as for instance with an ion engine (see Section 5.2), we would simply have $v_* = \eta v_e$. This is not the case for thermal engines, which will be examined in the following section. There we will see that the gas pressure at the chamber exit produces an additional thrust, the so-called pressure thrust. By the same token, we will see in Section 1.3.2 that for relativistic flight, relativistic effects have to be taken into account and, in addition, photons might also contribute to thrust. In all these cases, v_* takes on a more complex form.

In essence, one can state that for each type of engine, one has to investigate what the thrust-generating mechanisms are, how they act, and, by writing its total thrust in the form $F_* = \dot{m}_p v_*$ determine what the effective exhaust velocity of that engine is.

As we will see in Section 2.5.1, the effective exhaust velocity v_* is identical to and therefore can also be understood as "the achievable total impulse of an engine with respect to a given exhausted propellant mass m_p," called the mass-specific impulse: $v_* = I_{sp}$.

Generalized Thrust Equation

According to the above considerations, we generalize the above equations to

$$\boldsymbol{F}_* = \dot{m}\boldsymbol{v}_* \tag{1.1.2}$$

and for absolute values

$$F_* = -\dot{m}v_* = \dot{m}_p v_* \quad \textbf{propellant force (thrust) equation} \tag{1.1.3}$$

Equation (1.1.2) or (1.1.3), respectively, is of vital importance for astronautics, as it describes basic physical facts, just like every other physical relationship, relating just three parameters, such as $W = F \cdot s$ or $U = R \cdot I$. This is its statement: thrust is the product of exhaust velocity times mass flow rate. Only both properties together make up a powerful thruster. The crux of the propellant is not its "energy content" (actually, the energy to accelerate the propellant might be provided externally, which is the case with ion propulsions), but the fact that it possesses mass, which is ejected backward, and thus accelerates the rocket forward by means of conservation of momentum. The higher the mass flow rate, the larger the thrust. If "a lot of thrust" is an issue, for instance, during launch, when the thrust has to overcome the gravity pull of the Earth, and since the exhaust speed of engines is limited, you need thrusters with a huge mass flow rate. The more the better. Each of the five first stage engines of a Saturn V rocket had a mass flow rate of about 2.5 metric tons per second, in total 12.5 tons per second, to achieve the required thrust of 33 000 kN (corresponds to 3400 tons of thrust). This tremendous mass flow rate is exactly why, for launch, chemical thrusters are matchless up to now, and they will certainly continue to be so for quite some time.

1.1.3
Pressure Thrust

We now take the next step and refine our engine by taking into account that $\dot{m}_p$ is not just a stream of mass particles but gas particles in a jet engine. A jet engine is characterized as an engine that gains thrust through the repulsion of expelled gases (or fluids in general). This includes not only the classical thermal engines (see Chapter 4), resistojets, or arcjets working with neutral gases but also engines that work with "ion gases," i.e., plasma, such as ion jets (see Chapter 5) or Hall ion jets, where the ions interact via the Coulomb interaction and therefore also create pressure. Because all practical propulsion engines work with gases or plasmas, the considerations and results of this section apply to all today's engines, which is why we treat this as fundamental rocket knowledge.

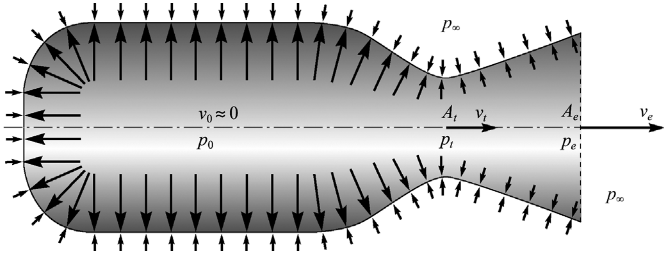

Figure 1.2 Pressure conditions inside and outside a jet engine chamber.

Gases are a loose accumulation of molecules, which, depending on temperature, display internal molecular motion and interact with each other, and thus generate pressure. On the other hand, the rocket at launch moves in an atmosphere whose gas molecules exert an external pressure. In order to understand what is the impact of the propellant gas pressure and external ambient pressure on the engine's thrust, let us have a look at the pressure conditions in a jet engine (see Figure 1.2).

Inside the combustion chamber and depending on the location within the chamber, we assume a varying internal pressure p_{int}, which exerts the force $d\boldsymbol{F}_{int} = p_{int} \cdot d\boldsymbol{A}$ on a wall segment $d\boldsymbol{A}$. In the area surrounding the chamber, we assume a constant external ambient pressure p_∞. Now quite generally, the total propellant force $\boldsymbol{F}_*$ generated by the chamber must be the sum of all effective forces acting on the entire engine wall with surface S

$$\boldsymbol{F}_* = \iint_S d\boldsymbol{F}_{eff} = \iint_S (p_{int} - p_\infty) \cdot d\boldsymbol{A} \tag{1.1.4}$$

The normal vector of the chamber surface can be split into two components: a radial $\boldsymbol{u}_r$ and an axial $\boldsymbol{u}_x$ component (Figure 1.3):

$$d\boldsymbol{A} = d\boldsymbol{A}_r + d\boldsymbol{A}_x = (\sin\theta \cdot \boldsymbol{u}_r + \cos\theta \cdot \boldsymbol{u}_x) \cdot dA$$

where the wall angle θ is the angle between surface normal and chamber axis. If the combustion chamber is axially symmetric, we have

$$\iint_S (p_{int} - p_\infty) \cdot d\boldsymbol{A}_r = 0$$

and, therefore, we only get axial contributions:

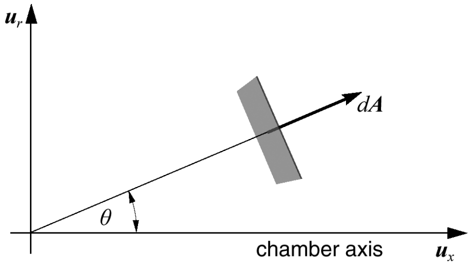

Figure 1.3 Definition of the wall angle with regard to the chamber axis.

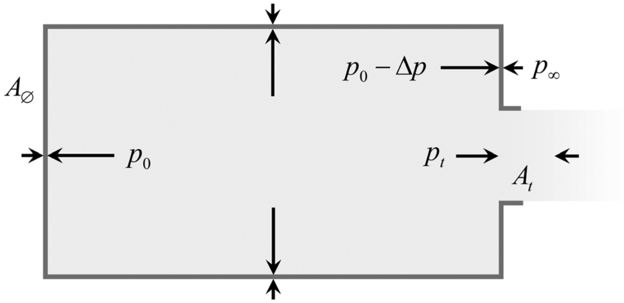

Figure 1.4 Pressure conditions in the idealized combustion chamber.

$$\boldsymbol{F}_* = \iint\limits_S (p_{int} - p_\infty) \cdot d\boldsymbol{A}_x = \boldsymbol{u}_x \iint\limits_S (p_{int} - p_\infty) \cos\theta \cdot dA \tag{1.1.5}$$

Maintaining the internal pressure conditions and thus without a change in thrust, we now deform the combustion chamber, so we get a rectangular combustion chamber (see Figure 1.4). Now that all wall angles are only $\theta = 0,\ 90^\circ,\ 180^\circ,\ 270^\circ$, the following is valid:

$$F_* = -\iint\limits_{A_\varnothing} (p_{int} - p_\infty)(-1) \cdot dA - \iint\limits_{A_\varnothing - A_t} (p_{int} - p_\infty) \cdot dA \tag{1.1.6}$$

where F_* now expresses the propellant force of the combustion chamber in forward direction, the direction in which the total force is effectively pushing.

As there is no wall at the throat with the surface A_t, no force can be exerted on it, and thus on the chamber's back side, the integral is limited to the surface $A_\varnothing - A_t$. The maximum combustion chamber pressure $p_{int} = p_0$ is on the front side of the chamber, where the gas is about at rest. Because the gas flow increases in the direction of the throat where it exits the chamber, the pressure at the rear of the chamber is reduced by a certain amount Δp: $p_{int} = p_0 - \Delta p(r)$, and due to the axial symmetry of the chamber, this pressure drop is also axially symmetrical; so at the throat, $p_{int} = p_0 - \Delta p(r) = p_t$ applies. So Eq. (1.1.6) reads as follows:

$$F_* = (p_0 - p_\infty)A_\varnothing - \iint\limits_{A_\varnothing - A_t} (p_0 - p_\infty) \cdot dA + \iint\limits_{A_\varnothing - A_t} \Delta p \cdot dA$$

As

$$\iint\limits_{A_\varnothing - A_t} (p_0 - p_\infty) \cdot dA = (p_0 - p_\infty)(A_\varnothing - A_t)$$

and

$$\iint\limits_{A_\varnothing - A_t} \Delta p \cdot dA = \iint\limits_{A_\varnothing} \Delta p \cdot dA - \iint\limits_{A_t} \Delta p \cdot dA = \iint\limits_{A_\varnothing} \Delta p \cdot dA - (p_0 - p_t)A_t$$

we get

$$F_* = (p_t - p_\infty)A_t + \iint_{A_\oslash} \Delta p \cdot dA \tag{1.1.7}$$

Let us have a closer look at the integral of the last equation. It describes a force that results from the pressure reduction along the rear combustion chamber wall. This pressure reduction is due to the propellant flow through the throat. This mass flow, of course, does not generate a sudden pressure drop at the rear wall, but rather a pressure gradient along the chamber axis. That is,

$$\iint_{A_\oslash} \Delta p \cdot d\boldsymbol{A} \rightarrow - \iiint_{chamber} \boldsymbol{\nabla} p \cdot dV$$

The pressure gradient corresponds to an acceleration field $d\boldsymbol{v}/dt$ of the mass flow. According to the Euler equation of hydrodynamics, they are intimately connected with each other via the mass density ϱ:

$$-\nabla p = \varrho \frac{d\boldsymbol{v}}{dt} \quad \textbf{Euler equation}$$

This equation expresses Newton's law in hydrodynamics. If we apply the Euler equation to the volume integral, we obtain

$$\iiint_{chamber} \boldsymbol{\nabla} p \cdot dV = - \iiint_{chamber} \frac{d\boldsymbol{v}}{dt}\frac{dm_p}{dV} dV = - \int_0^{\boldsymbol{v}_t} \dot{m}_p \cdot d\boldsymbol{v}$$

The velocity integral now ranges from the velocity at the front part of the chamber, where the pressure gradient and hence the drift velocity of the propellant is zero, to its throat, where the velocity takes on the exit value $\boldsymbol{v}_t$. According to the continuity equation (Eq. (1.1.12)), the mass flow rate $\dot{m}_p$ is invariant along the combustion chamber and also in the subsequent nozzle, and thus it is constant. So we find

$$\iint_{A_\oslash} \Delta p \cdot d\boldsymbol{A} = - \iiint_{chamber} \boldsymbol{\nabla} p \cdot dV = \dot{m}_p \int_0^{\boldsymbol{v}_t} d\boldsymbol{v} = \dot{m}_p \boldsymbol{v}_t \tag{1.1.8}$$

If we apply this result to Eq. (1.1.7), we get

$$F_* = \dot{m}_p v_t + (p_t - p_\infty)A_t$$

So far our considerations have been independent of the exact form of the combustion chamber, as long as it is axially symmetric. So we can consider the nozzle to be also a part of the combustion chamber. Then, all the parameters considered so far at the throat of the combustion chamber are also valid for the nozzle exit, i.e.,

$$\boxed{F_* = \dot{m}_p v_e + (p_e - p_\infty)A_e =: F_e + F_p} \tag{1.1.9}$$

We recover its vectorial form from the direction information in Eq. (1.1.5) by observing that the surface integral at the given conditions is negative and $\boldsymbol{u}_e = \boldsymbol{u}_x$

$$\boldsymbol{F}_* = \dot{m}\boldsymbol{v}_e - (p_e - p_\infty) A_e \boldsymbol{u}_e \tag{1.1.10}$$

where $\boldsymbol{u}_e$ is the unit vector of the exit surface in the direction of the exhaust jet with exhaust velocity v_e. The first term on the right side of the last two equations is the known momentum thrust F_e, while the second term is called **pressure thrust** F_p. This term, on one hand, is conclusive because it originates from the very special fact that the jet engine works with gases that produce pressure. On the other hand, and as according to Eq. (1.1.8), the momentum thrust here is also generated by a pressure on the chamber because of its internal pressure gradient. At the end, it is all pressure that accelerates the gas engine, and with it the rocket.

Effective Exhaust Velocity

If we compare Eq. (1.1.9) with Eq. (1.1.3), we can see that the effective exhaust velocity is made up by two contributions

$$v_* = v_e + (p_e - p_\infty) \frac{A_e}{\dot{m}_p} \qquad \textbf{effective exhaust velocity} \tag{1.1.11}$$

The expression "effective exhaust velocity" makes clear that it is not only about exhaust velocity v_e but also about modifying by a pressure thrust-equivalence exhaust velocity term. However, for a real engine chamber, the pressure thrust indeed is only a small contribution (see Eq. (1.1.14) and thereafter). For an ideally adapted nozzle with $p_e = p_\infty$, (see Section 4.2) it even vanishes.

1.1.4
Momentum versus Pressure Thrust

Ultimately, if it is only pressure that drives a jet engine, how does this fit together with the rocket principle discussed in Section 1.1, which was based on repulsion and not on pressure? And what is the physical meaning of pressure thrust? You often find the statement that pressure thrust occurs when the pressure at the exit (be it nozzle or chamber exit) hits the external pressure. The pressure difference at this point times the surface is supposed to be the pressure thrust. Though the result is right, the explanation is not. First, the exit pressure does not abruptly meet the external pressure. There is rather a smooth pressure transition from the exit pressure to the external pressure covering in principle an infinite volume behind the engine. Second, even if such a pressure difference could be traced back mathematically to a specific surface, this would not cause a thrust, because, as we will see later, the gas in the nozzle expands backward with supersonic speed, and such a gas cannot have a causal effect on the engine to exert a thrust on it.

For a true explanation, let us imagine for a moment and purely hypothetically a fully closed combustion chamber (see Figure 1.5) with the same pressure conditions as in the idealized combustion chamber with mass flow rate (see Figure 1.4). The

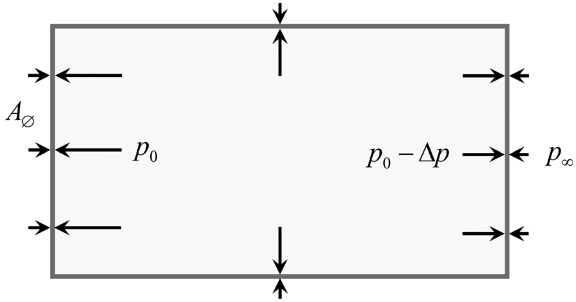

Figure 1.5 Pressure conditions of the idealized combustion chamber if it would be, hypothetically, fully closed.

surface force on the front side would be $F_{front} = (p_0 - p_\infty)A_\varnothing$, and $F_{rear} = (p_0 - \Delta p - p_\infty)A_\varnothing$ on the rear side. Hence, the net forward thrust would be $F_* = F_{front} - F_{rear} = \Delta p \cdot A_\varnothing$. Because the wall angle on the rear side is 0° and because of Eq. (1.1.8), this translates into $F_* = \Delta p \cdot A_\varnothing = \dot{m}_p v_t$. Therefore, we can say the following:

> The **momentum thrust** F_e physically results from the fact that, in a hypothetically closed engine chamber, due to the mass flow rate $\dot{m}_p$, there is a bigger chamber pressure on the front side compared to the Δp smaller pressure on the back side. This causes a net pressure force $\Delta p \cdot A_\varnothing$.

Ultimately, it is the Euler equation that relates the mass flow rate $\dot{m}_p$ with the pressure differences in the pressure chamber. In order to have the hypothetical gas flow indeed flowing, we need to make a hole with area A_t into the rear side (see Figure 1.4). Once this is done, the counterthrust at the rear side decreases by

$$\Delta F_{rear} = -(p_0 - \Delta p - p_\infty)A_t = -(p_t - p_\infty)A_t$$

which in turn increases the forward thrust by the same amount. But this is just the pressure thrust. Therefore,

> The **pressure thrust** F_p is the additional thrust that originates from the absence of the counterpressure force at the exit opening of the engine.

If the exit pressure happens to be equal to the external pressure, then the external pressure behaves like a wall, the pressure thrust vanishes, and we have an ideally adapted nozzle (see Section 4.2.1).

1.1.4.1 Momentum Thrust Revisited

The momentum thrust can also be described in a different mathematical form. Let us have a general look at the behavior of propellant gas perfusing an engine. A propellant

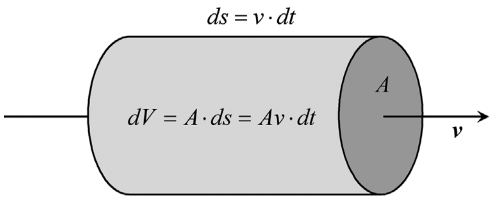

Figure 1.6 The volume dV that a mass flow with velocity v passes in time dt.

mass dm_p perfuses a given cross section of the engine with the area A and velocity v (see Figure 1.6). During the time interval dt, the volume of amount $dV = A \cdot ds = Av \cdot dt$ will have passed through it. Therefore,

$$dm_p = \varrho \cdot dV = \varrho Av \cdot dt$$

where ϱ is the mass density. From this, we derive the mass flow rate equation:

$$\boxed{\dot{m}_p = \varrho v A} \quad \textbf{continuity equation} \tag{1.1.12}$$

The continuity equation is a direct outcome of the transport of mass particles as a conserved quantity, which expresses the fact that the number of mass particles cannot increase or decrease, but can only move from place to place. This is exactly what the word "continuity" means.

We apply the continuity equation to a jet engine, which reads at the exit of the engine $\dot{m}_p = \varrho_e v_e A_e$. Applying this to Eq. (1.1.1) yields

$$F_e = \dot{m}_p v_e = \varrho_e A_e v_e^2 \tag{1.1.13}$$

This equation begs the question whether the momentum thrust of a jet engine is linearly or quadratically dependent on v_e. The answer depends on the engine in question. Depending on the engine type (e.g., electric or chemical engine), a change of its design in general will vary all parameters v_e and $\dot{m}_p, \varrho_e, A_e$ in a specific way. This is why the demanding goal of engine design is to tune all engine parameters, including v_e, such that the total thrust is maximized. Hence, it is not only v_e alone, which is decisive for the momentum thrust of a jet engine but also it is necessary to adjust all relevant engine parameters in a coordinated way.

Significance of Pressure Thrust

Given Eq. (1.1.13), we are able to qualitatively derive the contribution of moment and pressure thrust to the total thrust. We do so by rating it against the pressure-to-momentum ratio in vacuum:

$$\frac{F_p}{F_e} \le \left.\frac{F_p}{F_e}\right|_\infty = \frac{p_e A_e}{\varrho_e A_e v_e^2} = \frac{\varrho_e R_s T_e}{\varrho_e v_e^2} = \frac{R_s T_e}{v_e^2} \tag{1.1.14}$$

where we have made use of the ideal gas law Eq. (6.1.2) with the specific gas constant R_s. For thermal engines, as discussed in Chapter 4, we have $F_p/F_e|_\infty \approx$ 5–10%. For a so-called *ideally adapted nozzle* where $p_e = p_\infty$ (see Section 4.2.1), then of course

$F_p/F_e = 0$. Because the exhaust temperature T_e generally decreases with increasing v_e (cf. Eq. (4.1.6) for their general relation), we see that the pressure thrust becomes rapidly less important with increasing exhaust velocity. This will be particularly important for ion engines with exhaust velocity 10 times larger than for thermal engines.

1.2 Rocket Equation of Motion

Apart from its own thrust, external forces also determine the trajectory of a rocket. They are typically summarized to one external force F_{ext}:

$$\boldsymbol{F}_{ext} := \boldsymbol{F}_G + \boldsymbol{F}_D + \boldsymbol{F}_L \cdots \tag{1.2.1}$$

with

$\boldsymbol{F}_G$ = gravitational force
$\boldsymbol{F}_D$ = aerodynamic drag
$\boldsymbol{F}_L$ = aerodynamic lift.

For each of these external forces, a virtual point within the rocket can be assumed the external force effectively acts on (Figure 1.7). This point has a unique location with regard to the geometry of the rocket, and it is in general different for every type of forces. For instance, the masses of the rocket can be treated as lumped together in the center of mass where the gravitational force applies; the aerodynamic drag and lift forces effectively impact the spacecraft at the so-called center of pressure; and possible magnetic fields have still another imaginary point of impact. If the latter do not coincide with the center of mass, which in general is the case, the distance in between results in torques due to the inertial forces acting effectively at the center of mass. In this textbook, we disregard the resulting complex rotational movements, and we just assume that all the points of impact coincide with the center of mass or, alternatively, that the torques are compensated by thrusters.

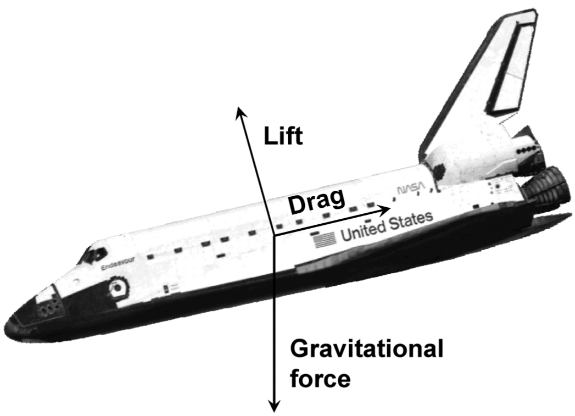

Figure 1.7 External forces acting on a Space Shuttle upon re-entry.

Newton's second law, Eq. (7.1.12), gives us an answer to the question of how the rocket will move under the influence of all non-inertial forces $\boldsymbol{F}_i$, including the rocket's thrust $\boldsymbol{F}_*$:

$$m\dot{\boldsymbol{v}} = \sum_{all\ i} \boldsymbol{F}_i$$

We therefore find the following equation of motion for the rocket:

$$m\dot{\boldsymbol{v}} = \boldsymbol{F}_* + \boldsymbol{F}_{ext}$$

and with Eq. (1.1.2), we finally obtain

$$\boxed{m\dot{\boldsymbol{v}} = \dot{m}\boldsymbol{v}_* + \boldsymbol{F}_{ext}} \qquad \textbf{rocket equation of motion} \qquad (1.2.2)$$

This is the key differential equation for the motion of the rocket. In principle, the speed can be obtained by a single integration step and its position by a double integration. Note that this equation not only applies to rockets but also to any type of spacecraft during ascent flight, re-entry, or when flying in space with or without propulsion.

1.3 Relativistic Rocket[1]

All what has been said here was based on Newton's classical mechanics. It holds as long as the speed of the rocket v is well below the speed of light c. We know from theory of special relativity, which Einstein developed at the beginning of the last century, that physics behaves differently if $v \approx c$. Many rockets eventually fly close to the speed of light? In order to find out, we need to know what is needed to get it close to the speed of light and how it performs there. But note that the need to apply relativistic physics depends on the precision that is needed to describe a given situation. A satellite navigation system in Earth orbit, for instance, needs a high-precision timekeeping system on board with a stability of less than $\Delta t/t \approx 10^{-12}$ that allows to determine a position on Earth with roughly 10 cm accuracy. At an orbital speed of $3.9\,\text{km}\,\text{s}^{-1}$, relativity contributes to the time deviation with $\Delta t/t = v^2/2c^2 \approx 8.5 \times 10^{-11}$ that is not negligible. Therefore, at much lower speeds, relativity must also be taken into account if the accuracy of the description is high.

Our goal here is to understand how relativity works for a spacecraft close to the speed of light and how this relates to classical mechanics at lower speeds. We start out by assuming a one-dimensional motion of the rocket, thrust direction, and hence acceleration along the x-axis. The main components of relativity will not be touched by this restriction. This implies that the position of a rocket in time can be

1) Section 1.3 is partly adapted from Walter (2006) with contributions from Westmorelend (2010).

appropriately described by the two vectors (x, t). We define two reference frames: the "primed" reference frame of an external inertial observer $O'(x', t')$ and the "unprimed" reference frame of the rocket under consideration $R(x, t)$, which is supposed to have an instantaneous velocity v relative to O'.

1.3.1
Space Flight Dynamics

For relativistic physics, it is important to note that among all existing reference frames, there is one preferred frame: the rest frame. This is the frame of the object under consideration in which it is at rest. Any other external observer having velocity v relative to this rest frame observes the properties of the object such as length, time, speed, and acceleration differently as the object itself. Since there may be an infinite number of observers and, therefore, many different views of the object properties, relativistic physics holds that only one has a proper view of the object: the object itself. In this sense, relativity is an absolute concept.

Relativistic physics, therefore, introduces the notion of "proper." In general, a "proper" measure of a quantity is that taken in the relevant instantaneous rest frame, thus also called proper reference frame. So "proper" is everything an astronaut experiences in his rocket. This is why we will not put a prime on such quantities and those as observed from outside will carry a prime. In general, the observed values depend on the reference frame with of course one exception: $v = v'$. Adopting this notion, what is of relevance first is how the proper measures relate to the measures of external observers.

Proper time (also called *eigentime*) τ is the time that the watch of an astronaut in a rocket shows. Special relativity holds that τ is related to the time t' of the external observer O' by

$$d\tau \equiv dt = \sqrt{1-\beta^2} \cdot dt' \tag{1.3.1}$$

where we adopt the convenient relativistic notations $\beta := v/c$ and $\gamma := 1/\sqrt{1-\beta^2}$. We will sometimes denote dt by $d\tau$ in order to point out that the proper time is meant. It should be noted that in special relativity, Eq. (1.3.1) holds for any condition of the rest frame even if it is accelerated, because, and contrary to common misjudgment, special relativity is not restricted to constant relative velocities or inertial reference frames.

Einstein pointed out that acceleration is an absolute concept: an astronaut does not experience rocket velocity in his rest frame, but he does so for acceleration. Let us assume that the astronaut experiences an acceleration α. Then, special relativity tells us that this is related to the acceleration a' as seen by an external observer through

$$\alpha = \gamma^3 a' \quad \textbf{proper acceleration} \tag{1.3.2}$$

Because acceleration is an absolute concept, we are apt to define

$$d\sigma := \alpha(t) \cdot dt \tag{1.3.3}$$

$d\sigma$ is an increase in speed as measured in the instantaneous rest frame. We integrate to get

$$\sigma(\tau) = \int_0^{\tau} \alpha(t) \cdot dt \qquad (1.3.4)$$

This equation tells us that σ is the integral of the acceleration as experienced in the proper reference frame and hence is the speed as experienced by an astronaut, who sees the outer world going by. Since this is the true meaning of proper, σ is a proper speed. In order to find the relation of this proper speed to the relative speed v, we apply Eqs. (1.3.1) and (1.3.2) to Eq. (1.3.4) and get

$$\sigma = \int_0^{t'} \gamma^2 a' \cdot dt'$$

As $a' = dv/dt' = c \cdot d\beta/dt'$, we find

$$\sigma = \int_0^{v} \frac{dv}{1-v^2/c^2} = c\int_0^{\beta} \frac{d\xi}{1-\xi^2} = c \cdot \text{arc}\tanh\beta \quad \textbf{proper speed}$$

or

$$\boxed{\beta = \frac{v}{c} = \tanh\left(\frac{\sigma}{c}\right)} \qquad (1.3.5)$$

It is now shown that the proper speed is proper in a more general sense. Let us consider a second rocket or any other object in space having the known speed u relative to the astronaut's R system. We want to know what its speed u' is as measured by O'. Special relativity tells us that

$$u' = \frac{u+v}{1+uv/c^2} = c\frac{\beta_u+\beta_v}{1+\beta_u\beta_v} \qquad (1.3.6)$$

The problem with this transformation equation is that it is not linear as in classical physics where the Galileo transformation $u' = u + v$ holds. In addition, Eq. (1.3.6) limits u' to the range $0 \le u' \le c$ if v starts out from below c. This can be seen immediately if one inserts even limiting velocities $u = c$. This is Einstein's famous law that nothing goes faster than the speed of light. It is exactly this non-linearity and limited range of values that cause problems when treating special relativity mathematically. We now apply Eq. (1.3.5) to Eq. (1.3.6) and find

$$\tanh(\sigma'_u/c) = \tanh(\sigma_u/c + \sigma_v/c)$$

where we have used the algebraic equation for any two values x and y:

$$\frac{\tanh x + \tanh y}{1 + \tanh x \cdot \tanh y} = \tanh(x+y)$$

As this must hold for any proper speed values, we find

$$\sigma'_u = \sigma_u + \sigma_v \tag{1.3.7}$$

i.e., proper speed recovers the linearity of speed transformation in special relativity.

According to Eq. (1.3.5), the proper speed goes to infinity if the externally observed speed goes to the speed of light. This is to say that from an astronaut's point of view, there is no speed limit. His subjective impression is that he can actually travel much faster than the speed of light. But of course he cannot travel faster than infinitely fast. This is the reason why the observer also sees a speed limit: the speed of light. So the ultimate reason why nothing can ever go faster than the speed of light is that no proper space traveler can ever go faster than infinitely fast. Note that from this point of view, photons always travel infinitely fast. They experience that any distance in the universe is zero: for them the universe is one point. Because their proper time is zero, one might say they do not even exist. But this would be wrong. They come into existence at one point in our universe, they transfer energy, momentum, angular momentum, and information to any other point in proper zero time, thereby causally linking any two parts in our universe and at the instance their work is done they are gone. This is why causality is the basic conservation law and hence the cement of our universe, and not the speed of light. The speed of light c may vary throughout our universe, but the fact that the proper time at $v = c$ is always zero and cannot become negative – implying that no inverse causality is possible – is firm.

In order to show that the concept of proper speed has relevance to the concept of classical speed, we finally show that for small speeds, the proper speed turns over into the classical concept of speed v for $v \to 0$

$$\sigma = c \cdot \text{arc}\tanh\beta = c\left(\beta + \frac{1}{3}\beta^3 + \cdots\right) \approx c\beta = v, \quad @ \quad v \to 0 \tag{1.3.8}$$

We summarize by noting that the proper speed exhibits four important properties: it is proper, it transforms linearly, its takes on real numbers, and it turns over into the classical concept of velocity at low speeds. This implies that it is a natural extension of the classical speed into special relativity and is mathematically integrable.

1.3.2 Relativistic Rocket Equation

With the concept of proper speed at hand, we start out to derive the relativistic rocket equation. We want to do this in its most general form. The two physically distinct rocket propulsion systems are mass propulsion and photon propulsion. We take both into account and assume that upon combustion, a portion ε of the propellant mass will be converted into energy with a certain efficiency η and that a portion δ of it expels the exhaust mass with velocity v_e, while the other portion $(1-\delta)$ is expelled as exhaust photons, and the rest is lost. Therefore, the overall energy scheme looks like Figure 1.8. In the rest frame R of the rocket, momentum conservation holds. Taking the momentums of both exhaust components and that of the rocket into account, we

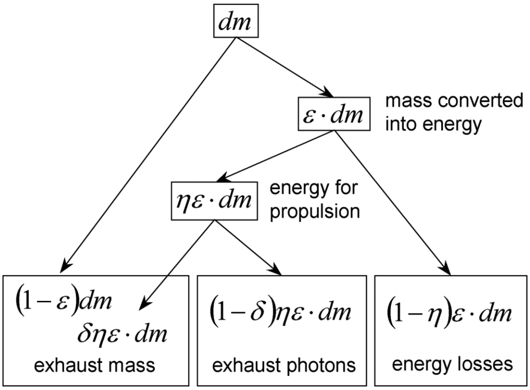

Figure 1.8 Energy scheme for a relativistic rocket with energy losses and expelled propulsion mass and photons.

can write

$$(1-\varepsilon)dm \cdot \gamma_e v_e + (1-\delta)\eta\varepsilon \cdot dm \cdot c + (m + dm)dv = 0$$

where the factor $\gamma_e = 1/\sqrt{1-v_e^2/c^2}$ takes into account the Lorentz factor of the ejected exhaust mass, and again we count dm negatively since m is the mass of the rocket. From the above equation, we find

$$dv = -v_* \frac{dm}{m} \tag{1.3.9}$$

where we have defined the effective exhaust velocity

$$v_* := (1-\varepsilon)\gamma_e v_e + (1-\delta)\eta\varepsilon c$$

Note that all terms in Eq. (1.3.9) are unprimed and are therefore terms measured in the proper reference frame including dv (observe that in this equation, the effective exhaust velocity is operationally defined as explicated in Section 1.1.2).

Now, in classical physics, the relation $dv = dv'$ holds and hence the equation can be readily integrated to yield the classical rocket equation $\Delta v = v_* \ln(m_0/m)$ (see Eq. (2.2.2)). But $dv = dv'$ is no longer valid for relativistic speeds. However, if we identify $dv = d\sigma$, we can again directly integrate to obtain

$$\boxed{\Delta\sigma = v_* \ln \frac{m_0}{m}} \quad \textbf{relativistic rocket equation} \tag{1.3.10}$$

So the relativistic rocket equation is to the utmost extent complementary to the classical rocket equation (2.2.2). In order to show that Eq. (1.3.10) is in accordance with today's more convenient form of the relativistic rocket equation, we apply Eq. (1.3.5) and the algebraic equation for the free variable x,

$$\text{arc}\tanh x = \ln \sqrt{\frac{1+x}{1-x}}$$

to Eq. (1.3.10) and, with Eq. (1.3.6), we derive the common form of the relativistic rocket equation

$$\frac{m_0}{m} = \left(\frac{1+\Delta\beta}{1-\Delta\beta}\right)^{\frac{1}{2\beta_*}} \tag{1.3.11}$$

with $\beta_* = v_*/c$.

From Eqs. (1.3.3) and (1.3.9), we can also derive the thrust F_* of the relativistic rocket in its rest frame

$$F_* = m\alpha = m\frac{d\sigma}{d\tau} = -v_*\frac{dm}{d\tau} = -\dot{m}v_* \quad \textbf{relativistic rocket thrust} \tag{1.3.12}$$

which is identical to the classical equation (1.1.3).

1.3.3
Exhaust Considerations

Because a portion of the converted energy propels the exhaust mass, the energy obtained from the propellant $dE_m = \delta\eta\varepsilon \cdot dm \cdot c^2$ has to equal the relativistic energy of the propelled mass $dm_e = (1-\varepsilon)dm$, i.e.,

$$dE_m = \delta\eta\varepsilon \cdot dm \cdot c^2 = \gamma_e dm_e \cdot c^2 - dm_e \cdot c^2 = (\gamma_e - 1)(1-\varepsilon)dm \cdot c^2 \tag{1.3.13}$$

This implies that for a given δ, η, the two terms γ_e (or β_e) and ε are interrelated, namely,

$$\varepsilon = \frac{\gamma_e - 1}{\delta\eta + \gamma_e - 1} \quad \text{or} \quad 1-\varepsilon = \frac{\delta\eta}{\delta\eta + \gamma_e - 1} \tag{1.3.14}$$

or

$$\beta_e = \sqrt{1 - \left[\frac{1-\varepsilon}{1-\varepsilon(1-\delta\eta)}\right]^2} \tag{1.3.15}$$

and

$$\gamma_e = \frac{1}{\sqrt{1-\beta_e^2}} = \frac{1-\varepsilon(1-\delta\eta)}{1-\varepsilon} \tag{1.3.16}$$

the other way around. We summarize by saying that internal energy considerations determine the relativistic exhaust mass velocity.

If we insert these results into the effective exhaust velocity from Eq. (1.3.9) we obtain

$$\begin{aligned} \beta_* &:= (1-\varepsilon)\gamma_e\beta_e + (1-\delta)\eta\varepsilon \\ &= \sqrt{\delta\eta\varepsilon(2-2\varepsilon+\delta\eta\varepsilon)} + (1-\delta)\eta\varepsilon \end{aligned} \quad \textbf{effective exhaust velocity} \tag{1.3.17}$$

For a rocket that exhausts just mass, $\delta = 1$, we find

$$\beta_* = \sqrt{\eta\varepsilon(2-2\varepsilon+\eta\varepsilon)} \quad @\,\delta = 1 \tag{1.3.18}$$

In case the rocket has 100% efficiency, $\eta = 1$, we find the expression

$$\beta_* = \sqrt{2\varepsilon-\varepsilon^2} \quad @\,\delta,\eta = 1$$

For a photon rocket, $\varepsilon = 1$ and $\delta = 0$, we get

$$\beta_* = \eta \tag{1.3.19}$$

1.3.3.1 Matter–Antimatter Annihilation Drive

As an example, let us assume a matter–antimatter annihilation rocket (rather like those in Figures 1.9 and 1.10). We assume that our rocket annihilates H_2 and anti-H_2 ($\bar{H}_2$) molecules stored as solid pellets in a storage tank below 14 K, the freezing temperature of hydrogen and hence also anti-hydrogen, typically at 1–2 K to avoid sublimation. In order to confine the neutral antimatter, either their diamagnetism would hold them together in a strong external magnetic field or they would be electrically charged and suspended in an array of electrostatic traps. Otherwise, we neglect all the technical obstacles that come along with such storing devices. Upon annihilation of an H and an $\bar{H}$ atom, each having a total rest mass of 938.8 MeV, 22.30% of them are converted into charged pions, 14.38% into neutral pions, and the electron and positron into two γ-rays. The charged pions can be deflected backward by a magnetic field to provide propulsion force. Let us assume that this can be done with 100% efficiency. The neutral pions are lost because after a 0.06 μm travel

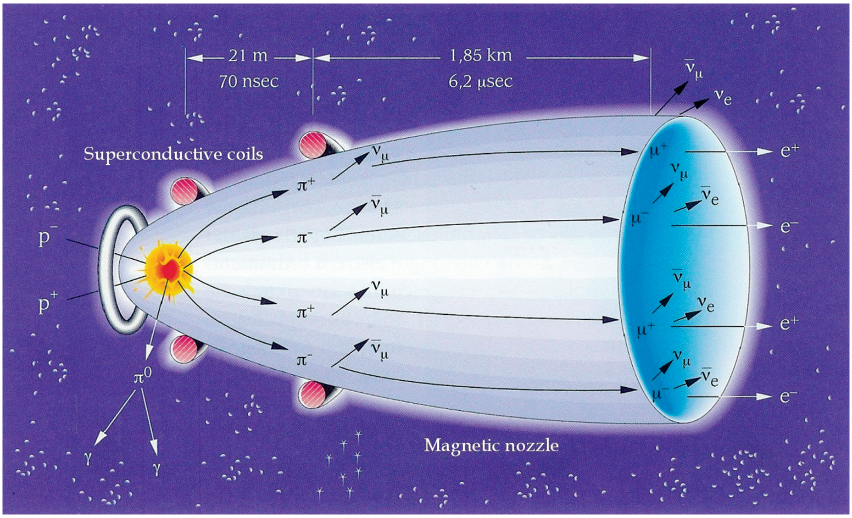

Figure 1.9 Working scheme of a matter–antimatter annihilation drive.

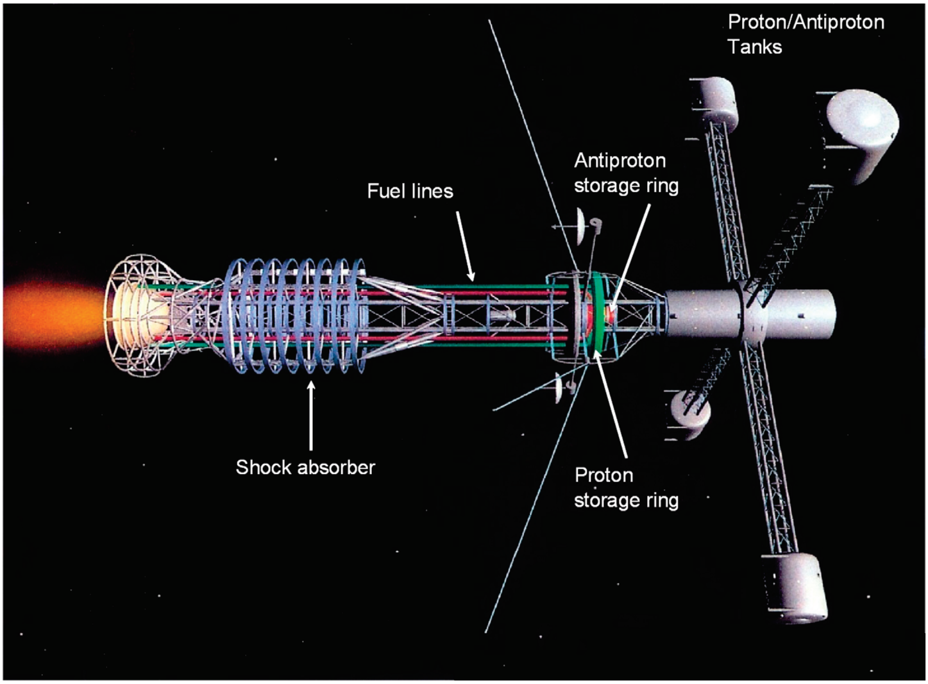

Figure 1.10 Artist view of a relativistic proton–antiproton annihilation drive rocket.

distance, they decay into 709.1 MeV γ-rays, which has to be considered as a major hazard to the crew. As long as the γ-rays cannot be directed backward as well (there seems to be no practical way of doing that), thus adding to the thrust via photonic propulsion, this drive is a purely mass-exhaust drive, $\delta = 1$.

So, effectively, we have 418.8 MeV of pion rest mass as propulsion mass, while the rest is converted into energy, i.e., $\varepsilon = 1 - 418.8/(2 \times 938.8) = 0.7769$. About 748.6 MeV of the energy goes into the kinetic energy of the pions, $\varepsilon\eta = 748.6/(2 \times 938.8)$ and therefore $\eta = 0.5132$, and the rest is lost. From Eq. (1.3.18), we then find with $\delta = 1$ an ultimate effective exhaust velocity of

$$\beta_* = 0.5804 \quad @ \quad H - \bar{H}\,\text{annihilation} \tag{1.3.20}$$

For a given total rocket mass at a given time, this can be used to calculate the travel speed at this instance from rocket equation (1.3.10) or (1.3.11).

1.3.4
External Efficiency

As for a classical rocket in Section 2.7, we want to derive the external rocket efficiency η_{ext} of a relativistic rocket that was defined by

$$\eta_{ext} := \frac{\text{rocket kinetic energy at burnout}}{\text{generated thrust energy}} =: \frac{E_{kin}}{E_*}$$

From Eq. (1.3.13) plus the photon energy, we have

$$E_* = (\gamma_e - 1)(1-\varepsilon)m_p c^2 + (1-\delta)\eta\varepsilon m_p c^2 = [(\gamma_e - 1)(1-\varepsilon) + (1-\delta)\eta\varepsilon]\left(\frac{m_i}{m} - 1\right)mc^2$$
$$= \frac{\eta(\gamma_e - 1)}{\delta\eta + \gamma_e - 1}\left(e^{\sigma/v_e} - 1\right)mc^2 = \eta\varepsilon\left(e^{\sigma/v_e} - 1\right)mc^2$$

In the second line, we have applied Eq. (1.3.14) and the relativistic rocket equation (1.3.10). And trivially,

$$E_{kin} = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$$

Employing the definition, we derive

$$\eta_{ext} = \frac{\gamma - 1}{\gamma_e - 1}\frac{\delta\eta + \gamma_e - 1}{\eta(e^{\sigma/v_*} - 1)} = \frac{1}{\eta\varepsilon}\frac{\gamma - 1}{e^{\sigma/v_*} - 1} \quad \textbf{relativistic external efficiency} \qquad (1.3.21)$$

For non-relativistic speeds, i.e., $\gamma \to 1 - v^2/2c^2$, $\sigma \to v$, and for $\delta = 1$, we recover the classical external rocket efficiency (see Eq. (2.5.10))

$$\eta_{ext} = \frac{(v/v_*)^2}{e^{v/v_*} - 1}$$

Note that while the external efficiency in the relativistic regime depends on the internal efficiency η, this does not hold for classical speeds.

1.3.5
Space–Time Transformations

It is an important and well-known feature of special relativity that observed values for space and time intervals depend on the reference frame of the external observer. This is what the word "relativity" actually refers to. With the concept of proper speed, it is easy to derive the space–time transformation equations between the proper (absolute) reference frame spacecraft and that of an external observer, which we now denote (σ, τ) and (x', v', t'), respectively. From Eqs. (1.3.1) and (1.3.5) and denoting $\phi := \sigma/c$, the so-called *rapidity*, we find

$$c \cdot dt' = \cosh\phi \cdot c\,d\tau$$

and because

$$dx' = v \cdot dt' = c \cdot \tanh\phi \cdot dt' = c \cdot \tanh\phi \cdot \cosh\phi \cdot d\tau = \sinh\phi \cdot c \cdot d\tau$$

we can write in shorthand vector notation

$$\begin{bmatrix} dx' \\ c \cdot dt' \end{bmatrix} = \begin{bmatrix} \sinh\phi \\ \cosh\phi \end{bmatrix} \cdot c \cdot d\tau \qquad (1.3.22)$$

Note that because $dx = 0$ for the rocket in the rest frame, we do not have a transformation matrix as in the general case. In order to derive the space–time transformations for any rocket–observer relation, we have to determine the rapidity (proper speed) and then solve the two differential equations (1.3.22). This will be done now for the two most simple cases.

1.3.5.1 **Cruising Rocket**

For a cruising (non-accelerated) rocket, $\cosh\phi = \cosh(\sigma/c) = \gamma = const$ and Eq. (1.3.22) can easily be integrated to give the well-known space–time transformation between inertial observers:

$$t' = \frac{1}{\gamma}\tau \tag{1.3.23a}$$

$$x' = \frac{v}{\gamma}\tau \tag{1.3.23b}$$

$$\frac{v}{c} = \tanh\frac{\sigma}{c} \tag{1.3.23c}$$

As an example: a rocket that travels 90% the speed of light as seen from an external observer or $\sigma = 1.47 \cdot c$ of proper speed would cross our Milky Way with diameter $d = 100\,000$ ly within $t' = d/v = 111\,000$ year or $\tau = 48\,000$ year in proper time.

1.3.5.2 **Constant-Acceleration Rocket**

If the acceleration α is constant, $\sigma = c\phi = \alpha\tau$. By integrating Eq. (1.3.22), we find

$$\frac{\alpha t'}{c} = \sinh\frac{\alpha\tau}{c} = \sinh\frac{\sigma}{c} \tag{1.3.24a}$$

Also, with Eq. (1.3.24a) and $\cosh x = \sqrt{1+\sinh^2 x}$, we have

$$\frac{\alpha x'}{c^2} = \cosh\frac{\alpha\tau}{c} - 1 = \sqrt{1+\left(\frac{\alpha t'}{c}\right)^2} - 1 \tag{1.3.24b}$$

Applying Eq. (1.3.24a) with $\tanh x = \sinh x/\sqrt{1+\sinh^2 x}$ to Eq. (1.3.5) yields

$$\frac{v}{c} = \tanh\frac{\sigma}{c} = \frac{\alpha t'}{\sqrt{c^2+(\alpha t')^2}} \tag{1.3.24c}$$

From Eq. (1.3.24b), we find

$$\left(x' + \frac{c^2}{\alpha}\right)^2 - (ct')^2 = \left(\frac{c^2}{\alpha}\right)^2$$

This denotes that the space–time trajectory of a rocket with constant acceleration is a hyperbola.

Let us reconsider the case of an ultimate manned $H-\bar{H}$ annihilation rocket with $\beta_* = 0.5804$, which we assume to cross the Milky Way ($x'_f = 100\,000$ ly) with a comfortable acceleration of $\alpha = 1g$. According to Eq. (1.3.24b), this would take only $\tau_f = 11.9$ year in proper time of an astronaut. His final proper speed would be $\sigma_f = \alpha\tau_f = 12.2c$ and the rocket's mass ratio can be calculated from Eq. (1.3.10) to be $m_i/m_f = 1.35 \times 10^9$. If the final spacecraft mass is, say, 100 metric tons (Space Shuttle), then the launch mass is $m_i = 1.35 \times 10^{14}$ kg. Moreover, if we assume that the $H, \bar{H}$ fuel is stored in liquid form with density 70 kg m^{-3}, the two tanks together must have dimensions of $14 \times 14 \times 14\,\text{km}^3$! Not to say anything about the engines that would have to propel such a gigantic space ship at 1g.

Problems

Problem 1.1 Balloon Propulsion

Consider a balloon that is propelled by exhausting its air with density $\varrho = 1.29$ g dm^{-3}. The balloon has a volume of 2 dm^3, the exit (throat) diameter is $A_t = 0.5\,\text{cm}^2$. Let us assume the balloon exhausts the gas with constant mass flow rate within 2 s. Show that the momentum thrust $F_e = 0.026$ N and the pressure thrust $F_p = 0.013$ N and hence the momentum thrust is roughly twice as big as the pressure thrust.

Hint: Observe that the exhaust velocity at the throat does not reach the speed of sound. Make use of the Bernoulli's equation $p + \frac{1}{2}\varrho v^2 = const.$

Problem 1.2 Nozzle Exit Area of an SSME

The thrust of a Space Shuttle main engine (SSME) is at 100% power level, 1.817×10^6 N at sea level, and 2.278×10^6 N in vacuum. By using only this information, derive that the nozzle exit area is $A_e = 4.55\,\text{m}^2$.