1.1 Short History

The theory of distributions, or of generalized functions, constitutes a chapter of functional analysis that arose from the need to substantiate, in terms of mathematical concepts, formulae and rules of calculation used in physics, quantum mechanics and operational calculus that could not be justified by classical analysis. Thus, for example, in 1926 the English physicist P.A.M. Dirac [1] introduced in quantum mechanics the symbol $\delta(x)$, called the Dirac delta function, by the formulae

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(x) dx = 1.$$
 (1.1)

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By this symbol, Dirac mathematically described a material point of mass density equal to the unit, placed at the origin of the coordinate axis.

We notice immediately that $\delta(x)$, called the impulse function, is a function not in the sense of mathematical analysis, as being zero everywhere except the origin, but that its integral is null and not equal to unity.

Also, the relations $x \delta(x) = 0$, $dH(x)/dx = \delta(x)$ do not make sense in classical mathematical analysis, where

$$H(x) = \begin{cases} 0 , & x < 0 \\ 1 , & x \ge 0 \end{cases}$$

is the Heaviside function, introduced in 1898 by the English engineer Oliver Heaviside.

The created formalism regarding the use of the function δ and others, although it was in contradiction with the concepts of mathematical analysis, allowed for the study of discontinuous phenomena and led to correct results from a physical point of view.

All these elements constituted the source of the theory of distributions or of the generalized functions, a theory designed to justify the formalism of calculation used in various fields of physics, mechanics and related techniques.

In 1936, S.L. Sobolev introduced distributions (generalized functions) in an explicit form, in connection with the study of the Cauchy problem for partial differential equations of hyperbolic type.

The next major event took place in 1950–1951, when L. Schwartz published a treatise in two volumes entitled "Théory des distributions" [2]. This book provided a unified and systematic presentation of the theory of distributions, including all previous approaches, thus justifying mathematically the calculation formalisms used in physics, mechanics and other fields.

Schwartz's monograph, which was based on linear functionals and on the theory of locally convex topological vector spaces, motivated further development of many chapters of mathematics: the theory of differential equations, operational calculus (Fourier and Laplace transforms), the theory of Fourier series and others.

Properties in the sense of distributions, such as the existence of the derivative of any order of a distribution and in particular of the continuous functions, the convergence of Fourier series and the possibility of term by term derivation of the convergent series of distributions, led to important technical applications of the theory of distributions, thus removing some restrictions of classical analysis.

The distribution theory had a significant further development as a result of the works developed by J. Mikusiński and R. Sikorski [3], M.I. Guelfand and G.E. Chilov [4, 5], L. Hörmander [6, 7], A. H. Zemanian [8], and so on.

Unlike the linear and continuous functionals method used by Schwartz to define distributions, J. Mikusiński and R. Sikorski introduced the concept of distribution by means of fundamental sequences of continuous functions.

This method corresponds to the spirit of classical analysis and thus it appears clearly that the concept of distribution is a generalization of the notion of function, which justifies the term generalized function, mainly used by the Russian school.

Other mathematicians, such as H. König, J. Korevaar, Sebastiano e Silva, and I. Halperin have defined the notion of distribution by various means (axiomatic, derivatives method, and so on).

Today the notion of distribution is generalized to the concept of a hyperfunction, introduced by M. Sato, [9, 10], in 1958. The hyperdistributions theory contains as special cases the extensions of the notion of distribution approached by C. Roumieu, H. Komatsu, J.F. Colombeu and others.

1.2

Fundamental Concepts and Formulae

For the purpose of distribution theory and its applications in various fields, we consider some function spaces endowed with a convergence structure, called fundamental spaces or spaces of test functions.

1.2.1

Normed Vector Spaces: Metric Spaces

We denote by Γ either the body \mathbb{R} of real numbers or the body \mathbb{C} of complex numbers and by \mathbb{R}_+ , \mathbb{R}^+ , \mathbb{N}_0 the sets $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{N}_0 = \{0, 1, 2, ..., n, ...\}$.

Let *E*, *F* be sets of abstract objects. We denote by $E \times F$ the direct product (Cartesian) of those two sets; where the symbol "×" represents the direct or Cartesian product.

Definition 1.1 The set *E* is called a vector space with respect to Γ , and is denoted by (E, Γ) , if the following two operations are defined: the sum, a mapping $(x, y) \rightarrow x + y$ from $E \times F$ into *E*, and the product with scalars from Γ , the mapping $(\lambda, x) \rightarrow \lambda x$ from $\Gamma \times E$ into *E*, having the following properties:

1. $\forall x, y \in E$, x + y = y + x; 2. $\forall x, y, z \in E$, (x + y) + z = x + (y + z); 3. $\exists 0 \in E$, $\forall x \in E$, x + 0 = x, (0 is the null element); 4. $\forall x \in E$, $\exists x' = -x \in E$, x + (-x) = 0; 5. $\forall x \in E$, $1 \cdot x = x$; 6. $\forall \lambda, \mu \in \Gamma$, $\forall x \in E$, $\lambda(\mu x) = (\lambda \mu)x$; 7. $\forall \lambda, \mu \in \Gamma$, $\forall x \in E$, $(\lambda + \mu)x = \lambda x + \mu x$; 8. $\forall \lambda \in \Gamma$, $\forall x, y \in E$, $\lambda(x + y) = \lambda x + \lambda y$.

The vector space (E, Γ) is real if $\Gamma = \mathbb{R}$ and it is complex if $\Gamma = \mathbb{C}$. The elements of (E, Γ) are called points or vectors.

Let *X* be an upper bounded set of real numbers, hence there is $M \in \mathbb{R}$ such that for all $x \in X$ we have $x \leq M$. Then there exists a unique number $M^* = \sup X$, which is called the lowest upper bound of *X*, such that

1. $\forall x \in X$, $x \le M^*$; 2. $\forall a \in \mathbb{R}$, $a < M^*$, $\exists x \in X$ such that $x \in (a, M^*]$.

Similarly, if *Y* is a lower bounded set of real numbers, that is, if there is $m \in \mathbb{R}$ such that for all $x \in Y$ we have $x \ge m$, then there exists a unique number $m^* = \inf X$, which is called the greatest lower bound of *Y*, such that

1. $\forall x \in Y$, $x \ge m^*$; 2. $\forall b \in \mathbb{R}$, $b > m^*$, $\exists x \in Y$ such that $x \in [m^*, b]$.

Example 1.1 The vector spaces \mathbb{R}^n , \mathbb{C}^n , $n \ge 2$ Let us consider the *n*-dimensional space $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (*n* times). Two elements $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, are said to be equal, x = y, if $x_i = y_i$, $i = \overline{1, n}$.

Denote $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$, $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$, $\alpha \in \mathbb{R}$, then \mathbb{R}^n is a real vector space, also called *n*-dimensional real arithmetic space.

The *n*-dimensional complex space \mathbb{C}^n may be defined in a similar manner. The elements of this space are ordered systems of *n* complex numbers. The sum and product operations performed on complex numbers are defined similarly with those in \mathbb{R}^n .

Definition 1.2 Let (X, Γ) be a real or complex vector space. A norm on (X, Γ) is a function $\|\cdot\| : X \to [0, \infty)$ satisfying the following three axioms:

1. $\forall x \in X$, ||x|| > 0 for $x \neq 0$, ||0|| = 0; 2. $\forall \lambda \in \Gamma$, $\forall x \in X$, $||\lambda x|| = |\lambda| ||x||$; 3. $\forall x, y \in X$, $||x + y|| \le ||x|| + ||y||$.

The vector space (X, Γ) endowed with the norm $\|\cdot\|$ will be called a normed vector space and will be denoted as $(X, \Gamma, \|\cdot\|)$.

The following properties result from the definition of the norm:

$$\begin{aligned} \|x\| &\ge 0, \quad \forall x \in X, \\ \|\|x_1\| - \|x_2\| &\le \|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, \\ \forall \alpha_i \in \Gamma, \quad \forall x_i \in X, \quad \|\alpha_1 x_1 + \dots + \alpha_n x_n\| &\le \|\alpha_1\| \|x_1\| + \dots + \|\alpha_n\| \|x_n\|. \end{aligned}$$

Definition 1.3 Let (X, Γ) be a vector space. We call an inner product on (X, Γ) a mapping $\langle \cdot, \cdot \rangle : E \to \Gamma$ that satisfies the following properties:

- 1. Conjugate symmetry: $\forall x \in X, \langle x, y \rangle = \overline{\langle y, x \rangle}$;
- 2. Homogeneity: $\forall \alpha \in \Gamma, \forall x, y \in E, \langle \alpha x, y \rangle = \alpha \langle y, x \rangle$;
- 3. Additivity: $\forall x, y, z \in X, \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;
- 4. Positive-definiteness: $\forall x \in X, \langle x, x \rangle \ge 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

An inner product space $(X, \langle \cdot, \cdot \rangle)$ is a space containing a vector space (X, Γ) and an inner product $\langle \cdot, \cdot \rangle$.

Conjugate symmetry and linearity in the first variable gives

$$\langle x, ay \rangle = \overline{\langle ay, x \rangle} = \overline{a} \overline{\langle y, x \rangle} = \overline{a} \langle x, y \rangle ,$$

$$\langle x, y + z \rangle = \overline{\langle y + z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle ,$$

so an inner product is a sesquilinear form. Conjugate symmetry is also called Hermitian symmetry.

In the case of $\Gamma = \mathbb{R}$, conjugate-symmetric reduces to symmetric, and sesquilinear reduces to bilinear. Thus, an inner product on a real vector space is a positive-definite symmetric bilinear form.

Proposition 1.1 In any inner product space $(X, \langle \cdot, \cdot \rangle)$ the Cauchy–Schwarz inequality holds:

$$|\langle x, y \rangle| \le \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle} , \quad \forall x, y \in X ,$$
(1.2)

with equality if and only if *x* and *y* are linearly dependent.

This is also known in the Russian mathematical literature as the Cauchy-Bunyakowski-Schwarz inequality.

Lemma 1.1 The inner product is antilinear in the second variable, that is $\langle x, y + z \rangle$ $= \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in \Gamma$ and $\langle x, ay \rangle = \overline{a} \langle x, y \rangle$.

Note that the convention in physics is often different. There, the second variable is linear, whereas the first variable is antilinear.

Definition 1.4 Let X be a nonempty set. We shall call metric (distance) on X any function $d: X \times X \to \mathbb{R}$, which satisfies the properties:

 D_1 $d(x, x) = 0, \forall x \in X; d(x, y) > 0, \forall x, y \in X, x \neq y,$ $D_2 \quad \forall x, y \in X, d(x, y) = d(y, x),$ $D_3 \quad \forall x, y, z \in X, d(x, z) < d(x, y) + d(y, z).$

The real number $d(x, y) \ge 0$ represents the distance between x and y, and the ordered pair (X, d) a metric space (whose elements are called points).

Let (*X*, *d*) be a metric space. We shall call an *open ball* in *X* a ball of radius r > 0centered at the point $x_0 \in X$, usually denoted $B_r(x_0)$ or $B(x_0; r)$, the set

$$B_r(x_0) = \{ x \in X \mid d(x, x_0) < r \} .$$
(1.3)

The closed ball, which will be denoted by $\overline{B}_r(x_0)$ is defined by

$$B_r(x_0) = \{ x \in X \mid d(x, x_0) \le r \} .$$
(1.4)

Note, in particular, that a ball (open or closed) always includes x_0 itself, since the definition requires r > 0. We shall call a sphere of radius r > 0 centered at the point $x_0 \in X$, usually denoted $S_r(x_0)$, the set

$$S_r(x_0) = \{ x \in X \mid d(x, x_0) = r \}.$$
(1.5)

Proposition 1.2 Any normed vector space is a metric space by defining the distance by the formula

$$d(x, y) = ||x - y||, \quad \forall x, y \in X.$$
(1.6)

Proposition 1.3 Any inner product space $(X, \langle \cdot, \cdot \rangle)$ is a normed vector space if we define the norm by

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad \forall x \in X.$$
(1.7)

An inner product space is also called a pre-Hilbert space, since its completion with respect to the metric induced by its inner product, is a Hilbert space.

The real vector space \mathbb{R}^n endowed with the inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i, x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n) \in \mathbb{R}^n$$
 (1.8)

is called the *n*-dimensional Euclidean real space.

The norm in \mathbb{R}^n is called *the Euclidean norm* and is defined as

$$\|x\| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} , \qquad (1.9)$$

whereas the metric associated to this norm is given by

$$d(x, y) = ||x - y|| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} .$$
(1.10)

1.2.2

Spaces of Test Functions

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ be a generic point in the *n*-dimensional Euclidean real space and let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ be a multiindex of order *n*; we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_n$ the length of the multiindex. If $\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n) \in \mathbb{N}_0^n$, then we use the following notations:

$$\alpha \le \beta \text{ if } \alpha_i \le \beta_i, i = \overline{1, n} ; \qquad (1.11)$$

$$\binom{\beta}{\alpha} = \frac{\beta!}{\alpha!(\beta - \alpha)!}, \quad \text{where } \alpha! = \alpha_1!\alpha_2!\dots\alpha_n!, \quad (1.12)$$

$$x^{a} = x_{1}^{a_{1}} x_{2}^{a_{2}} \dots x_{n}^{a_{n}} .$$
(1.13)

We denote by $D^{\alpha} f$ the partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$ of a function $f : \Omega \subset \mathbb{R}^n \to \Gamma$,

$$D^{\alpha} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}} f, \quad D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}, \quad D_j = \frac{\partial}{\partial x_j}, \quad j = \overline{1, n}.$$

If $|\alpha| = 0$, then $\alpha_i = 0$, $i = \overline{1, n}$, that is, $D^0 f = f$.

If the function f has continuous partial derivatives up to the order $|\alpha+\beta|$ inclusively, then

$$D^{\alpha+\beta} f = D^{\alpha}(D^{\beta} f) = D^{\beta}(D^{\alpha} f).$$

We shall denote by $C^m(\Omega)$ the set of functions $f : \Omega \subset \mathbb{R}^n \to \Gamma$ with continuous derivatives of order *m*, that is, $D^{\alpha} f$ is continuous on Ω for every α with $|\alpha| \leq m$. When m = 0 we have the set $C^0(\Omega)$ of continuous functions on Ω ; $C^{\infty}(\Omega)$ is the set of functions on Ω with continuous derivatives of all orders. Clearly, we have $C^{\infty}(\Omega) \subset C^m(\Omega) \subset C^0(\Omega)$.

These sets are vector spaces over Ω with respect to the usual definition of addition of functions and multiplication by scalars from Ω . The null element of these spaces is the identically zero function on Ω and it will be denoted by 0.

Definition 1.5 We call the support of the function $f : \mathbb{R}^n \to \Gamma$ the set

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^n, f(x) \neq 0\}}, \qquad (1.14)$$

hence the closure of the set of points where the function is not zero.

If $x_0 \in \text{supp}(f)$, then $\forall B_{x_0}(r), \exists x \in \mathbb{R}^n$ thus that $f(x) \neq 0$. In particular, if supp(f) is bounded, then, since supp(f) is a closed set, it is also compact.

Proposition 1.4 If $f, g : \mathbb{R}^n \to \Gamma$, then:

$$\operatorname{supp}(f+g) \subset \operatorname{supp}(f) \cup \operatorname{supp}(g)$$
, (1.15)

 $\operatorname{supp}(f \cdot g) \subset \operatorname{supp}(f) \cap \operatorname{supp}(g) , \qquad (1.16)$

$$\operatorname{supp}(\lambda f) = \operatorname{supp}(f), \quad \lambda \neq 0.$$
 (1.17)

Proof: If $x_0 \in \text{supp}(f + g)$, then $\forall B_r(x_0) \subset \mathbb{R}^n$, $\exists x \in B_r(x_0)$ such that $(f + g)(x) \neq 0$, from which results $f(x) \neq 0$ or $g(x) \neq 0$. Consequently, either $x_0 \in \text{supp}(f)$ or $x_0 \in \text{supp}(g)$, hence $x_0 \in \text{supp}(f) \cup \text{supp}(g)$. Regarding relation (1.16), we notice that $x_0 \in \text{supp}(f \cdot g)$ implies $(fg)(x) \neq 0$, $x \in B_r(x_0)$; hence $f(x) \neq 0$ and $g(x) \neq 0$. Consequently, $x_0 \in \text{supp}(f)$ and $x_0 \in \text{supp}(g)$, hence $\text{supp}(f) \cap \text{supp}(g)$. Because relation (1.17) is obvious, the proof is complete.

Proposition 1.5 If the functions $f, g \in C^{p}(\Omega), \Omega \subset \mathbb{R}^{n}$, then $fg \in C^{p}(\Omega)$ and we have

$$D^{\alpha}(f \cdot g) = \sum_{\beta + \gamma} \frac{\alpha!}{\beta! \gamma!} D^{\beta} f \cdot D^{\gamma} g , \quad D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} , \qquad (1.18)$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n, |\alpha| \le p$.

The proof of this formula is accomplished through induction.

Definition 1.6 A function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be uniformly continuous on *A* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $x, y \in A$ satisfying the condition $||x - y|| < \delta(\varepsilon)$ the inequality $|f(x) - f(y)| < \varepsilon$ holds.

We mention that a uniformly continuous function on $A \subset \mathbb{R}^n$ is continuous at each point of the set *A*. It follows that the continuity is a local (more precisely, pointwise) property of a function *f*, while the uniform continuity is a global property of *f*. In the study of the properties of spaces of test functions, the notion of uniformly convergent sequence plays an important role.

Definition 1.7 We consider the sequence of functions $(f_n)_{n\geq 1}$, $f_n : A \subset \mathbb{R}^n \to \mathbb{R}$ and the function $f : A \subset \mathbb{R}^n \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n\geq 1}, x \in A$ is uniformly convergent towards $f, x \in A$, and we write $f_n \xrightarrow{u} f, x \in A \subset \mathbb{R}^n$, if for every $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that for all $x \in A$ and all $n \ge N(\varepsilon)$ the inequality $|f_n(x) - f(x)| < \varepsilon$ holds.

In the case of uniform convergence, the natural number $N(\varepsilon)$ depends only on $\varepsilon > 0$, being the same for all $x \in A$, while in the case of pointwise convergence the natural number *N* depends on ε and $x \in A$. Therefore the uniform convergence implies pointwise convergence $f_n \stackrel{s}{\to} f$. The converse is not always true.

Definition 1.8 We say that the function $f : A \subset \mathbb{R}^n \to \mathbb{C}$ is absolutely integrable on *A* if the integral $\int_A |f(x)| dx$ is finite, hence $\int_A |f(x)| dx < \infty$. The integral can be considered either in the sense of Riemann, or in the sense of Lebesgue.

If the integral is considered in the sense of Lebesgue, then the existence of the integral $\int_A |f(x)| dx$ implies the existence of the integral $\int_A f(x) dx$.

The set of the Lebesgue integrable functions on *A* will be denoted $L^{1}(A)$.

If *f* is absolutely integrable on any bounded domain $A \subset \mathbb{R}^n$, then we say that *f* is a *locally integrable function*. We shall use $L^1_{loc}(A)$ to denote the space of locally integrable functions on *A*.

The set $A \subset \mathbb{R}^n$ is said to be *negligible* or of *null Lebesgue measure* if for any $\varepsilon > 0$ there is a sequence $(B_i)_{i\geq 1}, B_i \subset \mathbb{R}^n$, such that $\bigcup_{i=1}^{\infty} B_i \supset A$ and the summed volume of the open ball B_i is less than ε .

The function $f : A \subset \mathbb{R}^n \to \Gamma$ is said to be *null a.e.* (almost everywhere) on the set *A* if the set $\{x \in A, f(x) \neq 0\}$ is of null Lebesgue measure.

Thus, the functions $f, g : A \subset \mathbb{R}^n \to \Gamma$ are *a.e. equal* (almost everywhere equal), denoted by f = g *a.e.*, $x \in A$, if the set $\{x \in A, f(x) \neq g(x)\}$ is of null Lebesgue measure.

The function $f : A \subset \mathbb{R}^n \to \Gamma$ is *p*-integrable on $A, 1 \leq p < \infty$, if $|f|^p \in L^1(A)$. The set of *p*-integrable functions on A is denoted by $\mathcal{L}^p(A)$. In this set we can introduce the equivalence relation $f \sim g$ if f(x) = g(x) a.e. The set of all the equivalence classes is denoted by $L^p(A)$.

The space $L^{p}(A)$ is a vector space over Γ . The spaces $L^{p}(A)$ and $L^{q}(A)$ for which we have $p^{-1} + q^{-1} = 1$ are called *conjugate*. For these spaces, we have Hölder's inequality

$$\int_{A} |f(x)g(x)| \mathrm{d}x \leq \left(\int_{A} |f(x)|^{p} \mathrm{d}x\right)^{1/p} \cdot \left(\int_{A} |g(x)|^{q} \mathrm{d}x\right)^{1/q}.$$
(1.19)

In particular, for p = 2, we have q = 2, that is, $L^2(A)$ is self-conjugated and Schwarz's inequality holds

$$\int_{A} |f(x)g(x)| \mathrm{d}x \le \left(\int_{A} |f(x)|^2 \mathrm{d}x \right)^{1/2} \cdot \left(\int_{A} |g(x)|^2 \mathrm{d}x \right)^{1/2} \,. \tag{1.20}$$

The norm of the space $L^p(A)$ is defined as

$$\|f\|_{p} = \left(\int_{A} |f(x)|^{p} \mathrm{d}x\right)^{1/p} .$$
(1.21)

We notice that the space $L^p(A)$ is normed.

1.2.2.1 The Space $\mathcal{D}^m(\Omega)$

Definition 1.9 Let $\Omega \subset \mathbb{R}^n$ be a given compact set and consider the functions $\varphi : \mathbb{R}^n \to \Gamma$. The set of functions $\mathcal{D}^m(\Omega) = \{\varphi | \varphi \in C^m(\mathbb{R}^n), \operatorname{supp}(\varphi) \subset \Omega\}$ is called the space of test functions $\mathcal{D}^m(\Omega)$.

We notice that $\varphi \in C^m(\mathbb{R}^n)$ with $\operatorname{supp}(\varphi) \subset \Omega$ implies $\operatorname{supp}(D^a \varphi(x)) \subset \operatorname{supp}(\varphi) \subset$ $\Omega, |\alpha| \leq m$. Consequently, all functions $\varphi \in C^m(\Omega)$ together with all their derivatives up to order *m* inclusive are null outside the compact Ω . We notice that $\Omega^m(\Omega)$ is a vector space with respect to Γ . The null element of this space is the identically null function, denoted by 0, $\forall x \in \mathbb{R}^n$, $\varphi(x) = 0$.

Definition 1.10 We say that the sequence of functions $(\varphi_i)_{i>1} \subset \mathcal{D}^m(\Omega)$ converges towards $\varphi \in \mathcal{D}^m(\Omega)$, and we write $\varphi_i \xrightarrow{\mathcal{D}^m(\Omega)} \varphi$ if the sequence of functions $(D^{\alpha}\varphi_i(x))_{i\geq 1}$ converges uniformly towards $D^{\alpha}\varphi(x)$ in Ω , hence $D^{\alpha}\varphi_i(x) \xrightarrow{\mu}$ $D^{\alpha}\varphi(x), 0 \leq |\alpha| \leq m, \forall x \in \Omega.$

We note that the space $\mathcal{D}^m(\Omega)$ becomes a normed vector space if we define the norm by

$$\|\varphi\|_{\mathcal{D}^m} = \sup_{|\alpha| \le m, x \in \Omega} |\mathsf{D}^{\alpha}\varphi(x)| = \sup_{0 \le |\alpha| \le m} \sup_{x \in \Omega} |\mathsf{D}^{\alpha}\varphi(x)|, \alpha \in \mathbb{N}_0^n.$$
(1.22)

In particular, for m = 0, the space $\mathcal{D}^0(\Omega)$ will be denoted by $C_C^0(\Omega)$. This is the space of complex (real) functions of class $C^0(\mathbb{R}^n)$, the supports of which are contained in the compact set $\Omega \subset \mathbb{R}^n$. The test functions space $C^0_C(\Omega)$ is a normed vector space with the norm

$$\|\varphi\|_{C^0_{\mathcal{C}}} = \sup_{x \in \Omega} |\varphi(x)| .$$
(1.23)

The sequence $(\varphi_i)_{i\geq 1} \subset C^0_C(\Omega)$ converges towards $\varphi \in C^0_C(\Omega)$ if $\lim_i \sup_{x\in\Omega} \varphi_i$ $|\varphi_i - \varphi| = 0$, that is, if $(\varphi_i)_{i>1}$ converges uniformly towards φ in Ω .

An example of functions from the space $\mathcal{D}^m(\Omega)$ is the function

$$\varphi(x) = \begin{cases} \prod_{i=1}^{n} \sin^{m+1} \frac{x_i - a_i}{b_i - a_i} \pi, & x \in [a_1, b_1] \times \dots \times [a_n, b_n] = \prod_{i=1}^{n} \times [a_i, b_i] \\ 0, & x \notin [a_1, b_1] \times \dots \times [a_n, b_n] \end{cases}$$

where

$$\Omega \supset \prod_{i=1}^n \times [a_i, b_i].$$

It is immediately verified that $\varphi \in C^m(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi) = \prod_{i=1}^n \times [a_i, b_i]$. Also the function $\varphi : \mathbb{R} \to \mathbb{R}$, where

$$\varphi = \begin{cases} (x-a)^{\alpha}(b-x)^{\beta}, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}, \quad \alpha,\beta > m,$$

is a function from $\mathcal{D}^m([c, d]), [c, d] \supset [a, b]$, because $\varphi \in C^m([c, d])$ and supp $(\varphi) = [a, b]$.

Let us consider the sequence of functions $(\varphi_n)_{n>1} \subset \mathcal{D}^m(\Omega)$, defined by

$$\varphi_n(x) = \begin{cases} \frac{1}{n} \sin^{m+1} \frac{x+a}{2a} \pi, & x \in [-a, a], \\ 0, & x \notin [-a, a]. \end{cases}$$

We have $\operatorname{supp} \varphi_n(x) = [-a, a] = \Omega$ for any *n*. This sequence, with its derivatives up to order *m* inclusive, converges uniformly towards zero in Ω . So we can write $\varphi_n(x) \xrightarrow{\mathcal{D}(\Omega)} 0$ in Ω .

Even if the sequence of functions

$$\varphi_n(x) = \begin{cases} \frac{1}{n} \sin^{m+1} \frac{a+x/n}{2a} \pi, & \frac{x}{n} \in [-a, a], \\ 0, & \frac{x}{n} \notin [-a, a], \end{cases}$$

converges uniformly towards zero, together with all their derivatives up to order *m* inclusive, it is not convergent towards zero in the space $\mathcal{D}^m(\Omega)$. This is because $\sup p[\varphi_n(x)] = [-na, na]$, thus the supports of the functions $\varphi_n(x)$ are not bounded when $n \to \infty$, hence $\varphi_n(x)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, are not test functions from $\mathcal{D}^m(\Omega)$.

1.2.2.2 The Space $\mathcal{D}(\Omega)$

Definition 1.11 Let $\Omega \subset \mathbb{R}^n$ be a given compact set and consider the functions $\varphi : \mathbb{R}^n \to \Gamma$. The set of functions

$$\mathcal{D}(\Omega) = \{\varphi | \varphi \in C^{\infty}(\mathbb{R}^n), \operatorname{supp}(\varphi) \subset \Omega\}$$

is called the space of test functions $\mathcal{D}(\Omega)$.

The space $\mathcal{D}(\Omega)$ is a vector space over Γ like $\mathcal{D}^m(\Omega)$.

Definition 1.12 We say that the sequence $(\varphi_i)_{i\geq 1} \subset \mathcal{D}(\Omega)$ converges towards $\varphi \in \mathcal{D}(\Omega)$, and we write $\varphi_i \xrightarrow{\mathcal{D}(\Omega)} \varphi$, if the sequence of derivative $(D^{\alpha}\varphi_i(x))_{i\geq 1}$ converges uniformly towards $D^{\alpha}\varphi(x)$ in Ω , $\forall \alpha \in \mathbb{N}_0^n$, hence $D^{\alpha}\varphi_i(x) \xrightarrow{u} D^{\alpha}\varphi(x)$, $\forall x \in \Omega, \forall \alpha \in \mathbb{N}_0^n$.

We remark that the test space $\mathcal{D}(\Omega)$ is not a normed vector space.

Example 1.2 If $\Omega = \{x | x \in \mathbb{R}^n, \|x\| \le 2a\}$, then the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, having the expression

$$\varphi(x) = \begin{cases} \exp\left(-\frac{a^2}{a^2 - \|x\|^2}\right), & \|x\| < a\\ 0, & \|x\| \ge a \end{cases}, \quad a > 0, \qquad (1.24)$$

is an element of the space $\mathcal{D}(\Omega)$, since $\varphi \in C^{\infty}(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi) = \{x | x \in \mathbb{R}^n,$ $||x|| < a \} \subset \Omega.$

The sets Ω and supp(φ) are compact sets of \mathbb{R}^n , representing closed balls with centers at the origin and radii 2*a* and *a*, respectively.

Unlike the function φ , the function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\psi(x) = \begin{cases} 0, & x \le 0, \\ \exp(-x^2), & x > 0, \end{cases}$$
(1.25)

does not belong to the space $\mathcal{D}(\Omega)$.

This function is infinitely differentiable, so $\psi \in C^{\infty}(\mathbb{R}^n)$, but the support is not a compact set because $\operatorname{supp}(\psi) = (0, \infty)$.

1.2.2.3 The Space *E*

Definition 1.13 The functions set

$$\mathcal{E} = \{\varphi | \varphi : \mathbb{R}^n \to \Gamma, \varphi \in C^{\infty}(\mathbb{R}^n)\}.$$
(1.26)

having arbitrary support is called the space of test functions $\mathcal{E} = \mathcal{E}(\mathbb{R}^n)$.

With respect to the usual sum and scalar product operation, the space \mathcal{E} is a vector space over Γ .

Thus, the functions $\varphi(x) = 1$, $\varphi(x) = x^2$, $\varphi(x) = \exp(x^2)$, $x \in \mathbb{R}$ are elements of $\mathcal{E}(\mathbb{R}^n).$

As regards the convergence in the space \mathcal{E} this is given:

Definition 1.14 The sequence $(\varphi_i)_{i\geq 1} \subset \mathcal{E}$ is said to converge towards $\varphi \in \mathcal{E}$, and we write $\varphi_i \xrightarrow{\mathcal{E}} \varphi$, if the sequence of functions $(D^{\alpha}\varphi_i)_{i\geq 1} \subset \mathcal{E}$ converges uniformly towards $D^{\alpha} \varphi(x) \in \mathcal{E}$ on any compact of \mathbb{R}^n , $\forall \alpha \in \mathbb{N}_0^n$, that is, $D^{\alpha} \varphi_i \xrightarrow{u} D^{\alpha} \varphi$.

The function (1.25) belongs to the space \mathcal{E} since $\psi \in C^{\infty}(\mathbb{R}^n)$, its supports being the unbounded set $(0, \infty)$.

1.2.2.4 The Space D (the Schwartz Space)

Definition 1.15 The space $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ consists of the set of functions

$$\mathcal{D} = \{\varphi | \varphi : \mathbb{R}^n \to \Gamma, \varphi \in C^{\infty}(\mathbb{R}^n), \operatorname{supp}(\varphi) = \Omega = \operatorname{compact}\}.$$
(1.27)

Since $\forall \varphi \in \mathcal{D}$, it belongs to a certain $\mathcal{D}(\Omega)$, it follows that \mathcal{D} is the reunion of spaces $\mathcal{D}(\Omega)$ over the compacts $\Omega \subset \mathbb{R}^n$. Consequently, we can write the following relations:

$$\mathfrak{D} = igcup_{\Omega} \mathfrak{D}(\Omega), \mathfrak{D}(\Omega) \subset \mathfrak{D} \subset \mathcal{E} \;.$$

With respect to the usual sum and scalar product operations, \mathcal{D} is a vector space on Γ , its null element being the identically zero function. The support of this function is the empty set.

The convergence in the space $\mathcal D$ is defined as:

Definition 1.16 The sequence of functions $(\varphi_i)_{i\geq 1} \subset \mathcal{D}$ converges towards $\varphi \in \mathcal{D}$, and we write $\varphi_i \xrightarrow{\mathcal{D}} \varphi$, if the following conditions are satisfied:

- 1. $\forall i \in \mathbb{N}$, there is a compact $\Omega \subset \mathbb{R}^n$ such that $\operatorname{supp}(\varphi_i)$, $\operatorname{supp}(\varphi) \subset \Omega$;
- 2. $\forall \alpha \in \mathbb{N}_0^n, \mathbb{D}^{\alpha} \varphi_i \text{ converges uniformly towards } \mathbb{D}^{\alpha} \varphi \text{ on } \Omega \text{ , that is, } \mathbb{D}^{\alpha} \varphi_i \xrightarrow{u} \mathbb{D}^{\alpha} \varphi$ on Ω .

Thus, the convergence in the space \mathcal{D} is reduced to the convergence in the space $\mathcal{D}(\Omega)$.

The vector space $\mathcal{D}(\mathbb{R}^n)$ endowed with the convergence structure defined above is called the space of test functions or the Schwartz space. Every element of the space \mathcal{D} will be called a test function.

Example 1.3 The function $\varphi_a : \mathbb{R}^n \to \mathbb{R}$, a > 0, defined by

$$\varphi_{a}(x) = \begin{cases} \exp\left(-\frac{a^{2}}{a^{2} - \|x\|^{2}}\right), & \|x\| < a, \\ 0, & \|x\| \ge a, \end{cases}$$
(1.28)

is an element of $\mathcal{D}(\mathbb{R}^n)$, since $\varphi_a \in C^{\infty}(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi_a) = \{x | x \in \mathbb{R}^n, ||x|| \le a\} = \operatorname{compact}$.

Example 1.4 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\varphi(x) = \begin{cases} \exp\left(-\frac{|ab|}{(x-a)(b-x)}\right), & x \in (a,b), \\ 0, & x \notin (a,b). \end{cases}$$
(1.29)

It is noted that $\varphi \in C^{\infty}(\mathbb{R})$ has compact support [*a*, *b*]. At the points *a* and *b*, the function φ and with its derivatives of any order are zero. Consequently, $\varphi \in \mathcal{D}(\mathbb{R})$. The graph of the function is shown in Figure 1.1.





Also, the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, where

$$\varphi(x_1,\ldots,x_n) = \begin{cases} \prod_{i=1}^n \exp\left(-\frac{|a_i b_i|}{(x_i - a_i)(b_i - x_i)}\right), & x_i \in (a_i, b_i), \\ 0, & x_i \notin (a_i, b_i), \end{cases}$$
(1.30)

is a function of the space $\mathcal{D}(\mathbb{R}^n)$, with the compact support $\Omega_n = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Example 1.5 Let $(\varphi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R})$ be a sequence of functions

$$\varphi_n(x) = \frac{1}{n} \varphi_a(x) = \begin{cases} \frac{1}{n} \exp\left(-\frac{a^2}{a^2 - x^2}\right), & |x| < a, a > 0, \\ 0, & |x| \ge a, a > 0. \end{cases}$$
(1.31)

We have $\varphi_n \xrightarrow{\mathcal{D}(\mathbb{R})} 0$, that is, the sequence $(\varphi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R})$ converges towards $\varphi = 0 \in \mathcal{D}(\mathbb{R})$ in the space $\mathcal{D}(\mathbb{R})$, because $\forall n \in \mathbb{N}$, $\operatorname{supp}(\varphi_n) \subset \operatorname{supp}(\varphi_a) = \operatorname{compact} \operatorname{and} (\operatorname{d}^a/\operatorname{d} x^a)\varphi_n(x) \xrightarrow{u} 0, \forall a \in \mathbb{N}_0, |x| \leq a.$

Definition 1.17 We say that the function $\psi : \mathbb{R}^n \to \Gamma$ is a multiplier for the space \mathcal{D} if $\forall \varphi \in \mathcal{D}$ the mapping $\varphi \to \psi \varphi$ is continuous from \mathcal{D} in \mathcal{D} .

Hence, if ψ is a multiplier for space \mathcal{D} , then $\psi \varphi \in \mathcal{D}, \forall \varphi \in \mathcal{D}$ and $\varphi_i \xrightarrow{\mathcal{D}} \varphi$ implies $\psi \varphi_i \xrightarrow{\mathcal{D}} \psi \varphi$.

We can easily check that any function $\psi \in C^{\infty}(\mathbb{R}^n)$ is a multiplier for space \mathcal{D} .

Indeed, since $\psi \in C^{\infty}(\mathbb{R}^n)$ and $\varphi \in C^{\infty}(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we apply formula (1.18) and have

$$D^{\alpha}(\psi\varphi) \in \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D^{\beta} \psi D^{\gamma} \varphi, D^{\alpha} = D_{1}^{\alpha_{1}} \dots D_{n}^{\alpha_{n}}, \alpha = (\alpha_{1}, \dots, \alpha_{n}) \in \mathbb{N}_{0}^{n},$$
(1.32)

from which it results that $\psi \varphi \in C^{\infty}(\mathbb{R}^n)$.

On the other hand, we have $\operatorname{supp}(\psi \varphi) \subset \operatorname{supp}(\psi) \cap \operatorname{supp}(\varphi) \subset \operatorname{supp}(\varphi) = \Omega =$ compact.

Next, we show that $\varphi_i \xrightarrow{\mathcal{D}} \varphi$ implies $\psi \varphi_i \xrightarrow{\mathcal{D}} \psi \varphi$. From the expression of the derivative $D^{\alpha}(\psi \varphi)$ it results

$$|\mathsf{D}^{lpha}\psi(arphi_{i}-arphi)|\leq \sum_{\|\gamma\|\leq\|lpha\|}A_{\gamma}|\mathsf{D}^{\gamma}(arphi_{i}-arphi)|$$
 , $A_{\gamma}>0$ constants .

Since $D^{\alpha}(\varphi_i - \varphi) \xrightarrow{\mathcal{D}} 0$, we obtain $|D^{\alpha}\psi(\varphi_i - \varphi)| \xrightarrow{\mathcal{D}} 0$, hence $\psi\varphi_i \xrightarrow{\mathcal{D}} \psi\varphi$.

Theorem 1.1 The partition of unity If $\varphi \in \mathcal{D}$ and U_i , i = 1, 2, ..., p, are open and bounded sets, which form a finite covering of the support function φ , then there exist the functions $e_i \in \mathcal{D}$, i = 1, 2, ..., p, with the properties:

1.
$$e_i(x) \in [0, 1]$$
, $\operatorname{supp}(e_i) \subset U_i$;
2. $\sum_{i=1}^p e_i(x) = 1, x \in \operatorname{supp}(\varphi)$;
3. $\varphi(x) = \sum_{i=1}^p e_i(x)\varphi(x)$.

We note that the partition theorem is frequently used to demonstrate the local properties of distributions, as well as the operations with them.

1.2.2.5 The Space S (the Space Functions which Decrease Rapidly)

Definition 1.18 We call the test function space $S = S(\mathbb{R}^n)$ the set of functions $\varphi : \mathbb{R}^n \to \Gamma$, infinitely differentiable, which for $||x|| \to \infty$ approach zero together with all their derivatives of any order, faster than any power of $||x||^{-1}$.

If $\varphi \in S$, then $\forall k \in \mathbb{N}$ and $\forall \beta \in \mathbb{N}_0^n$ we have

$$\lim_{\|x\|\to\infty} \|x\|^k \mathsf{D}^\beta \varphi = 0$$

This means that $\forall \varphi \in S$, we have $\varphi \in C^{\infty}(\mathbb{R}^n)$ and $\forall \alpha, \beta \in \mathbb{N}_0^n$, $\lim_{\|x\| \to \infty} |x^{\alpha} D^{\beta} \varphi| = 0$, that is, $|x^{\alpha} D^{\beta} \varphi| < C_{\alpha,\beta}$, where $C_{\alpha,\beta}$ are constants.

Example 1.6 An example of a function in S is $\varphi(x) = \exp(-a||x||^2)$, a > 0, $x \in \mathbb{R}^n$. On the other hand, the function $\varphi(x) = \exp(-x)$, $x \in \mathbb{R}$, does not belong to the space $S(\mathbb{R})$, since $\lim_{\|x\|\to-\infty} |x^{\alpha}\varphi^{(n)}(x)| = \lim_{x\to-\infty} |x|^{\alpha} \exp(-x) = \infty$, $\forall \alpha \in \mathbb{N}$, although $\lim_{\|x\|\to+\infty} |x^{\alpha}\varphi^{(n)}(x)| = \lim_{x\to+\infty} |x|^{\alpha} \exp(-x) = 0$, $\forall \alpha \in \mathbb{N}_0$.

Also, the functions $\varphi_1(x) = \exp(x)$, $\varphi_2(x) = \exp(-|x|)$, $x \in \mathbb{R}$ do not belong to the space $S(\mathbb{R})$ because the function $\varphi_1(x)$ does not tend to zero when $x \to \infty$, and the function $\varphi_2(x)$ is not differentiable at the origin.

Obviously, the space S is a vector space over Γ , having as null element $\varphi = 0, \forall x \in \mathbb{R}^n$. Between the spaces $\mathcal{D}, S, \mathcal{E}$ there exist the relations $\mathcal{D} \subset S \subset \mathcal{E}$.

Definition 1.19 Let $\varphi \in S$ and consider the sequence $(\varphi_i)_{i \ge 1} \subset S$. We say that the sequence of functions $(\varphi_i)_{i>1}$ converges towards φ and write $\varphi_i \xrightarrow{S} \varphi$ if

$$\forall \alpha, \beta \in \mathbb{N}_0^n, x^{\beta} \mathbb{D}^{\alpha} \varphi_i \xrightarrow{u} x^{\beta} \mathbb{D}^{\alpha} \varphi , \quad x \in \mathbb{R}^n .$$
(1.33)

Consequently, if $\varphi_i \xrightarrow{S} \varphi$, then $\forall \alpha, \beta \in \mathbb{N}_0^n$ on any compact from \mathbb{R}^n we have $x^\beta D^\alpha \varphi_i \xrightarrow{u} x^\beta D^\alpha \varphi$.

Comparing the convergence of the spaces \mathcal{D} and \mathcal{S} , $\mathcal{D} \subset \mathcal{S}$, we can state:

Proposition 1.6 The convergence in space \mathcal{D} is stronger than the convergence in space \mathcal{S} .

Indeed, if $\varphi_i \xrightarrow{\mathcal{D}} \varphi$, then there is $\mathcal{D}(\Omega) \subset \mathcal{D}$ so that $\varphi_i \xrightarrow{\mathcal{D}(\Omega)} \varphi$, hence $x^{\beta} D^{\alpha} \varphi_i$ converges uniformly towards $x^{\beta} D^{\alpha} \varphi$ on any compact from \mathbb{R}^n , that is, $\varphi_i \xrightarrow{S} \varphi$.

Proposition 1.7 The space \mathcal{D} is dense in \mathcal{S} .

This means that $\forall \varphi \in S$ there is $(\varphi_i)_{i>1} \subset \mathcal{D}$ such that $\varphi_i \xrightarrow{S} \varphi$. Also, we can prove that the space \mathcal{D} is dense in \mathcal{E} .

Regarding the multipliers of the space S, we note that not every infinitely differentiable function is a multiplier.

Thus, the function $a(x) = \exp(||x||^2)$ belongs to the class $C^{\infty}(\mathbb{R}^n)$, but it is not a multiplier of the space S, because considering $\varphi(x) = \exp(-\|x\|^2) \in S$, we then have $a(x)\varphi(x) \equiv 1 \notin S$.

We note O_M the functions of class $C^{\infty}(\mathbb{R}^n)$ such that the function and all its derivatives do not increase at infinity faster than a polynomial does, hence if $\psi \in$ O_M , then we have

$$\forall \alpha \in \mathbb{N}_0^n, |\mathsf{D}^{\alpha}\psi| \le c_{\alpha}(1+\|x\|)^{m_{\alpha}}, \tag{1.34}$$

where $c_a > 0$, $m_a \ge 0$ are constants.

It follows that O_M is the space of multipliers for S, because if $\psi \in O_M$ and $\forall \varphi \in S$, then $\psi \varphi \in S$ and $\varphi_i \xrightarrow{S} \varphi$ involve $\psi \varphi_i \xrightarrow{S} \psi \varphi$.

Thus, the functions $f_1(x) = \cos x$, $f_2(x) = \sin x$, P(x) (polynomial in *x*), $x \in \mathbb{R}$, are multipliers for the space $S(\mathbb{R})$.

Consequently, if $\varphi \in S$ then $\forall \alpha, \beta \in \mathbb{N}_{0}^{n}$, $x^{\beta} D^{\alpha} \varphi \in S$ is bounded and integrable on \mathbb{R}^n , hence $\mathcal{S} \subset L^p$, $p \geq 1$.

The spaces of functions with convergence $\mathcal{D}^{m}(\Omega), \mathcal{D}(\Omega), \mathcal{D}, \mathcal{E}$ and \mathcal{S} will be called test function spaces, and the functions of these spaces, test functions.

Let Φ be a test function space, so $\Phi \in \{\mathcal{D}^m(\Omega), \mathcal{D}(\Omega), \mathcal{D}, \mathcal{E}, \mathcal{S}\}$.

We note that the function $h(x) = e^x$, $x \in \mathbb{R}$ is not a multiplier of the space $\mathcal{S}(\mathbb{R})$, because it increases to infinity faster than a polynomial.

1.2.3

Spaces of Distributions

The concept by which one introduces the notion of distribution is the linear functional one. This method, used by Schwartz, has been proved useful, with wide applications in various fields of mathematics, mechanics, physics and technology.

Let (E, Γ) , (Y, Γ) be two vector spaces over the same scalar body Γ and let $X \subset E$ be a subspace of (E, Γ) . We shall call the mapping $T : X \to Y$ operator defined on X with values in Y. The value of the operator T at the point $x \in X$ will be denoted by $(T, x) = T(x) = y \in Y$.

Definition 1.20 The operator $T : X \to Y$ is called linear if and only if

$$T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2), \forall a_1, a_2 \in \Gamma, \forall x_1, x_2 \in X.$$
(1.35)

Thus, if we denote $E = C^n(\Omega)$ and $Y = C^0(\Omega)$, $\Omega \subset \mathbb{R}$, then the application $T: E \to Y$ defined by

$$(T, f) = a_0 D^n f + a_1 D^{n-1} f + \dots + a_{n-1} D f + a_n f , \qquad (1.36)$$

where $f(x) \in E$, $D^k = d^k/dx^k$, $a_k(x) \in C^0(\Omega)$, k = 0, 1, 2, ..., n is a linear operator on *E*.

The operator (1.36) expressed by means of derivatives D^{j} is called linear differential operator with variable coefficients or polynomial differential operator and we also note P(D).

The operator $T : C^0[a, b] \to C^1[a, b]$ defined by

$$(T, f) = \int_{a}^{x} f(t) dt , \quad x \in [a, b], \qquad (1.37)$$

is an integral operator. It is shown that it is an integral operator.

A particular class of operators is formed by functionals. Thus, if the domain *Y* in which the linear operator *T* takes values is Γ , $Y = \Gamma$, then the operator

$$T: X \subset E \to \Gamma \tag{1.38}$$

will be called functional.

The functional *T* will be called real or complex as its value (T, x) at the point $x \in X$ is a real or complex number.

We say that the functional (1.38) is linear if it satisfies the condition of linearity of an operator (1.35).

Definition 1.21 A continuous linear functional defined on a space of test functions $\Phi \in \{\mathcal{D}^m(\Omega), \mathcal{D}, \mathcal{D}, \mathcal{E}, \mathcal{S}\}$ is called distribution.

This definition involves the fulfillment of the following conditions:

- To any function φ ∈ Φ we associate according to some rule *f*, a complex number (*f*, φ) ∈ Γ;
- 2. $\forall \lambda_1, \lambda_2 \in \Gamma, \forall \varphi_1, \varphi_2 \in \Phi, (f, \lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1(f, \varphi_1) + \lambda_2(f, \varphi_2);$
- 3. If $(\varphi_i)_{i\geq 1} \in \Phi$, $\varphi \in \Phi$ and $\varphi_i \xrightarrow{\Phi} \varphi$, then $\lim_i (f, \varphi_i) = (f, \varphi)$.

The first condition expresses the fact that it is a functional, the second condition corresponds to the linearity of the functional, whereas the third condition expresses its continuity.

The set of distributions defined on Φ is denoted by Φ' and can be organized as a vector space over the field of scalars Γ .

For this purpose, we define the sum of two distributions and the product of a distribution with a scalar as follows:

$$\forall f , g \in \Phi', \quad \forall \varphi \in \Phi , (f + g, \varphi) = (f, \varphi) + (g, \varphi), \tag{1.39}$$

$$\forall \alpha \in \Gamma , \forall \varphi \in \Phi , \quad \forall f \in \Phi' , (\alpha f, \varphi) = \alpha(f, \varphi) .$$
(1.40)

It can be verified immediately that the functional $\alpha f + \beta g$ is linear and continuous, hence it is a distribution from Φ' .

Definition 1.22 Let $f \in \Phi'$ and consider the sequence $(f_i)_{i\geq 1} \subset \Phi'$. We say that the sequence $(f_i)_{i\geq 1}$ converges towards the distribution f and we shall write $\lim_{i} f_{i} = f$ if and only if $\forall \varphi \in \Phi$ we have $\lim_{i} (f_{i}, \varphi) = (f, \varphi)$.

This convergence is called weak convergence.

The vector space of distributions Φ' endowed with the structure of weak convergence is called distributions space and will be noted by Φ' .

It can be shown that the space Φ' is a complete space with respect to the weak convergence introduced.

If the sequence of distributions $(f_i)_{i\geq 1} \subset \Phi'$ is such that, for any $\varphi \in \Phi$ the numerical sequence (f_i, φ) has a limit, then there is a single distribution $f \in \Phi'$ for which we have $\lim_{i} (f_i, \varphi) = (f, \varphi)$.

The linearity and continuity properties of a distribution allow us to state:

Proposition 1.8 Let $f \in \Phi'(\mathbb{R}^n)$ be the distribution and $\varphi_a(x) \in \Phi(\mathbb{R}^n)$ the test function, depending on the parameter $a \in I \subset \mathbb{R}$. If $\partial \varphi_a(x) / \partial a$ exists and $\partial \varphi_a(x)/\partial a \in \Phi$, and $(\varphi_{a+h}(x) - \varphi_a(x))/h \xrightarrow{\Phi}{\to}_{h\to 0}(\partial)/(\partial a)\varphi_a(x), \forall a \in I \subset \mathbb{R}$, then the following relation occurs

$$\frac{\mathrm{d}}{\mathrm{d}a}(f(x),\varphi_a(x)) = \left(f(x),\frac{\partial\varphi_a(x)}{\partial a}\right), \quad a \in I \subset \mathbb{R}.$$
(1.41)

As an application of this proposition, we have:

Proposition 1.9 Let $\varphi \in \Phi = \mathcal{D}(\mathbb{R}^n)$ and the distribution $f \in \Phi' = \mathcal{D}'(\mathbb{R}^n)$. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(f(x_1,\ldots,x_n),\varphi\left(\frac{x_1}{a},\ldots,\frac{x_n}{a}\right)\right) = \left(f(x),-\sum_{i=1}^n \frac{x_i}{a^2}\frac{\partial}{\partial x_i}\varphi\left(\frac{x}{a}\right)\right), \quad a > 0.$$
(1.42)

Let $f \in \Phi'$ be the distribution and the function $\psi \in C^{\infty}(\mathbb{R}^n)$, multiplier of the test function space Φ . Then the product ψ *f* is defined by the formula

$$(\psi f, \varphi) = (f, \psi \varphi), \quad \forall \varphi \in \Phi$$
 (1.43)

Obviously, $(\psi f) \in \Phi'$ is a distribution, because ψ being the multiplier for Φ we have $\psi \phi \in \Phi$.

Various spaces of distributions are obtained by customizing the test function space Φ . Thus, the distributions defined on \mathcal{D} are called Schwartz distributions and we note $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$.

If $\Phi = \mathcal{D}^m$ then the distribution \mathcal{D}'^m are called distributions of finite order $\leq m$, and the distributions defined on $\mathcal{D}^0 = C^0_C(\mathbb{R}^n)$ are called measures.

Also, the distributions defined on the test functions space $S = S(\mathbb{R}^n)$ are called tempered distributions.

Because $\mathcal{D} \subset \mathcal{S}$ and the convergence in the space \mathcal{D} is stronger than the convergence in the space S, then between the spaces of distributions S' and D' the relation $\mathcal{S}' \subset \mathcal{D}'$ occurs.

Because $S \subset \mathcal{E}$ and the convergence of a sequence from S implies the convergence in the space \mathcal{E} , it follows that between the spaces \mathcal{E}' and S' there exists the relation $\mathcal{E}' \subset S'$.

Consequently, any distribution from \mathcal{E}' is a distribution with compact support and at the same time it is a tempered distribution.

Thus, the dependence of the spaces of test functions $\mathcal{D}, \mathcal{E}, \mathcal{S}$ is $\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}$ and between the corresponding spaces of distributions occur the inclusions $\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$.

Let $\varphi \in \Phi$ be a complex-valued function test, hence $\varphi(x) \in \mathbb{C}$, $f \in \Phi'$ representing a complex-valued distribution, that is, $(f, \varphi) \in \mathbb{C}$. Then, the product of, complex distribution f and, complex function $a : \mathbb{R}^n \to \mathbb{C}$ is defined by the relation

$$(a f, \varphi) = \overline{a}(f, \varphi) = (f, \overline{a}\varphi) , \qquad (1.44)$$

with the assumption that $\overline{a}\varphi \in \Phi$, where \overline{a} represents the complex conjugate of the function *a*.

We note that to the complex-valued distribution f can be associated a complex conjugate distribution \overline{f} by the relation

$$(\overline{f},\varphi) = \overline{(f,\overline{\varphi})} . \tag{1.45}$$

As well, to each locally integrable complex-valued function $f \in L^1_{loc}(\mathbb{R}^n)$ corresponds a distribution from \mathcal{D}' , $T_f = f \in \mathcal{D}'$, defined by the formula

$$(f,\varphi) = \int_{\mathbb{R}^n} \overline{f(x)}\varphi(x) \mathrm{d}x , \qquad (1.46)$$

where $\varphi \in \mathcal{D}$ represents a complex-valued function test.

An important distribution in mathematical physics is the Dirac delta distribution $\delta_a = \delta(x - a), x, a \in \mathbb{R}^n$, which can be defined on any test function space by the relation

$$(\delta_a, \varphi(x)) = \varphi(a), \forall \varphi \in \Phi .$$
(1.47)

One can easily verify, taking into account the uniform convergence properties, that the functional δ_a defined by (1.47) is a distribution.

We say that the Dirac delta distribution δ_a is concentrated at the point $a \in \mathbb{R}^n$. If the distribution δ_a is defined on the space $\mathcal{D}^0(\mathbb{R}^n) = C^0_C(\mathbb{R}^n)$ of continuous functions with compact support and if $\psi \in C^0(\mathbb{R}^n)$, then the product $\psi(x)\delta(x-a)$ makes sense and we can write

$$\forall \varphi \in \mathcal{D}^0, \quad (\psi(x)\delta(x-a), \varphi(x)) = (\delta(x-a), \varphi(x)\psi(x)) = \psi(a)\varphi(a).$$
(1.48)

Instead, the functional *T* defined on the space of test functions Φ by the formula

$$(T,\varphi(x)) = |\varphi(a)|, \varphi \in \Phi , \qquad (1.49)$$

is not a distribution from Φ' , because although the functional *T* is continuous, it is not linear.

An important class distribution are the distributions of function type or regular distributions, which are generated by locally integrable functions.

We shall show now that to every locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$ corresponds a distribution from $\mathcal{D}'(\mathbb{R}^n)$ denoted by T_f or f, if it does not lead to confusion.

We consider the functional $T_f : \mathcal{D} \to \Gamma$ defined by the formula

$$(T_f,\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \varphi \in \mathcal{D} .$$
(1.50)

The linearity of the functional being obvious, we show its continuity. If the sequence $\varphi_i \xrightarrow{\mathcal{D}} \varphi$, then there is the compact set Ω so that the sequence $\varphi_i \xrightarrow{\mathcal{D}(\Omega)} \varphi$; it results that $\operatorname{supp}(\varphi_i) \subset \Omega$, $\operatorname{supp}(\varphi) \subset \Omega$.

Taking into account (1.50), we get

$$|(T_f,\varphi_i) - (T_f,\varphi)| = |(T_f,\varphi_i - \varphi)| \le \sup_{\Omega} |\varphi_i - \varphi| \int_{\Omega} |f(x)| \mathrm{d}x .$$
(1.51)

Because $\varphi_i \xrightarrow{\mathcal{D}(\Omega)} \varphi$ we have $\lim_i \sup_{\Omega} |\varphi_i - \varphi| = 0$, hence $\lim_i (T_f, \varphi_i) =$ (T_f, φ) , which reflects the continuity of T_f .

Therefore, the functional T_f associated by (1.50) with the locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$ is a distribution on \mathcal{D}' , called a function type distribution or regular distribution.

In general, if $f \in L^1_{loc}$ such that $|f(x)| \leq A ||x||^k$, k > 0, for $||x|| \to \infty$, and if $\varphi \in S$, that is, $|\varphi(x)| \leq B ||x||^{-(k+2)}$, then the functional T_f given by (1.50) defines a regular distribution on the space of test functions S, because $|f(x)\varphi(x)| \leq$ $AB ||x||^{-2}$ and thus the integral (1.50) exists.

Consequently, T_f is a regular tempered distribution, from which $T_f \in S'$.

The distributions which cannot be represented in the integral form (1.50) are called singular distributions.

Such distributions cannot be identified with locally integrable functions. For example, the Dirac delta distribution δ_a defined by (1.47) is a singular distribution.

The function $H : \mathbb{R}^n \to \mathbb{R}$ where

$$H(x) = \begin{cases} 1 & \text{for } x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0 , \\ 0 & \text{otherwise} , \end{cases}$$
(1.52)

is called the Heaviside function and obviously generates a distribution of function type that we denote by *H* and which acts according to the rule

$$(H(x),\varphi(x)) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \varphi(x_1, x_2, \dots, x_n) \mathrm{d}x_1 \dots \mathrm{d}x_n , \quad \varphi \in \mathcal{D} .$$
(1.53)

We remark that this regular distribution can be represented as

$$H(x_1,...,x_n) = H(x_1)...H(x_n)$$
, (1.54)

where $H(x_i)$ represents the Heaviside distribution of one variable, namely

$$H(x_i) = \begin{cases} 0, & x_i < 0 \\ 1, & x_i \ge 0 \end{cases}, \quad x_i \in \mathbb{R} .$$
(1.55)

Proposition 1.10 Let $(f_i)_{i\geq 1} \subset L^1_{loc}(\mathbb{R}^n)$ be a sequence of locally integrable functions, uniformly convergent towards the function $f : \mathbb{R}^n \to \Gamma$ on any compact $\Omega \subset \mathbb{R}^n$; then $f \in L^1_{loc}(\mathbb{R}^n)$ and $T_{f_i} \xrightarrow{\mathcal{D}'} T_f$.

Proof: Since f_i converges uniformly to f on any compact Ω , then $f \in L^1_{loc}$ and $\int_{\Omega} f_i dx \to \int_{\Omega} f dx$. For $\forall \varphi \in \mathcal{D}$ with $supp(\varphi) \subset \Omega$ we have

$$|(T_{f_i}, \varphi) - (T_f, \varphi)| \leq \int_{\Omega} |\varphi| \cdot |f_i - f| dx$$

$$\leq \operatorname{mes}(\Omega) \sup_{\Omega} |\varphi(x)| \cdot \sup_{\Omega} |f_i(x) - f(x)| , \qquad (1.56)$$

where $\operatorname{mes}(\Omega)$ denotes the measure of Ω . Since $\operatorname{mes}(\Omega)$, $\sup_{\Omega} |\varphi(x)|$ are bounded and $\lim_{i} \sup_{\Omega} |f_{i}(x) - f(x)| = 0$, it follows that $\lim_{i} (T_{f_{i}}, \varphi) = (T_{f}, \varphi)$, that is, $T_{f_{i}}$ converges towards T_{f} on \mathcal{D}' .

1.2.3.1 Equality of Two Distributions: Support of a Distribution

Definition 1.23 The distribution $f \in \mathcal{D}'$ is said to be null on the open set $A \subset \mathbb{R}^n$ if $\forall \varphi \in \mathcal{D}$ with supp $(\varphi) \subset A$ we have $(f, \varphi) = 0$; we write $f = 0, x \in A$.

Also, we say that the distributions $f, g \in \mathcal{D}'$ are equal on the open set A, and we write $f = g, x \in A$, if $\forall \varphi \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subset A$ we have $(f - g, \varphi) = 0$. Hence, in particular, f = g on \mathbb{R}^n if the condition

$$(f,\varphi) = (g,\varphi), \forall \varphi \in \mathcal{D}$$
, (1.57)

is satisfied.

Definition 1.24 We call support of the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ and we note supp(f) the complement of the reunion of open sets which nullify the distribution f.

If the support of a distribution is bounded, and since it is closed, then we say that the distribution is with compact support.

Hence, if $x_0 \in \text{supp}(f)$, then the distribution f is not nullified on any open neighborhood of x_0 .

If $x_0 \notin \text{supp}(f)$, then there exists a neighborhood of point x_0 where f = 0.

For example, for the Dirac delta distribution δ_a given by formula (1.47) it follows: supp $(\delta_a) = \{a\}$, so δ_a is a distribution with compact support, the support being formed from a single point $a \in \mathbb{R}^n$ in which we say that the distribution is concentrated.

From the definition of equality of two distributions on an open set it results that $\delta_a = 0$ for $x \neq a$.

Indeed, $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ with the property $a \notin \operatorname{supp}(\varphi)$ we have $(\delta_a, \varphi) = \varphi(a) = 0$, hence the distribution δ_a is zero on the set $A = \mathbb{R}^n - \{a\}$.

The complement of it is $C_{\mathbb{R}^n}A = \{a\}$, that is, $\operatorname{supp}(\delta_a) = \{a\}$. In other words, the distribution δ_a does not vanish on any neighborhood of the point $a \in \mathbb{R}^n$, hence $\delta_a = 0, x \neq a$.

From the physical point of view, the distribution δ_a expresses the density of a material point of mass equal to the unit and placed at the point $a \in \mathbb{R}^n$.

An important property of the distribution $\delta_a = \delta(x-a), x \in \mathbb{R}^n$, called the filter property of the Dirac delta distribution, is given by the relation

$$\psi(x)\delta(x-a) = \psi(a)\delta(x-a), \qquad (1.58)$$

where ψ is a continuous function in the vicinity of the origin.

Indeed, we have

$$\begin{aligned} (\psi(x)\delta(x-a),\varphi(x)) &= (\delta(x-a),\varphi(x)\psi(x)) = \varphi(a)\psi(a) \\ &= \psi(a)(\delta(x-a),\varphi(x)) = (\psi(a)\delta(x-a),\varphi) , \quad \forall \varphi \in \mathcal{D} , \end{aligned}$$
(1.59)

from which follows (1.58).

We will show now that the Dirac delta distribution δ_a is a singular distribution that cannot be identified with a locally integrable function.

Indeed, otherwise there is $f \in L^1_{loc}$ such that

$$\int_{\mathbb{R}^n} f(x)\varphi(x)dx = (\delta(x-a),\varphi(x)) = \varphi(a) , \quad \forall \varphi \in \mathcal{D} .$$
(1.60)

Because $\varphi \in \mathcal{D}$ is arbitrary, in its place we consider the function $||x - a||^2 \psi(x)$, where $\psi \in \mathcal{D}$ is arbitrary; from (1.60) we obtain

$$\int_{\mathbb{R}^n} f(x)\psi(x)\|x-a\|^2 \mathrm{d}x = 0, \forall \psi \in \mathcal{D} .$$
(1.61)

It follows that $||x - a||^2 f(x) = 0$ almost everywhere on \mathbb{R}^n , which implies f(x) = 0 a.e., from which we have $f(x)\varphi(x) = 0$ a.e. But $f(x)\varphi(x) = 0$ a.e. involves $\int_{\mathbb{R}^n} f(x)\varphi(x) dx = 0$, which contradicts relation (1.60).

If T_f is a distribution of function type generated by a continuous function f, then their supports coincide, that is, we have

$$\operatorname{supp}(T_f) = \operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^n, f(x) \neq 0\}} .$$
(1.62)

Regarding the Heaviside function $H(x), x \in \mathbb{R}^n$, defined by (1.52), we have $\operatorname{supp}(f) = \mathbb{R}^n_+ = [0, \infty) \times [0, \infty] \times \cdots \times [0, \infty) = \operatorname{supp}(T_H)$, where $T_H \in \mathcal{D}'(\mathbb{R}^n)$ represents the distribution generated by the Heaviside function, which is a locally integrable function.

Let the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0, x \in \mathbb{R} \setminus \{x_1\}$.

This function is piecewise continuous and its support is

$$supp(f) = \overline{\{x \in \mathbb{R}, f(x) \neq 0\}} = \overline{\{x_1\}} = \{x_1\}.$$
 (1.63)

The support of the distribution function type $T_f \in \mathcal{D}'(\mathbb{R})$, generated by the function f, is the empty set \emptyset , that is, $\operatorname{supp}(T_f) = \emptyset$. Indeed, $\forall \varphi \in \mathcal{D}(\mathbb{R})$ we have $(T_f, \varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx = 0$, hence $T_f = 0$ on \mathbb{R} .

An analogue of the Dirac delta distribution δ_a is the distribution $\delta_s = \delta(S)$, where $S \subset \mathbb{R}^n$ is a piecewise smooth hypersurface.

The functional $\delta_S : \mathcal{D} \to \mathbb{C}$, acting according to the formula

$$(\delta_S, \varphi) = \int_S \varphi(x) \mathrm{d}S, \forall \varphi \in \mathcal{D} , \qquad (1.64)$$

represents the Dirac delta distribution concentrated on the hypersurface *S*, where d*S* is the differential area element on $S \subset \mathbb{R}^n$.

For any $\varphi \in \mathcal{D}$ whose support does not contain points from *S*, the distribution δ_S is null, that is, $\delta_S = 0, x \notin S$. The support of this distribution is the set of all points of *S*.

From the physical point of view, the distribution δ_S expresses a mass density equal to unity, distributed on the hypersurface *S*.

For this reason, the distribution δ_S is called Dirac delta distribution concentrated on $S \subset \mathbb{R}^n$.

If $S = S_1 \cup S_2$, then from (1.63) we obtain

$$\delta_{S_1 \cup S_2} = \delta_{S_1} + \delta_{S_2} \,. \tag{1.65}$$

Indeed, we have

$$(\delta_{S_1 \cup S_2}, \varphi) = \int_{S_1 \cup S_2} \varphi(x) dS = \int_{S_1} \varphi(x) dS_1 + \int_{S_2} \varphi(x) dS_2$$

= $(\delta_{S_1} + \delta_{S_2}, \varphi)$, $\forall \varphi \in \mathcal{D}$, (1.66)

from which, on the basis of the equality of two distributions, we get (1.65).

Obviously, supp $(\delta_S) = S$ because if $x \notin S$ then $\delta_S = 0$. In general, if f is a piecewise continuous function, given on the surface S, we have

$$(f\delta_{S},\varphi) = \int_{S} f(x)\varphi(x)dS, \forall \varphi \in \mathcal{D}(\mathbb{R}^{n}).$$
(1.67)

In addition to the distribution $\delta_S \in \mathcal{D}'(\mathbb{R}^n)$ concentrated on piecewise smooth surface $S \subset \mathbb{R}^n$, the distribution $\delta_{S \times \mathbb{R}} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ associated to the surface $S \subset \mathbb{R}^n$ and to the temporal variable $t \in \mathbb{R}$ is important in mechanics.

This distribution is defined by the formula

$$(\delta_{S \times \mathbb{R}}, \varphi(x, t)) = \int_{\mathbb{R}} dt \int_{S} \varphi(x, t) dS, \forall \varphi \in \mathcal{D}(\mathbb{R}^{n} \times \mathbb{R}), \qquad (1.68)$$

where \int_{S} is the surface integral, and d*S* the differential element of area on $S \subset \mathbb{R}^{n}$.

For the continuous real function $f(x, t) \in C^0(\mathbb{R}^n \times \mathbb{R})$ the distribution f(x, t) $\delta_{S \times \mathbb{R}} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ acts according to the formula

$$(f(x,t)\delta_{S\times\mathbb{R}},\varphi(x,t)) = (\delta_{S\times\mathbb{R}},f(x,t)\varphi(x,t)) = \int_{\mathbb{R}} dt \int_{S} f(x,t)\varphi(x,t)dS .$$
(1.69)

We note that $\delta_{S \times \mathbb{R}} = 0$, for $x \notin S$, $\forall t \in \mathbb{R}$, hence supp $(\delta_{S \times \mathbb{R}}) = S \times \mathbb{R}$.

In general, the local nonintegrable functions cannot be associated with distributions. However, in some cases, by the regularization process, we can correspond local nonintegrable functions with distributions, on which we can apply linear differential operators.

To illustrate this point we will consider the following.

Example 1.7 Let $\lambda \in \mathbb{R}$ and the function $f_{\lambda} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where

$$f_{\lambda}(x) = \frac{\cos \lambda x}{x} . \tag{1.70}$$

We show that the functional $T_{f_{\lambda}} : \mathcal{D} \to \mathbb{C}$ defined by the formula

$$(T_{f_{\lambda}}, \varphi) = \text{p.v.} \int_{-\infty}^{\infty} \frac{\cos \lambda x}{x} \varphi(x) \mathrm{d}x, \varphi \in \mathcal{D}(\mathbb{R})$$
 (1.71)

is a distribution of first order which satisfies the relation

$$\lim_{\lambda \to +\infty} f_{\lambda}(x) = 0 , \qquad (1.72)$$

where the notation p.v. represents the Cauchy principal value. The distribution T_{f_2} will be denoted as p.v. $(\cos \lambda x)/x \in \mathcal{D}'(\mathbb{R})$.

Proof: We note that the function f_{λ} is not integrable in the neighborhood of the origin, hence $f_{\lambda} \notin L^{1}_{loc}(\mathbb{R})$ and the integral (1.71) is considered in the sense of Cauchy principal value; we thus have

p.v.
$$\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x} \varphi(x) dx = \lim_{\epsilon \to +0} \left[\int_{-\infty}^{-\epsilon} \frac{\cos \lambda x}{x} \varphi(x) dx + \int_{\epsilon}^{\infty} \frac{\cos \lambda x}{x} \varphi(x) dx \right].$$
(1.73)

Since $x \to (\cos \lambda x)/x$ is an odd function we obtain

p.v.
$$\int_{-\infty}^{\infty} \frac{\cos \lambda x}{x} dx = 0.$$
(1.74)

Therefore, relation (1.71) can be written as

$$(T_{f_{\lambda}}, \varphi) = \text{p.v.} \int_{-\infty}^{\infty} \cos \lambda x \frac{[\varphi(x) - \varphi(0)]}{x} dx , \quad \varphi \in \mathcal{D}(\mathbb{R}) .$$
 (1.75)

 $T_{f_{\lambda}}$ is obviously a linear functional.

We shall prove its continuity. Applying the mean value formula, we can write

$$\varphi(x) - \varphi(0) = x \varphi'(\xi_x) , \quad \xi_x \in (0, x) , \quad \text{or} \quad \xi_x \in (x, 0) .$$
 (1.76)

Therefore, considering supp(φ) \subset [-a, a], a > 0, from (1.75) we obtain

$$\left| \left(T_{f_{\lambda}}, \varphi \right) \right| = \left| \text{p.v.} \int_{-a}^{a} \cos \lambda x \frac{\left[\varphi(x) - \varphi(0) \right]}{x} \mathrm{d}x \right|$$
$$\leq \text{p.v.} \int_{-a}^{a} \left| \cos \lambda x \right| \left| \frac{\varphi(x) - \varphi(0)}{x} \right| \mathrm{d}x \leq 2a \sup_{x \in [-a,a]} \left| \varphi'(x) \right|. \quad (1.77)$$

Hence, $\forall \varphi \in \mathcal{D}(\mathbb{R})$ with supp $(\varphi) \subset [-a, a]$ and we have

$$\left|\left(T_{f_{\lambda}},\varphi\right)\right| \le c \sup_{x \in [-a,a]} |\varphi'(x)|, c = 2a ; \qquad (1.78)$$

the relation shows that the linear functional $T_{f_{\lambda}}$ defined by (1.71) is a first-order distribution, hence $T_{f_{\lambda}} = f_{\lambda} \in \mathcal{D}'^1(\mathbb{R})$.

The distribution p.v. $(\cos \lambda x)/x$ is a regularization of the function $f_{\lambda} = (\cos \lambda x)/x$.

We note that for $\varphi \in \mathcal{D}(\mathbb{R})$ one can write

$$\left(x \text{ p.v. } \frac{\cos \lambda x}{x}, \varphi\right) = \left(\text{p.v. } \frac{\cos \lambda x}{x}, x\varphi\right)$$
$$= \text{p.v. } \int_{\mathbb{R}} \varphi(x) \cos(\lambda x) dx = \int_{\mathbb{R}} \cos(\lambda x) \varphi(x) dx = (\cos \lambda x, \varphi(x)), \quad (1.79)$$

from which we obtain

$$x \text{ p.v.} \frac{\cos \lambda x}{x} = \cos \lambda x$$
 (1.80)

Hence, for $x \neq 0$ the distribution p.v. $(\cos \lambda x)/x$ coincides with the function $(\cos \lambda x)/x$.

To show that f_{λ} converges to zero on $\mathcal{D}'(\mathbb{R})$ when $\lambda \to +\infty$ we note that we can write

$$\varphi(x) - \varphi(0) = \int_{0}^{x} \varphi'(t) \mathrm{d}t \ . \tag{1.81}$$

Making the change of variable t = x u, relation (1.81) becomes

$$\varphi(x) - \varphi(0) = x \int_{0}^{1} \varphi'(x u) du .$$
(1.82)

We denote

$$\psi(x) = \int_{0}^{1} \varphi'(xu) du , \qquad (1.83)$$

which is a function from $\mathcal{D}(\mathbb{R})$, because $\varphi \in \mathcal{D}(\mathbb{R})$ and $\operatorname{supp}(\psi) \subset [-a, a]$.

Taking into account (1.75) and (1.83), we obtain

$$(T_{f_{\lambda}}, \varphi) = \text{p.v.} \int_{-a}^{a} \psi(x) \cos(\lambda x) dx$$

= $\int_{-a}^{a} \psi(x) \cos(\lambda x) dx$, $\forall \varphi \in \mathcal{D}$, $\text{supp}(\varphi) \subset [-a, a]$. (1.84)

Integrating by parts, we have

$$(T_{f_{\lambda}},\varphi) = \frac{1}{\lambda} \left\{ (\psi(x)\sin(\lambda x))|_{-a}^{a} - \int_{-a}^{a}\sin(\lambda x) \cdot \psi'(x)dx \right\} , \qquad (1.85)$$

from which the inequality $|(T_{f_{\lambda}}, \varphi)| \leq A/|\lambda|$, where A is a positive constant which depends on a > 0; therefore, $\lim_{\lambda \to \infty} (T_{f_{\lambda}}, \varphi) = 0, \forall \varphi \in \mathcal{D}, \operatorname{supp}(\varphi) \subset [-a, a]$, hence $\lim_{\lambda \to \infty} T_{f_{\lambda}} = 0$.

The last relation shows that the family of distributions $T_{f\lambda} = f_{\lambda}$ converges to zero on $\mathcal{D}'(\mathbb{R})$ when $\lambda \to \pm \infty$.

Example 1.8 We consider the function $f(x) = 1/x^2$, $x \in \mathbb{R} \setminus \{0\}$, to which we assign the functional p.v. $1/x^2:\mathcal{D}\rightarrow\mathbb{C}$ defined by the relation

$$\left(p.v. \frac{1}{x^2}, \varphi\right) = p.v. \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{x^2} dx$$
$$= \lim_{\epsilon \to +0} \left[\int_{-\infty}^{-\epsilon} \frac{\varphi(x) - \varphi(0)}{x^2} dx + \int_{\epsilon}^{\infty} \frac{\varphi(x) - \varphi(0)}{x^2} dx \right]. \quad (1.86)$$

Let us show that the functional p.v. $1/x^2$ is a second-order distribution from $\mathcal{D}'(\mathbb{R})$.

Because the linearity of the functional is evident we shall test only its continuity. Thus, taking into account that $\varphi(x) - \varphi(0) = x\varphi'(0) + x^2\varphi''(\xi_x)/2$, $\xi_x \in (0, x)$ or $\xi_x \in (x, 0)$, we have

$$\left(\text{p.v.}\ \frac{1}{x^2},\varphi\right) = \text{p.v.}\ \int_{\mathbb{R}} \left(\frac{\varphi'(0)}{x} + \frac{\varphi''(\xi_x)}{2}\right) \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}} \varphi''(\xi_x) \mathrm{d}x , \qquad (1.87)$$

because p.v. $\int_{\mathbb{R}} (dx/x) = 0.$

Consequently, considering $\operatorname{supp}(\varphi) \subset [-a, a], a > 0$, the previous relation becomes

$$\left| \left(\text{p.v. } \frac{1}{x^2}, \varphi \right) \right| = \frac{1}{2} \left[\int_{-a}^{a} \varphi^{\prime\prime}(\xi_x) \mathrm{d}x \right] \le a \sup_{x \in [-a,a]} |\varphi^{\prime\prime}(x)| , \qquad (1.88)$$

from which results the continuity of the functional p.v. $1/x^2$ and that p.v. $1/x^2$ is a second-order distribution from $\mathcal{D}'(\mathbb{R})$.

For $\forall \varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\left(x^{2} \text{p.v. } \frac{1}{x^{2}}, \varphi\right) = \left(\text{p.v. } \frac{1}{x^{2}}, x^{2}\varphi\right) = \text{p.v. } \int_{\mathbb{R}} \varphi(x) dx = \int_{\mathbb{R}} \varphi(x) dx = (1, \varphi),$$
(1.89)

hence x^2 p.v. $1/x^2 = 1$, which shows that, except at the origin, the distribution p.v. $1/x^2$ coincides with the function $1/x^2$.

We associate to the function $f = 1/x^2$, $x \in \mathbb{R} \setminus \{0\}$, the functional $Pf1/x^2$: $\mathcal{D}(\mathbb{R}) \to \mathbb{C}$, called a pseudofunction, defined by the relation

$$\left(Pf\frac{1}{x^2},\varphi\right) = \lim_{\varepsilon \to +0} \left[\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x^2} dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x^2} dx - 2\frac{\varphi(0)}{\varepsilon}\right].$$
 (1.90)

One can show as in the previous case that $Pf1/x^2$ is a second-order distribution from $\mathcal{D}'(\mathbb{R})$ and that $x^2 Pf1/x^2 = 1$. Also, the distributions PfH(x)/x and PfH(-x)/x are defined by the relations

$$\left[Pf\frac{H(x)}{x},\varphi\right] = \lim_{\varepsilon \to +0} \left[\int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x} dx + \varphi(0)\ln\varepsilon\right],$$
(1.91)

$$\left[Pf\frac{H(-x)}{x},\varphi\right] = \lim_{\varepsilon \to +0} \left[\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx - \varphi(0) \ln \varepsilon\right].$$
(1.92)

One easily verifies the relation p.v. 1/x = Pf H(x)/x + Pf H(-x)/x.

We remark that the concept of finite part of an integral and the concept of pseudofunction were introduced by J. Hadamard.

1.2.4 Characterization Theorems of Distributions

To test whether a functional defined on the test space Φ is a distribution, we check the linearity and continuity of the functional. In general, the linearity of the functional is easy to verify, but we have difficulties verifying the continuity, because it involves the use of the convergence introduced on the test space Φ .

In the following we give a condition equivalent to the continuity of the linear functional defined on Φ , which is particularly useful in applications, which can be considered also as definition of the distribution of the space Φ' [6, 11, 12].

Theorem 1.2 The linear functional $T : \mathcal{D} \to \Gamma$ is a distribution of \mathcal{D}' if and only if for any compact $\Omega \subset \mathbb{R}^n$ there exist the constants $C(\Omega)$ and $m(\Omega) \in \mathbb{N}_0$ such that

$$|(T,\varphi)| \le C(\Omega) \sup_{\substack{|\alpha| \le m(\Omega)\\ x \in \Omega}} |D^{\alpha}\varphi|, \quad \forall \varphi \in \mathcal{D}(\Omega).$$
(1.93)

Theorem 1.3 The linear functional $T : \mathcal{D} \to \Gamma$ is a distribution of \mathcal{D}'^m if and only if, for any compact $\Omega \subset \mathbb{R}^n$, there exists the constant $c(\Omega) > 0$ such that

$$|(T,\varphi)| \le C(\Omega) \sup_{\substack{|\alpha| \le m \\ x \in \Omega}} |\mathcal{D}^{\alpha}|\varphi, \quad |\forall \varphi \in \mathcal{D}(\Omega).$$
(1.94)

Theorem 1.4 The linear functional $T : \mathcal{D} \to \Gamma$ is a measure if and only if any are the compact $\mathcal{Q} \subset \mathbb{R}^n$ there exists the constant $c(\mathcal{Q}) > 0$ such that

$$|(T,\varphi)| \le c(\Omega) \sup_{x \in \Omega} |\varphi(x)| , \quad \forall \varphi \in \mathcal{D}(\Omega) .$$
(1.95)

Below, we give some applications of the characterization theorems of distributions.

Example 1.9 We consider the function $f \in L^1_{loc}(\mathbb{R}^n)$ and the functional $T_f : \mathcal{D} \to \mathbb{C}$ defined by the formula

$$(T_f,\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x , \quad \varphi \in \mathcal{D} .$$
(1.96)

Using Theorem 1.2 of the characterization of distributions of \mathcal{D}' , we shall show that the functional T_f associated to the locally integrable function f is a distribution of \mathcal{D}' , referred to as distribution of function type.

The functional T_f is, obviously, linear and we have

$$|(T_f,\varphi)| \leq \int_{\mathbb{R}^n} |f(x)| |\varphi(x)| \mathrm{d}x \leq \sup_{x \in \mathbb{R}^n} |\varphi(x)| \int_{\mathrm{supp}(\varphi)} |f(x)| \mathrm{d}x .$$
(1.97)

For $\varphi \in \mathcal{D}(\Omega) \subset \mathcal{D}$, the previous relation becomes

$$|(T_f,\varphi)| \le \sup_{x \in \Omega} |\varphi(x)| \int_{\Omega} |f(x)| \mathrm{d}x .$$
(1.98)

Putting $c(\Omega) = \int_{\Omega} |f(x)| dx$, we have

$$|(T_f,\varphi)| \le c(\Omega) \sup_{x \in \Omega} |\varphi(x)| , \quad \forall \varphi \in \mathcal{D}(\Omega) .$$
(1.99)

If $f \neq 0$, and taking into account Theorem 1.4, it follows that T_f is a zero-order distribution and that the constant $c(\Omega) > 0$ depends on the compact $\Omega \in \mathbb{R}^n$. Hence, $T_f = f \in \mathcal{D}'^0$.

For f = 0 we have $c(\Omega) = 0$, and relation (1.99) becomes $|(T_f, \varphi)| \le 0 \le \sup_{x \in \Omega} |\varphi(x)|$, which shows that, in this case, any positive number can be considered as constant $c(\Omega)$.

In conclusion, any locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$ can be identified with a zero-order distribution, hence $T_f = f \in \mathcal{D}'^0$.

In general, let $f \in L^1_{loc}(\Omega)$, $\Omega \in \mathbb{R}^n$ be a compact set, and the linear functional

$$(T_f,\varphi) = \int_{\Omega} f(x) \mathrm{D}^{\alpha} \varphi(x) \mathrm{d}x , \quad \forall \varphi \in \mathcal{D}(\Omega) , \quad \alpha \in \mathbb{N}_0^n , \qquad (1.100)$$

where D^{α} is the operator of derivation.

For $\forall \varphi \in \mathcal{D}(\Omega_1) \subset \mathcal{D}(\Omega), \forall \Omega_1 \text{ compact set and for } \Omega_1 \subset \Omega \text{ from (1.100), we obtain}$

$$|(T_f,\varphi)| \le \sup_{x \in \Omega_1} |\mathsf{D}^{\alpha}\varphi(x)| \cdot \int_{\Omega_1} |f(x)| \mathrm{d}x , \qquad (1.101)$$

hence,

$$|(T_f,\varphi)| \le c \sup_{x \in \Omega_1} |\mathsf{D}^{\alpha}\varphi(x)|, \quad \forall \varphi \in \mathcal{D}(\Omega_1),$$
(1.102)

where $c = \int_{\Omega} |f(x)| dx$.

From relation (1.102) it follows that the linear functional T_f associated with the function $f \in L^1_{loc}(\Omega)$, by formula (1.96), is a distribution of function type of order $k = |\alpha|$.

Example 1.10 Let $\delta_a : C^0_{\mathbb{C}}(\mathbb{R}^n) \to \mathbb{C}$, $a \in \mathbb{R}^n$, defined by the formula

$$(\delta_a, \varphi) = \varphi(a) , \quad \varphi \in C^0_C(\mathbb{R}^n) ,$$
 (1.103)

where $C_{\rm C}^0 = \mathcal{D}^0(\mathbb{R}^n)$ is the continuous function space with compact support. Obviously, $\delta_a = \delta(x - a)$ represents the Dirac delta distribution concentrated at the point $a \in \mathbb{R}^n$. Using Theorem 1.4 we shall show that functional δ_a is a zero-order distribution of \mathcal{D}' .

The linearity of the functional δ_a is evident. To prove the continuity of the functional, from (1.103) we have

$$|(\delta_a, \varphi)| = |\varphi(a)| \le \sup_{x \in \Omega} |\varphi(x)|, \quad \forall \varphi \in C^0_{\mathbb{C}}(\Omega).$$
(1.104)

This relation shows that the linear functional δ_a is a zero-order distribution, because k = 0 and $c(\Omega) = 1$.

Example 1.11 Let the functional δ_s be associated to the piecewise smooth hypersurface $S \subset \mathbb{R}^n$, by the formula

$$(\delta_S, \varphi) = \int_S \varphi(x) \mathrm{d}S , \quad \varphi \in \mathcal{D} .$$
(1.105)

The functional $\delta_{S}: \mathcal{D} \to \Gamma$ is called the Dirac delta distribution concentrated on S.

We will show that δ_{S} is a zero-order distribution. Since the linearity of the functional δ_S results from the surface integral linearity, we shall show the continuity of it.

From (1.105), we obtain

$$|(\delta_{S},\varphi)| \leq \sup_{x \in \Omega} |\varphi(x)| \cdot \int_{S \cap \operatorname{supp}(\varphi)} \mathrm{d}S , \quad \forall \varphi \in \mathcal{D}(\Omega) .$$
(1.106)

Noting $c(\Omega) = \int_{S \cap \text{supp}(\omega)} dS$, the previous relation becomes

$$|(\delta_{S},\varphi)| \leq c \sup_{x \in \Omega} |\varphi(x)| , \quad \forall \varphi \in \mathcal{D}(\Omega) ,$$
(1.107)

which shows that δ_s is a zero-order distribution.

Theorem 1.5 The linear functional $T : S \to \Gamma$ is a distribution of S' if and only if there exist the constant c > 0 and the integers $m, \ell \in \mathbb{N}_0$ such that

$$|(T,\varphi)| \le c \sup_{\substack{|\alpha| \le m \\ x \in \mathbb{R}^n}} |(1+\|x\|^2)^{\varepsilon} D^{\alpha} \varphi(x)|, \quad \forall \varphi \in \mathcal{S}.$$

$$(1.108)$$

Theorem 1.6 The linear functional $T : \mathcal{E} \to \Gamma$ is a distribution of \mathcal{E}' if and only if there exist a compact $\Omega \subset \mathbb{R}^n$ and the constants c > 0, $m \in \mathbb{N}_0$ such that

$$|(T,\varphi)| \le c \sup_{|\alpha| \le m} \sup_{x \in \Omega} |D^{\alpha}\varphi(x)|, \quad \forall \varphi \in \mathcal{E}(\mathbb{R}^n).$$
(1.109)

Example 1.12 Let $\delta_a^{(p)} : S(\mathbb{R}^n) \to \Gamma$ be a functional, $p \in \mathbb{N}$, $a \in \mathbb{R}^n$, defined by the formula

$$\left(\delta_a^{(p)},\varphi\right) = (-1)^p \mathcal{D}^a \varphi(a) , \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n) , \quad |\alpha| = p , \qquad (1.110)$$

where D^{α} is the derivation operator.

The linearity of the functional results from the relation

$$\left(\delta_{a}^{(p)}, \alpha_{1}\varphi_{1} + \alpha_{2}\varphi_{2} \right) = (-1)^{p} D^{\alpha} (\alpha_{1}\varphi_{1} + \alpha_{2}\varphi_{2})$$

= $\alpha_{1}(-1)^{p} D^{\alpha}\varphi_{1} + \alpha_{2}(-1)^{p} D^{\alpha}\varphi_{2} = \alpha_{1} \left(\delta_{a}^{(p)}, \varphi_{1} \right) + \alpha_{2} \left(\delta_{a}^{(p)}, \varphi_{2} \right) ,$ (1.111)

 $\forall \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^n)$, and $\forall \alpha_1, \alpha_2 \in \Gamma$.

Regarding the continuity of the functional $\delta_a^{(p)}$ on $\mathcal{S}(\mathbb{R}^n)$ we have, from (1.110),

$$\left| \left(\delta_a^{(p)}, \varphi \right) \right| = \left| \mathsf{D}^a \varphi(a) \right| \le \sup_{\substack{x \in \Omega \\ |\alpha| = p}} \left| \mathsf{D}^a \varphi(x) \right|, \quad \forall \varphi \in S(\mathbb{R}^n) , \tag{1.112}$$

which, on the basis of Theorem 1.5, show that the linear functional $\delta_a^{(p)}$ is continuous, hence $\delta_a^{(p)} \in \mathcal{E}'$.

A particular class of tempered distributions consists of locally integrable functions with slow growth to infinity, that is, the functions $f \in L^1_{loc}(\mathbb{R}^n)$ that satisfy to infinity the relation $|f(x)| \le c(1 + ||x||)^k$, $c \ge 0$, $k \ge 0$.

In this case we associate the functional $T_f = f$ defined by the formula

$$(f,\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \varphi \in \mathcal{S} , \qquad (1.113)$$

to the function *f*, from which we obtain

$$|(f,\varphi)| \leq \int_{\mathbb{R}^n} |f(x)| |\varphi(x)| \mathrm{d}x \leq \int_{\mathbb{R}^n} c(1+\|x\|)^k |\varphi(x)| \mathrm{d}x \leq A \sup_{x \in \mathbb{R}^n} |\varphi| , \quad A > 0 ,$$
(1.114)

because $(1 + ||x||)^k |\varphi(x)| \in L^1(\mathbb{R}^n)$.

Since the linear functional f defined by (1.113) is bounded, according to Theorem 1.5 of characterization of distributions of S', it means that $T_f = f$ is a distribution of function type of S'.

Also, the absolutely integrable functions, $f \in L^1(\mathbb{R}^n)$, and the functions with polynomial growth $f(x) = x^m = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}, x \in \mathbb{R}^n, m_i \ge 0$, generate temperate distribution of function type on $S'(\mathbb{R}^n)$.

Conversely, the locally integrable function $f(x) = e^x \cos e^x$, $x \in \mathbb{R}$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, although it is not with slow growth (polynomial) generates a temperate distribution of function type, which is defined by the formula

$$(f(x),\varphi(x)) = \int_{\mathbb{R}} e^x \cos e^x \varphi(x) dx , \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) .$$
(1.115)

Indeed, the integral on the right-hand side does exist and we have

$$\begin{aligned} \left| \int_{\mathbb{R}} e^{x} \cos e^{x} \varphi(x) dx \right| &= \left| \int_{\mathbb{R}} \varphi(x) (\sin e^{x})' dx \right| \\ &= \left| \varphi(x) \sin e^{x} \right|_{-\infty}^{\infty} - \int_{\mathbb{R}} \sin e^{x} \varphi'(x) dx \right| = \left| \int_{\mathbb{R}} \varphi'(x) \sin e^{x} dx \right| \\ &\leq \int_{\mathbb{R}} |\varphi'(x)| dx = \int_{\mathbb{R}} (1+x^{2}) |\varphi'| (1+x^{2})^{-1} dx \leq A \sup_{x \in \mathbb{R}} (1+x^{2}) |\varphi'| , \quad (1.116) \end{aligned}$$

where we take into consideration that $\varphi(x) \sin e^x \in \mathcal{S}(\mathbb{R})$, hence $\lim_{|x|\to\infty} \sin e^x \varphi(x) = 0$ and $\exists \int_{\mathbb{R}} |\varphi'(x)| dx < \infty$. According to Theorem 1.5, $f \in \mathcal{S}'(\mathbb{R})$.

1.3 Operations with Distributions

1.3.1 The Change of Variables in Distributions

In geometry, mechanics and mathematical analysis the transformations of independent variables are frequently used [13], so as to simplify the calculations and interpretation of the results.

These changes, applied to functions, lead to new functions.

The methodology of these changes of variables can be extended from functions to distributions.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an application defined by the relation x = h(u), hence

$$x_i = h_i(u_1, \dots, u_n), \quad i = \overline{1, n},$$
 (1.117)

which represents a transformation from Cartesian coordinates $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ to the coordinates $(u_1, u_2, ..., u_n) \in \mathbb{R}^n$.

We see that the functions h_i , $i = \overline{1, n}$, are of class $C^{\infty}(\mathbb{R}^n)$ and that the punctual transformation is bijective. Therefore, the transformation (1.117) allows for the inverse punctual transform $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$, defined by the formula

$$u = h^{-1}(x) \Leftrightarrow u_i = h_i^{-1}(x_1, \dots, x_n).$$
 (1.118)

The Jacobians of the transformations *T* and *T*⁻¹ are $\partial(x)/\partial(u)$ and $\partial(u)/\partial(x)$ for which we have $\partial(x)/\partial(u) = (\partial(u)/\partial(x))^{-1}$, $(\partial(u)/\partial(x)) \neq 0$.

To see how to approach the definition of the change of variables, we shall consider the case of a locally integrable function which can be identified with a distribution of function type.

Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We have

$$(f(h(u)),\varphi(u)) = \int_{\mathbb{R}^n} f(h(u))\varphi(u)du = \int_{\mathbb{R}^n} f(x)\varphi(h^{-1}(x)) \left| \frac{\partial(u)}{\partial(x)} \right| dx . \quad (1.119)$$

This equality can be transcribed as

$$(f(x), \psi(x)) = (f(h(u)), \varphi(u)), \qquad (1.120)$$

where $\psi(x) \in \mathcal{D}(\mathbb{R}^n)$, and has the expression

$$\psi(x) = \varphi(h^{-1}(x)) \left| \frac{\partial(u)}{\partial(x)} \right| .$$
(1.121)

Noting (Tf)(u) = f(h(u)), relation (1.120) becomes

$$(f(x), \psi(x) = ((Tf)(u), \varphi(u))$$
(1.122)

and sets the dependence between the function given in the variable u, namely f(h(u)), and its correspondent (Tf)(x), obtained by means of the punctual transformation (1.117).

Since ψ and φ are functions on *D*, relation (1.122) is adopted for defining the change of variables in the case of distributions.

Definition 1.25 Let $f(x) \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution in the variable $x \in \mathbb{R}^n$. Then, the corresponding distribution in the variable $u \in \mathbb{R}^n$, defined by the transformation (1.117), will be denoted $(Tf)(u) \in \mathcal{D}'(\mathbb{R}^n)$ and is given by the formula

$$(f(x), \psi(x)) = ((Tf)(u), \varphi(u)), \quad \varphi \in \mathcal{D}(\mathbb{R}^n), \quad (1.123)$$

where $\psi(x) \in \mathcal{D}(\mathbb{R}^n)$ and has the expression

$$\psi(x) = \varphi(u(x)) \frac{1}{|\partial(x)/\partial(u)|} .$$
(1.124)

We note that if the punctual transformation is not bijective, hence $\partial(x)/\partial(u) = 0$ at some points, then the change of variable formula (1.123) is inapplicable. Such cases will be analyzed for the transition to spherical coordinates on \mathbb{R}^n and for the transition to cylindrical coordinates on \mathbb{R}^3 .

To illustrate the change of variables for the Dirac delta distribution $\delta_0 = \delta(x_1, \dots, x_n)$ concentrated at the origin. According to formula (1.123), we have

$$(\delta(x), \psi(x)) = (\delta(h(u)), \varphi(u)) = \psi(0) , \quad \varphi \in \mathcal{D} , \qquad (1.125)$$

where

$$\psi(x) = \frac{\varphi(u(x))}{|\partial(h)/\partial(u)|}, \quad \frac{\partial(h)}{\partial(u)} = \frac{\partial(h_1, \dots, h_n)}{\partial(u_1, \dots, u_n)}.$$
(1.126)

From (1.126), we obtain

$$\psi(0) = \varphi(u_0) \left[\left| \frac{\partial(h)}{\partial(u)} \right|_{u=u_0} \right]^{-1} , \qquad (1.127)$$

where $0 = h(u_0)$ and x = h(u), $h \in C^{\infty}(\mathbb{R}^n)$, is the punctual bijective transformation.

Taking into account (1.125) and (1.127), we can write

$$(\delta(x),\psi(x)) = (\delta(h(u)),\varphi(u)) = \left(\frac{\delta(u-u_0)}{|\partial(h)/\partial(u)|_{u=u_0}},\varphi(u)\right), \quad (1.128)$$

from which we obtain the formula

$$(T\delta)(u) = \delta(h(u)) = \delta(h_1(u), \dots, h_n(u)) = \frac{\delta(u - u_0)}{|\partial(h)/\partial(u)|_{u = u_0}} .$$
(1.129)

In particular, for the Dirac delta distribution of a variable, we obtain, from (1.129),

$$\delta(h(u)) = \frac{\delta(u - u_0)}{|h'(u_0)|} , \quad u \in \mathbb{R} .$$
(1.130)

Thus, we have $\delta(e^{au} - 1) = \delta(u)/|a|$, $a \neq 0$, because $x = h(u) = e^{au} - 1$ and $h(u) = 0 \Rightarrow u = 0$, $\partial h/\partial u = ae^{u}$.

We note that formula (1.130) can be generalized to the case in which the equation h(u) = 0 allows a finite or infinite number of simple roots, that is, $\alpha_1, \alpha_2, \ldots, \alpha_p, \ldots$ and $h'(\alpha_i) \neq 0$.

By definition, we write

$$\delta(h(u)) = \sum_{p} \frac{\delta(u - u_{p})}{|h'(u_{p})|} .$$
(1.131)

For example

$$\delta(u^2 - a^2) = \frac{1}{2a} [\delta(u - a) + \delta(u + a)], \quad a > 0, \qquad (1.132)$$

$$\delta(\cos u) = \sum_{n} \delta\left(u - (2n+1)\frac{\pi}{2}\right) . \tag{1.133}$$

Let us look at the case of the punctual transformation $x = \rho \cos \theta$, $\gamma = \rho \sin \theta$, $\rho \ge 0$, $\theta \in [0, 2\pi)$, which expresses the transition from Cartesian coordinates $(x, \gamma) \in \mathbb{R}^2$ to polar coordinates (ρ, θ) , and the Jacobian of the transformation is $J(\rho, \theta) = \partial(x, \gamma)/\partial(\rho, \theta) = \rho$.

In all points $(x, y) \in \mathbb{R}^2$ where $J(\rho, \theta) = \rho \neq 0$, the considered transformation is locally bijective, that is, except at the origin (0, 0) for which $\rho = 0$, and θ is arbitrary. Consequently, the considered punctual transformation is inapplicable to the Dirac delta distribution $\delta(x, y) \in \mathcal{D}'(\mathbb{R}^2)$, which is concentrated (has the support) at the point (0, 0), but it may be applied to the Dirac delta distribution $\delta(x - a, y - b)$ which has the support at the point $(a, b) \neq (0, 0)$.

Passing to polar coordinates in the formulae (1.123) and (1.124), we can write

$$(\delta(x-a, y-b), \psi(x, y)) = \psi(a, b) = (\delta(\rho \cos \theta - a, \rho \sin \theta - b), \varphi(\rho, \theta)),$$
(1.134)

where $\psi(x, y) = \varphi(\rho, \theta) / \rho, \varphi \in \mathcal{D}(\mathbb{R}^2)$.

From here, we obtain $\psi(a, b) = \varphi(\rho_0, \theta_0)/\rho_0\rho_0$, because (ρ_0, θ_0) represents the polar coordinates of the point (a, b), hence $a = \rho_0 \cos \theta_0$, $b = \rho_0 \sin \theta_0$, $\rho_0 = \sqrt{a^2 + b^2}$, $\theta_0 \in [0, 2\pi)$, $\tan \theta_0 = b/a$.

Ultimately, we obtain the relation

$$\psi(a,b) = \frac{\varphi(\rho_0,\theta_0)}{\rho_0} = \left(\frac{\delta(\rho-\rho_0,\theta-\theta_0)}{\rho_0}, \quad \varphi(\rho,\theta)\right)$$
$$= \left(\delta(\rho\cos\theta - a,\rho\sin\theta - b),\varphi(\rho,\theta)\right), \quad (1.135)$$

resulting in the formula

$$\delta(\rho\cos\theta - a, \rho\sin\theta - b) = \frac{\delta(\rho - \rho_0, \theta - \theta_0)}{\rho_0}.$$
(1.136)

Forwards, we shall treat the transition from the Cartesian coordinates $(x, y, z) \in \mathbb{R}^3$ to the spherical coordinates (r, φ, θ) , given by the formulae

$$x = r \sin \varphi \cos \theta$$
, $y = r \sin \varphi \sin \theta$, $z = r \cos \varphi$, (1.137)

where $r \ge 0$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi)$.

The Jacobian of the transformation is $J(r, \varphi, \theta) = \partial(x, y, z)/\partial(r, \varphi, \theta) = r^2 \sin \varphi$ and shows that the punctual transformation is locally bijective everywhere on \mathbb{R}^3 with the exception of the points on the *Oz*-axis, for which $\varphi = 0$ or $\varphi = \pi$. At the origin (0, 0, 0) we can consider r = 0 and φ arbitrary.

Consequently, the transition to spherical coordinates cannot be achieved for the Dirac delta distribution $\delta(x, y, z) \in \mathcal{D}'(\mathbb{R}^3)$ concentrated at the point (0, 0, 0).

For the Dirac delta distribution $\delta(x - a, y - b, z - c)$ concentrated at the point $(a, b, c) \notin Oz$, where $a^2 + b^2 > 0$, we can apply the considered transformation since $J(r, \varphi, \theta) \neq 0$. On the basis of the formulae (1.123) and (1.124), we have

$$\begin{aligned} (\delta(x-a, y-b, z-c), \psi(x, y, z)) \\ &= (\delta(r\sin\varphi\cos\theta - a, r\sin\varphi\sin\theta - b, r\cos\varphi - c), \varphi(r, \varphi, \theta)), \end{aligned}$$
(1.138)

where $\psi(x, y, z) = \varphi(r, \varphi, \theta) / r^2 \sin \varphi, \varphi \in \mathcal{D}(\mathbb{R}^3)$.

If $(r_0, \varphi_0, \theta_0)$ represents the spherical coordinates of the point $(a, b, c) \notin Oz, a^2 + b^2 > 0$, then we have the relations

$$a = r_0 \sin \varphi_0 \cos \theta_0 , \quad b = r_0 \sin \varphi_0 \sin \theta_0 , \quad c = r_0 \cos \varphi_0 , \quad (1.139)$$

$$r_{0} = \sqrt{a^{2} + b^{2} + c^{2}}, \tan \theta_{0} = \frac{b}{a}, \ \theta_{0} \in [0, 2\pi), \ \varphi_{0} = \arccos \frac{c}{r_{0}}, \ \varphi_{0} \in [0, \pi],$$
(1.140)

and thus we obtain

$$\psi(a, b, c) = \frac{\varphi(r_0, \varphi_0, \theta_0)}{r^2 \sin \varphi_0} = \left(\frac{\delta(r - r_0, \varphi - \varphi_0, \theta - \theta_0)}{r_0^2 \sin \varphi_0}, \varphi(r, \varphi, \theta)\right)$$
$$= \left(\delta(r \sin \varphi \cos \theta - a, r \sin \varphi \sin \theta - b, r \cos \varphi - c), \varphi(r, \varphi, \theta)\right).$$
(1.141)

There results the formula

$$\delta(r\sin\varphi\cos\theta - a, r\sin\varphi\sin\theta - b, r\cos\varphi - c) = \frac{\delta(r - r_0, \varphi - \varphi_0, \theta - \theta_0)}{r_0^2\sin\varphi_0}.$$
(1.142)

The punctual transformation (ρ , θ , z) \rightarrow (x, y, z), given by formulae

$$x = \rho \cos \theta , \quad y = \rho \sin \theta , \quad z = z , \quad \rho \ge 0 , \quad \theta \in [0, 2\pi) , \quad z \in \mathbb{R} ,$$
(1.143)

expresses the transition from Cartesian coordinates $(x, y, z) \in \mathbb{R}^3$ to cylindrical coordinates $(\rho, \theta, z) \in \mathbb{R}^3$.

The Jacobian of this transformation is

$$J(\rho, \theta, z) = \frac{\partial(x, y, z)}{\partial(\rho, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -\rho \sin \theta & 0 \\ \sin \theta & \rho \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$
(1.144)

Consequently, the punctual transformation is locally bijective everywhere on \mathbb{R}^3 with the exception of the points on the *Oz*-axis, where $\rho = 0$.

Thus, for the Dirac delta distribution $\delta(x, y, z)$ concentrated at the point (0, 0, 0), we cannot apply the considered punctual transformation, but we may apply it to the Dirac delta distribution $\delta(x - a, y - b, z - c)$, concentrated at the point $(a, b, c) \notin Oz$, hence $a^2 + b^2 > 0$.

We can write

$$\begin{aligned} (\delta(x-a, y-b, z-c), \psi(x, y, z)) \\ &= (\delta(\rho\cos\theta - a, \rho\sin\theta - b, z-c), \varphi(\rho, \theta, z)) , \end{aligned}$$
 (1.145)

where $\psi(x, y, z) = (\varphi(\rho, \theta, z) / J(\rho, \theta, z)) = (\varphi(\rho, \theta, z) / \rho), \varphi \in \mathcal{D}(\mathbb{R}^3).$

Noting with (ρ_0, θ_0, z_0) the cylindrical coordinates of the point $(a, b, c) \notin Oz$, we have $a = \rho_0 \cos \theta_0$, $b = \rho_0 \sin \theta_0$, $c = z_0$, $\rho_0 = \sqrt{a^2 + b^2}$, $\tan \theta_0 = b/a$, $\theta_0 \in [0, 2\pi]$, $z_0 = c \in \mathbb{R}$ and thus we obtain

$$\psi(a, b, c) = \frac{\varphi(\rho_0, \theta_0, z_0)}{\rho_0} = \left(\frac{\delta(\rho - \rho_0, \theta - \theta_0, z - z_0)}{\rho_0}, \varphi(\rho, \theta, z)\right)$$
$$= \left(\delta(\rho \cos \theta - a, \rho \sin \theta - b, z - c), \varphi(\rho, \theta, z)\right).$$
(1.146)

It results in the formula

$$\frac{\delta(\rho - \rho_0, \theta - \theta_0, z - c)}{\rho_0} = \delta(\rho \cos \theta - a, \rho \sin \theta - b, z - c).$$
(1.147)

1.3.2

Translation, Symmetry and Homothety of Distributions

In applications, it is important to consider the punctual linear transformation defined by the equation

$$x = au + b, \quad x, u, b \in \mathbb{R}^n , \tag{1.148}$$

that is,

$$x_i = \sum_{k=1}^n a_{ik} u_k + b_i , \qquad (1.149)$$

where the transformation matrix is

$$a = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \det a \neq 0,$$
(1.150)

while a^{-1} is the inverse matrix.

Because det $a \neq 0$, the linear transformation (1.148) is bijective and therefore the conditions of application of the formula (1.123) are satisfied.

Consequently, if $f(x) \in \mathcal{D}'(\mathbb{R}^n)$, in accordance with (1.123), then we have

$$(f(x), \psi(x)) = (f(au + b), \varphi(u)), \qquad (1.151)$$

$$\psi(x) = \frac{\varphi(u(x))}{|\det a|} = \frac{\varphi(a^{-1} \cdot (x-b))}{|\det a|}, \qquad (1.152)$$

because $\partial(x)/\partial(u) = \det a$.

By customizing the formula (1.151), we obtain the transformations: translation, symmetry, homothety.

Translation Let $a = (\delta_{ij})$ be the diagonal matrix, where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1, 2, \dots, n$$
(1.153)

represents Kronecker's symbol.

In this case, the linear transformation (1.148) takes the particular form x = u + b which represents the translation of the variable u by the vector $b \in \mathbb{R}^n$ and for which det a = 1.

As a consequence, the formula (1.151) becomes

$$(f(x), \psi(x)) = (f(u+b), \varphi(u)), \quad \varphi \in \mathcal{D},$$
(1.154)

where $\psi(x) = \varphi(x - b)$.
We shall note with τ_b the symbol of the operator of translation by the vector $b \in \mathbb{R}^n$. Then $\tau_b f = f(x + b)$, $\tau_b \varphi = \varphi(x + b)$ represent the translation, by the vector *b*, of the distribution *f* and of the test function $\varphi \in \mathcal{D}$, respectively. With the formula (1.154) it can be written

$$(\tau_b f, \varphi) = (f, \tau_{-b} \varphi) , \qquad (1.155)$$

that is,

$$(f(x+b),\varphi(x)) = (f(x),\varphi(x-b)).$$
(1.156)

The equivalent formulae (1.155) and (1.156) express the translation formula of the distribution $f(x) \in \mathcal{D}'(\mathbb{R}^n)$. From (1.156), there results the manner of application of the translation operator τ_b , namely

$$\tau_{\alpha+\beta}f = \tau_{\alpha}(\tau_{\beta}f) = \tau_{\beta}(\tau_{\alpha}f), \qquad (1.157)$$

hence

$$\tau_{a+\beta} = \tau_a(\tau_\beta) . \tag{1.158}$$

For the Dirac delta distribution we have

$$(\delta(x-a),\varphi(x)) = (\delta(x),\varphi(x+a)) = \varphi(a) , \quad \varphi \in \mathcal{D} , \qquad (1.159)$$

hence

$$(\tau_{-a}\delta,\varphi) = (\delta,\tau_a\varphi), \quad \tau_{-a}\delta = \delta(x-a) = \delta_a.$$
 (1.160)

If $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in C^{\infty}(\mathbb{R}^n)$, then we have the relation

$$\tau_b(\psi f) = (\tau_b \psi)(\tau_b f), \quad b \in \mathbb{R}^n.$$
(1.161)

Indeed, we have

$$(\tau_b(\psi f),\varphi) = (f,\psi\tau_{-b}\varphi) = (f,\tau_{-b}(\varphi\tau_b\psi)) = (\tau_b f,\varphi\tau_b\psi), \qquad (1.162)$$

hence

$$(\tau_b(\psi f), \varphi) = (\tau_b f \tau_b \psi, \varphi) , \qquad (1.163)$$

from which results the relation.

By means of translation, we can define the periodic distributions.

Let $f \in \mathcal{D}'(\mathbb{R}^n)$. We say that the distribution f is a periodic distribution if there is $T \in \mathbb{R}^n$, $T \neq 0$, with the property $\tau_T f = f$. The vector $T \in \mathbb{R}^n$ is called the distribution period.

Based on this definition, any periodic distribution $f \in \mathcal{D}'$ satisfies the relation

$$(f(x), \varphi(x)) = (\tau_T f, \varphi) = (f(x), \varphi(x - T)).$$
(1.164)

It is immediately verified that the periodic distributions satisfy the relation

$$\tau_T f = \tau_{-T} f \,. \tag{1.165}$$

Indeed, we have $(\tau_T f, \varphi) = (f, \tau_{-T}(\tau_T \varphi)) = (\tau_{-T} f, \tau_{-T} \varphi) = (f, \tau_{-T} \varphi)$, which gives the required relation.

Symmetry towards the origin of the coordinates The symmetry towards the origin of the function $f : \mathbb{R}^n \to \Gamma$ will be noted f^{ν} and defined by the relation

$$(f^{\nu})(x) = f(-x), \quad x \in \mathbb{R}^n.$$
 (1.166)

From (1.166) we obtain the properties

$$(f^{\nu})^{\nu} = f$$
, $supp(f^{\nu}) = supp(f)$. (1.167)

If $f \in \mathcal{D}'(\mathbb{R}^n)$, then the symmetry of this distribution is given by the relation

$$(f^{\nu},\varphi) = (f,\varphi^{\nu}), \quad \varphi \in \mathcal{D},$$
(1.168)

hence

$$(f(-x),\varphi(x)) = (f(x),\varphi(-x)).$$
(1.169)

This formula is obtained from (1.151) by considering $a = (-\delta_{ij})$ and b = 0, for which det $a = (-1)^n$.

Therefore, the formula (1.151) takes the form $(f(x), \psi(x)) = (f(-u), \varphi(u))$, $\psi(x) = \varphi(-x)$, that is,

$$(f(x),\varphi(-x)) = (f(-x),\varphi(x)), \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$
(1.170)

We say that the distribution f is even if $(f, \varphi^{\nu}) = (f, \varphi)$ and odd if $(f, \varphi^{\nu}) = -(f, \varphi)$.

For example, in the case of the Dirac delta distribution $\delta(x) \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$(\delta(-x),\varphi(x)) = (\delta(x),\varphi(-x)) = \varphi(0) = (\delta(x),\varphi(x)) , \quad \forall \varphi \in \mathcal{D} , \quad (1.171)$$

which leads to the relation $\delta(-x) = \delta(x)$, which shows that the Dirac delta distribution $\delta(x)$ is an even distribution.

If $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in C^{\infty}(\mathbb{R}^n)$, then we have

$$\left(\psi f\right)^{\nu} = \psi^{\nu} \cdot f^{\nu} . \tag{1.172}$$

Indeed, $((\psi f)^{\nu}, \varphi) = (\psi f, \varphi^{\nu}) = (f, \psi \varphi^{\nu}) = (f, (\varphi \psi^{\nu})^{\nu})$, hence

$$((\psi f)^{\nu}, \varphi) = (f^{\nu}, \varphi \psi^{\nu}) = (f^{\nu} \psi^{\nu}, \varphi), \quad \forall \varphi \in \mathcal{D}.$$

$$(1.173)$$

Homothety The transformation through homothety is obtained from the linear transformation (1.148) considering b = 0 and the matrix transformation of the form $a = (a_{ij} \delta_{ij})$.

By specifications, the homothety transformation takes the form

$$x_i = a_{ii}u_i , \quad i = \overline{1, n} , \qquad (1.174)$$

and the determinant of the transformation has the value det $a = a_{11}a_{22} \dots a_{nn}$.

Taking into account (1.174) and the formula (1.151), the homothety transformation of the distribution $f(x) \in \mathcal{D}'(\mathbb{R}^n)$ is given by

$$(f(x), \psi(x)) = (f(au), \varphi(u)), \qquad (1.175)$$

where

$$\psi(x) = \frac{\varphi(a^{-1} \cdot x)}{|\det a|}, \quad \det a = \prod_{i=1}^{n} a_{ii} \neq 0,$$
(1.176)

that is, we have

$$\frac{1}{|\det a|}(f(x),\varphi(a^{-1}\cdot x)) = (f(au),\varphi(u)).$$
(1.177)

In particular, if $a_{ii} = \beta \neq 0$, $i = \overline{1, n}$, then det $a = \beta^n$, and

$$x = au \Leftrightarrow x = \beta u$$
, $a^{-1}x = \frac{1}{\beta}x$; (1.178)

thus, the formula (1.177) becomes

$$(f(\beta x), \varphi(x)) = \frac{1}{|\beta|^n} \left(f(x), \varphi\left(\frac{x}{\beta}\right) \right) .$$
(1.179)

For the Dirac delta distribution $\delta = \delta(x) \in \mathcal{D}(\mathbb{R}^n)$ we obtain

$$(\delta(\beta x),\varphi(x)) = \frac{1}{|\beta|^n} \left(\delta(x),\varphi\left(\frac{x}{\beta}\right)\right) = \frac{\varphi(0)}{|\beta|^n} = \left(\frac{\delta(x)}{|\beta|^n},\varphi(x)\right) , \quad (1.180)$$

leading to the relation

$$\delta(\beta x) = \frac{1}{|\beta|^n} \delta(x) . \tag{1.181}$$

The homothety transformation allows for the introduction of the notion of homogeneous distribution.

Let $f(x) \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution and $\alpha > 0$. We say that the distribution f is homogeneous and of degree $\lambda \in \mathbb{R}$ if it satisfies the relation

$$f(\alpha x) = \alpha^{\lambda} f(x) . \tag{1.182}$$

Substituting this in (1.179), we obtain the formula for characterizing the homogeneous distributions of degree λ , namely

$$(f(x),\varphi(x)) = \alpha^{-n-\lambda} \left(f(x),\varphi\left(\frac{x}{\alpha}\right) \right) , \quad \varphi \in \mathcal{D} .$$
(1.183)

Taking into account (1.181), it results that the Dirac delta distribution $\delta(x) \in$ $\mathcal{D}'(\mathbb{R}^n)$ is a homogeneous distribution of degree -n.

Obviously, the homogeneous and locally integrable functions in the ordinary sense will be particular cases of homogeneous distributions of function type having the degree of homogeneity equal to the locally integrable function.

1.3.3

Differentiation of Distributions

Among the distribution operations, the operation of derivation has a special importance because of its effectiveness. Unlike the functions that do not always allow for derivatives, the distributions have derivatives of any order. Therefore, any locally integrable function considered as a regular distribution and, in particular, the continuous functions will have derivatives of any order in the sense of distributions.

This essentially changes the issues of the series of functions and of the Fourier series; this is because on the space of distributions any convergent series of locally integrable function can be differentiated term by term and the Fourier series are always convergent.

To find the natural way of introducing the concept of derivative, we consider the distribution function $f \in C^1(\mathbb{R})$, which obviously generates a regular distribution $T_f = f \in \mathcal{D}'(\mathbb{R})$.

To the function $f' \in C^0(\mathbb{R})$ also corresponds a regular distribution $T_{f'} = f' \in \mathcal{D}'(\mathbb{R})$, which is defined by

$$(f',\varphi) = \int_{\mathbb{R}} f'(x)\varphi(x)dx , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}) .$$
(1.184)

Integrating by parts, we can write

$$(f',\varphi) = f(x)\varphi(x)|_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)\varphi'(x)dx , \qquad (1.185)$$

but as the function φ has compact support, the first term on the right-hand side is zero and thus we obtain

$$(f',\varphi) = -(f,\varphi'), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$
 (1.186)

Relation (1.186) is adopted for the definition of the first-order derivative of a distribution from $\mathcal{D}'(\mathbb{R})$.

Hence, if $f \in \mathcal{D}'(\mathbb{R})$, then the functional f' given by (1.186) is called a derivative of the distribution *f*.

It is immediately verified that the new functional f', defined on $\mathcal{D}(\mathbb{R})$, is linear and continuous, hence $f' \in \mathcal{D}'(\mathbb{R})$ is a distribution.

Thus, if H is the Heaviside function of one variable, then we have, in the sense of distributions,

$$\frac{\mathrm{d}H(x)}{\mathrm{d}x} = \delta(x) , \qquad (1.187)$$

where $\delta(x) \in \mathcal{D}'(\mathbb{R})$ is the Dirac delta distribution concentrated at the origin. Indeed, since

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0, \end{cases}$$
(1.188)

we have

$$\left(\frac{\mathrm{d}H}{\mathrm{d}x},\varphi\right) = -(H(x),\varphi'(x)) = -\int_{\mathbb{R}} H(x)\varphi'(x)\mathrm{d}x$$
$$= -\int_{0}^{\infty} \varphi'(x)\mathrm{d}x = \varphi(0) = (\delta(x),\varphi(x)), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}), \quad (1.189)$$

resulting in the formula (1.187).

We note that, in the ordinary sense, the function *H* is differentiable everywhere, except the point x = 0, where there is a first type discontinuity.

Noting with d/dx the derivative in the classical sense, we have dH/dx = 0, $x \neq 0$.

We note that, due to the translation operator τ_h , the derivative f' of the distribution $f \in \mathcal{D}'(\mathbb{R})$ is given by the following limit on the space \mathcal{D}'

$$f'(x) = \lim_{h \to 0} \frac{\tau_h f - f}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} .$$
(1.190)

This definition of the derivative of a distribution coincides with the classical one. Indeed, because of the continuity of the functional *f*, we have

$$\left(\frac{f(x+h)-f(x)}{h},\varphi(x)\right) = \left(f(x),\frac{\varphi(x-h)-\varphi(x)}{h}\right) \underset{h\to0}{\to} (f(x),-\varphi'(x))$$
(1.191)

giving

$$\lim_{h \to 0} \frac{\tau_h f - f}{h} = f'(x) . \tag{1.192}$$

As a generalization of the differentiation formula (1.186) of one-variable distributions, we have the following definition.

Definition 1.26 Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and $D^{\alpha} = D^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ the derivative operator of order $|\alpha| = \sum_{i=1}^n \alpha_i$. We call the derivative of order $|\alpha|$ of the distribution *f*, the distribution denoted D^{α} *f* and given by the relation

$$(\mathsf{D}^{\alpha} f, \varphi) = (-1)^{|\alpha|} (f, \mathsf{D}^{\alpha} \varphi) , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^{n}) .$$

$$(1.193)$$

We note that this definition is correct, because the functional given on $D^{\alpha} f$ by the formula (1.193) is linear and continuous, which is easily checked. On the other hand, formula (1.193) contains, as a particular case, formula (1.186).

We emphasize that the distributions' derivative does not depend on the order of derivation, so that there is the relation

$$D^{\alpha+\beta} f = D^{\alpha}(D^{\beta} f) = D^{\beta}(D^{\alpha} f), \quad f \in \mathcal{D}'.$$
(1.194)

Let $H(x, y), (x, y) \in \mathbb{R}^2$, be the Heaviside function on \mathbb{R}^2 , namely:

$$H(x) = \begin{cases} 1 , & x \ge 0, y \ge 0 , \\ 0 , & \text{otherwise} . \end{cases}$$
(1.195)

We have

$$\frac{\partial^2 H(x, y)}{\partial x \partial y} = \delta(x, y), \qquad (1.196)$$

where $\delta(x, y) \in \mathcal{D}'(\mathbb{R}^2)$ is the Dirac delta distribution concentrated at the origin. Indeed, $\forall \varphi \in \mathcal{D}$ we can write

$$\begin{bmatrix} \frac{\partial^2 H}{\partial x \partial y}, \varphi(x, y) \end{bmatrix} = \left(H, \frac{\partial^2 \varphi}{\partial x \partial y} \right) = \iint_{\mathbb{R}^2} H(x, y) \frac{\partial^2 \varphi(x, y)}{\partial x \partial y} dx dy$$
$$= \int_0^\infty \int_0^\infty \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial y} \right) dx dy = \int_0^\infty dy \left(\frac{\partial \varphi}{\partial y} \right) \Big|_0^\infty = -\int_0^\infty \frac{\partial \varphi(0, y)}{\partial y} dy$$
$$= -\varphi(0, y) \Big|_0^\infty = \varphi(0, 0) = (\delta(x, y), \varphi(x, y))$$
(1.197)

giving the formula (1.196).

The general formula can be proved in the same way, that is,

$$\frac{\partial^n H(x_1, x_2, \dots, x_n)}{\partial x_1 \partial x_2 \dots \partial x_n} = \delta(x_1, \dots, x_n) , \qquad (1.198)$$

where $\delta(x_1, ..., x_n) = \delta \in \mathcal{D}'(\mathbb{R}^n)$ is the Dirac distribution and H(x) is the Heaviside function

$$H(x) = \begin{cases} 1, & x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0, \\ 0, & \text{otherwise}. \end{cases}$$
(1.199)

Below we will denote D^{α} the derivation operator in the sense of distributions and \tilde{D}^{α} the derivation operator in the usual sense, where it exists for the regular distributions.

Thus, in the case of the Heaviside function, we have

$$\frac{\partial^n H(x_1,\ldots,x_n)}{\partial x_1\ldots\partial x_n} = 0 , \quad \frac{\partial^n H(x_1,\ldots,x_n)}{\partial x_1\ldots\partial x_n} = \delta(x_1,x_2,\ldots,x_n) , \quad (1.200)$$

also,

~

$$(\mathrm{D}^{\alpha}\,\delta(x-a),\varphi(x)) = (-1)^{|\alpha|}(\mathrm{D}^{\alpha}\,\varphi)(a)\,,\quad\forall\varphi\in\mathcal{D}(\mathbb{R}^{n})\,,\tag{1.201}$$

where $\delta_a = \delta(x - a)$ is the Dirac delta distribution concentrated at the point $a \in \mathbb{R}^n$.

Now, we have the function $a \in C^{\infty}(\mathbb{R}^n)$ and the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$; the derivation formula of a product, known in the classical case, remains valid, that is, we have

$$\frac{\partial}{\partial x_i}(af) = \frac{\partial a}{\partial x_i}f + a\frac{\partial f}{\partial x_i}, \quad i = 1, 2, \dots, n.$$
(1.202)

For distributions defined on \mathcal{D}^m and \mathcal{S} the formula is the same so long as the product between a function and a distribution makes sense.

Example 1.13 Let the function $f(x) = \ln |x|, x \in \mathbb{R} \setminus \{0\}$. This function is locally integrable, so $f \in L^1_{loc}(\mathbb{R})$. To the considered function we assign the linear functional defined by the formula

$$(\ln |x|, \varphi(x)) = \int_{\mathbb{R}} \varphi(x) \ln |x| dx , \quad \varphi \in \mathcal{D}(\mathbb{R}) .$$
(1.203)

We shall show that the functional $T_f = \ln |x|$ is a distribution on $\mathcal{D}'(\mathbb{R})$. We have

$$(\ln |x|, \varphi) = \int_{\mathbb{R}} (x) \ln |x| dx = \lim_{\epsilon \to +0} \left[\int_{-\infty}^{-\epsilon} \varphi(x) \ln |x| dx + \int_{\epsilon}^{\infty} \varphi(x) \ln |x| dx \right]$$
$$= \lim_{\epsilon \to +0} \int_{\epsilon}^{\infty} [\varphi(x) - \varphi(-x)] \ln x dx .$$
(1.204)

Considering supp(φ) \subset [-a, a], a > 0, then we can write

$$|(\ln |x|, \varphi)| \leq \lim_{\varepsilon \to +0} \int_{\varepsilon}^{a} |\varphi(x) - \varphi(-x)| |\ln x| dx .$$
(1.205)

Because

$$\lim_{x \to +0} [\varphi(x) - \varphi(-x)] \ln x = 2 \lim_{x \to +0} \left[\varphi'(\xi_x) x \ln x \right] = 2\varphi'(0) \lim_{x \to +0} (x \ln x) = 0 ,$$
(1.206)

we have

$$|(\ln |x|, \varphi)| \le 2 \sup_{x \in [0,a]} |x \ln x| \sup_{x \in [0,a]} |\varphi'(x)|, \qquad (1.207)$$

hence

$$|(\ln |x|, \varphi)| \le c \sup_{x \in [0,a]} |\varphi'(x)| , \text{ where } c = 2 \sup_{x \in [0,a]} |x \ln x| .$$
 (1.208)

Consequently, the functional T_f is continuous and as it is obviously a linear functional, it results that $T_f = \ln |x|$ is a first-order distribution on $\mathcal{D}'(\mathbb{R})$.

The derivative of this function is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = \mathrm{p.v.}\ \frac{1}{x}\ . \tag{1.209}$$

Indeed, we have

$$\begin{aligned} ((\ln|x|)',\varphi) &= -(\ln|x|,\varphi') = -\int_{\mathbb{R}} \varphi'(x) \ln|x| dx \\ &= -\lim_{\epsilon \to +0} \int_{\epsilon}^{\infty} [\varphi'(x) - \varphi'(-x)] \ln x dx \\ &= \lim_{\epsilon \to 0} [\varphi(\epsilon) - \varphi(-\epsilon)] \ln \epsilon + \lim_{\epsilon \to +0} \int_{\epsilon}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx . \end{aligned}$$
(1.210)

Because $\lim_{\varepsilon \to 0} [\varphi(\varepsilon) - \varphi(-\varepsilon)] \ln \varepsilon = 0$ and

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx = \left(\text{p.v. } \frac{1}{x}, \varphi \right) = \text{p.v. } \int_{\mathbb{R}} \frac{\varphi(x)}{x} dx$$

we can write

$$((\ln |x|)', \varphi) = \left(\text{p.v. } \frac{1}{x}, \varphi\right)$$

from which we get the formula $(\ln |x|)' = p.v. 1/x$.

Example 1.14 Let the distribution p.v. $1/x^2 \in \mathcal{D}'(\mathbb{R})$, defined by the formula

$$\left(\text{p.v.}\ \frac{1}{x^2},\varphi\right) = \text{p.v.}\ \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{x^2} dx = \lim_{\varepsilon \to +0} \int_{|x| \ge \varepsilon} \frac{\varphi(x) - \varphi(0)}{x^2} dx, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$
(1.211)

The relation d/dx p.v. $1/x = -p.v. 1/x^2$, where the distribution p.v. 1/x is defined by

$$\left(\text{p.v.}\;\frac{1}{x},\varphi\right) = \text{p.v.}\;\int_{\mathbb{R}}\frac{\varphi(x)}{x}\mathrm{d}x = \lim_{\varepsilon \to +0}\int_{|x| \ge \varepsilon}\frac{\varphi(x)}{x}\mathrm{d}x\;.$$
(1.212)

Indeed, since $\forall \varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{p.v.}\,\frac{1}{x},\varphi\right) = -\left(\mathrm{p.v.}\,\frac{1}{x},\varphi'\right) = -\mathrm{p.v.}\int_{\mathbb{R}}\frac{\varphi'(x)}{x}\mathrm{d}x = -\lim_{\varepsilon \to +0}\int_{|x| \ge \varepsilon}\frac{\varphi'(x)}{x}\mathrm{d}x \ .$$
(1.213)

Integrating by parts, we obtain

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{p.v.} \frac{1}{x}, \varphi \end{pmatrix} = -\lim_{\varepsilon \to +0} \left[\frac{\varphi(x)}{x} \Big|_{-\infty}^{-\varepsilon} + \frac{\varphi(x)}{x} \Big|_{\varepsilon}^{\infty} + \int_{|x| \ge \varepsilon} \frac{\varphi(x)}{x^2} \mathrm{d}x \right]$$
$$= -\lim_{\varepsilon \to +0} \left[-\frac{\varphi(-\varepsilon)}{\varepsilon} - \frac{\varphi(\varepsilon)}{\varepsilon} - \frac{\varphi(0)}{x} \Big|_{-\infty}^{-\varepsilon} - \frac{\varphi(0)}{x} \Big|_{\varepsilon}^{\infty} + \int_{|x| \ge \varepsilon} \frac{\varphi(x) - \varphi(0)}{x^2} \mathrm{d}x \right].$$
(1.214)

Therefore, we can write

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} \text{ p.v. } \frac{1}{x}, \varphi\right) = \lim_{\epsilon \to +0} \left[\frac{\varphi(\epsilon) - \varphi(0)}{\epsilon} - \frac{\varphi(-\epsilon) - \varphi(0)}{-\epsilon}\right] - \lim_{\epsilon \to +0} \int_{|x| \ge \epsilon} \frac{\varphi(x) - \varphi(0)}{x^2} \mathrm{d}x \;. \tag{1.215}$$

Because $\lim_{\epsilon \to +0} ((\varphi(\epsilon) - \varphi(0))/\epsilon - (\varphi(-\epsilon) - \varphi(0))/(-\epsilon)) = \varphi'(0) - \varphi'(0) = 0$ the previous relation becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{p.v.} \frac{1}{x}, \varphi\right) = -\lim_{\epsilon \to +0} \int_{|x| \ge \epsilon} \frac{\varphi(x) - \varphi(0)}{x^2} \mathrm{d}x = \left(-\mathrm{p.v.} \frac{1}{x^2}, \varphi\right) , \quad (1.216)$$

resulting in d/dx (p.v. 1/x) = -p.v. $1/x^2$.

1.3.3.1 **Properties of the Derivative Operator**

Let D^{α} be the derivative operator in the sense of distributions, and τ_{a} , ν the operator of translation and the symmetry operator, respectively.

The following properties occur:

$$D^{\alpha}(\lambda f + \mu g) = \lambda D^{\alpha} f + \mu D^{\alpha} g, \quad f, g \in \mathcal{D}', \quad \lambda, \mu \in \Gamma , \qquad (1.217)$$

$$f_i \xrightarrow{\mathcal{D}'} f \Rightarrow \mathsf{D}^a f_i \xrightarrow{\mathcal{D}'} \mathsf{D}^a f$$
, $(f_i)_{i \in \mathbb{N}} \subset D'$, $f \in \mathcal{D}'$, (1.218)

$$\operatorname{supp}(\mathsf{D}^{\alpha} f) \subset \operatorname{supp}(f), \quad f \in \mathcal{D}', \tag{1.219}$$

$$D^{\alpha}(\tau_{a}f) = \tau_{a}(D^{\alpha}f), \quad \forall a \in \mathbb{R}^{n}, \quad D^{\alpha}(f^{\nu}) = (-1)^{|\alpha|}(D^{\alpha}f)^{\nu}, \quad f \in \mathcal{D}'.$$
(1.220)

The first relation expresses the linearity of the operator of derivation and its demonstration is easy. Relation (1.218) shows the continuity of the operator and, for its justification, we can write $(D^{\alpha} f_i, \varphi) = (-1)^{|\alpha|} (f_i, D^{\alpha} \varphi).$

On the basis of the completeness theorem of the space \mathcal{D}' , we obtain

$$\lim_{i} (D^{\alpha} f_{i}, \varphi) = (-1)^{|\alpha|} \lim_{i} (f_{i}, D^{\alpha} \varphi) = (-1)^{|\alpha|} (f, D^{\alpha} \varphi) = (D^{\alpha} f, \varphi) , \quad (1.221)$$

hence $\lim_{i} D^{\alpha} f_{i} = D^{\alpha} f$. For the proof of formula (1.219), we allow for f = 0, $x \in A \subset \mathbb{R}^{n}$. Then, for any $\varphi \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subset A$, we have $(D^{\alpha} f, \varphi) = (-1)^{|\alpha|} (f, D^{\alpha} \varphi) = 0$, because $\operatorname{supp}(D^{\alpha} \varphi) \subset A$. From the last equality, it follows $D^{\alpha} f = 0, x \in A$, and considering the complementaries of the sets on which the distributions f and $D^{\alpha} f$ vanished, we get the required relation.

Example 1.15 We have

$$\begin{aligned} f(x) &= H(x) , \quad \operatorname{supp}(f) = [0, \infty] \\ f'(x) &= \delta(x) , \quad \operatorname{supp}(f') = \{0\} \end{aligned} \Rightarrow \begin{aligned} \sup p(f') &= \sup p(\delta) = \{0\} \\ &\subset \operatorname{supp}(f) = [0, \infty] . \end{aligned}$$
 (1.222)

As to formulae (1.220), the first equation shows the commutativity of the derivation operator with respect to the operator of translation τ_a , $a \in \mathbb{R}^n$.

For this, we can write

$$(D^{\alpha}(\tau_{a}f),\varphi) = (-1)^{|\alpha|}(f,\tau_{-a}D^{\alpha}\varphi) = (-1)^{|\alpha|}(f,D^{\alpha}(\tau_{-a}\varphi)), \qquad (1.223)$$

because $D^{\alpha}(\tau_a \varphi) = \tau_a(D^{\alpha} \varphi)$.

The second formula of (1.220) expresses the relation between the derivation operator and the symmetry operator v with respect to the origin of the coordinates.

Because $\forall \varphi \in \mathcal{D}$, we have $D^{\alpha}(\varphi^{\nu}) = (-1)^{|\alpha|} (D^{\alpha} \varphi)^{\nu}$, hence we can write

$$(D^{\alpha}(f^{\nu}),\varphi) = (f, D^{\alpha}(\varphi^{\nu})) = ((-1)^{|\alpha|} (D^{\alpha} f)^{\nu}, \varphi) , \qquad (1.224)$$

which proves the required formula.

Proposition 1.11 Let the function $\psi \in C^{\infty}(\mathbb{R}^n)$, the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, and $\alpha \in \mathbb{N}_0^n$. Thus, the formula occurs

$$D^{\alpha}(\psi f) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} D^{\beta} \psi \cdot D^{\gamma} f = \sum_{\beta \le \alpha} \frac{\alpha!}{\beta! (\alpha - \beta)!} D^{\beta} \psi \cdot D^{\alpha - \beta} f.$$
(1.225)

The formula (1.225) represents the *Leibniz's formula for the derivation of a product in the space of distributions*.

In applications, the point of interest is the calculation of the derivatives of the function type distributions which have discontinuities of the first type, distributed at some points or on certain types of manifolds of \mathbb{R}^{n} .

We can state the following.

Proposition 1.12 Let *f* be a function of the class $C^1(\mathbb{R})$ except at the point x_0 , where it has a discontinuity of the first order with the jump

$$s_0(f) = f(x_0 + 0) - f(x_0 - 0)$$
, (1.226)

where $f(x_0 + 0) = \lim_{x \to x_0+0} f(x)$, $f(x_0 - 0) = \lim_{x \to x_0-0} f(x)$. The formula occurs

$$f' = \tilde{f}' + s_0(f)\delta_{x_0}.$$
(1.227)

Proof: For any $\varphi \in \mathcal{D}(\mathbb{R})$, we can write

$$(f',\varphi) = -(f,\varphi') = -\lim_{\epsilon \to +0} \left[\int_{-\infty}^{x_0-\epsilon} f\varphi' dx + \int_{x_0+\epsilon}^{\infty} f\varphi' dx \right]$$
$$= s_0(f)\varphi(x_0) + (\tilde{f}',\varphi) = (\tilde{f}' + (s_0 f)\delta_{x_0},\varphi) .$$
(1.228)

According to the adopted convention, f' = df/dx is the derivative in the sense of distributions and $\tilde{f}' = \tilde{d} f/dx$ represents the function type distribution corresponding to the derivative *f* in the ordinary sense.

Generalizing the formula (1.227), we obtain the following.

Corollary 1.1 Let $f \in C^1(\mathbb{R})$ except for the points x_i , i = 1, 2, ..., p, where it has discontinuities of the first type with the jumps $s_{x_i}(f) = f(x_i + 0) - f(x_i - 0)$, then

$$f' = \tilde{f}' + \sum_{i=1}^{p} s_{x_i}(f) \delta_{x_i} .$$
(1.229)

Corollary 1.2 Let $f \in C^{\ell}(\mathbb{R})$ except the point x_0 where both the function and its derivatives up to order $\ell - 1$ have discontinuities of the first type with the jumps $s_{x_0}(\tilde{f}^{(i)})$, corresponding to the function $\tilde{f}^{(i)}$, $i = 0, 1, 2, \dots, \ell - 1$. Then, we have the formula

$$f^{(p)} = \tilde{f}^{(p)} + \sum_{i=0}^{p-1} s_{x_0}(\tilde{f}^{(i)}) \delta_{x_0}^{(p-i-1)}, \quad p = 1, 2, \dots, \ell.$$
(1.230)

This last formula is obtained from (1.227) by successive derivation.

Example 1.16 Let the operator be $P(D) = (d^2/dx^2) + (d/dx) - 2$ and the function type distribution

$$f(x) = \begin{cases} -e^{x}/3, & x < 0, \\ -e^{-2x}/3, & x \ge 0. \end{cases}$$
(1.231)

The relation then follows

$$P(D)f = f'' + f' - 2f = \delta(x).$$
(1.232)

We note that f is a continuous function on \mathbb{R} , hence $f \in C^0(\mathbb{R})$ and, in accordance with the formula (1.227), we have

$$f'(x) = \tilde{f}'(x) = \begin{cases} -e^x/3, & x < 0, \\ 2e^{-2x}/3, & x > 0. \end{cases}$$
(1.233)

The distribution $f' = \tilde{f}'$ is of function type which has at the origin a discontinuity of the first order with the jump $s_0(\tilde{f}') = (2/3) - (-1/3) = 1$.

Taking into account that

$$\tilde{f}'' = \frac{\tilde{d}}{dx}f' = \frac{\tilde{d}^2}{dx^2}f = \begin{cases} -e^x/3, & x < 0, \\ -4e^{-2x}/3, & x > 0, \end{cases}$$
(1.234)

and applying (1.227), we obtain $f'' = \tilde{f}'' + \delta(x)$, from which we obtain that P(D) $f = \delta(x)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u : \Omega \times [0, \infty) \to \mathbb{R}$ a real function of the class $C^2(\Omega \times [0, \infty))$. We define the function type distribution $u^* \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, that is,

$$u^*(x,t) = \begin{cases} u(x,t), & (x,t) \in \Omega \times [0,\infty), \\ 0, & \text{otherwise}. \end{cases}$$
(1.235)

This function type distribution can be written in the form

$$u^{*}(x,t) = \chi(x)H(t)u(x,t), \quad (x,t) \in \mathbb{R}^{n} \times \mathbb{R},$$
 (1.236)

where

$$\chi(x) = \begin{cases} 1, & x \in \Omega \subset \mathbb{R}^n, \\ 0, & x \notin \Omega, \end{cases}$$
(1.237)

is the characteristic function corresponding to the domain $\Omega \subset \mathbb{R}^n$, while *H* is the Heaviside function.

We can state the following.

Proposition 1.13 The formulae

$$\frac{\partial u^*}{\partial t} = \frac{\bar{\partial} u^*}{\partial t} + u_0^*(x) \times \delta(t) , \qquad (1.238)$$

$$\frac{\partial^2 u^*}{\partial t^2} = \frac{\tilde{\partial}^2 u^*}{\partial t^2} + \dot{u}_0^*(x) \times \delta(t) + u_0^*(x) \times \delta(t) , \qquad (1.239)$$

exist for the function type distribution $u^* \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, where

$$u_{0}^{*}(x) = u^{*}(x,t)|_{t=+0} = \chi(x)u_{0}(x), \quad \dot{u}_{0}^{*}(x) = \left.\frac{\tilde{\partial}u^{*}(x,t)}{\partial t}\right|_{t=+0} = \chi(x)\dot{u}_{0}(x),$$
(1.240)

and where the conditions $u_0(x) = u(x,t)|_{t=+0} \in C^0(\Omega), \ \dot{u}_0(x) = [\tilde{\partial}u(x,t)/\partial t]|_{t=+0} \in C^0(\Omega)$ are satisfied.

Proof: For any $\varphi(x, t) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ we have

$$\begin{pmatrix} \frac{\partial u^*}{\partial t}, \varphi \end{pmatrix} = -\left(u^*, \frac{\tilde{\partial}\varphi}{\partial t}\right) = -\int_{\mathbb{R}^n \times \mathbb{R}} u^*(x, t) \frac{\tilde{\partial}\varphi}{\partial t} dx dt$$
$$= -\int_{\mathbb{R}^n} dx \int_{\mathbb{R}} u^*(x, t) \frac{\tilde{\partial}\varphi}{\partial t} dt = -\int_{\mathbb{R}^n} \chi(x) dx \int_{\mathbb{R}} H(t) u(x, t) \frac{\tilde{\partial}\varphi}{\partial t} dt .$$
(1.241)

On the other hand, we can write

$$\int_{\mathbb{R}} H(t)u(x,t)\frac{\tilde{\partial}\varphi}{\partial t}dt = \int_{0}^{\infty} u(x,t)\frac{\tilde{\partial}\varphi}{\partial t}dt = u\varphi|_{0}^{\infty} - \int_{0}^{\infty} \frac{\tilde{\partial}u}{\partial t}\varphi(x,t)dt$$
$$= -u(x,0+0)\varphi(x,0) - \int_{0}^{\infty} \frac{\tilde{\partial}u}{\partial t}\varphi(x,t)dt = -u_{0}(x)\varphi(x,0) - \int_{0}^{\infty} \frac{\tilde{\partial}u}{\partial t}\varphi(x,t)dt .$$
(1.242)

Thus, the previous relation becomes

$$\left(\frac{\partial u^*}{\partial t},\varphi\right) = \int_{\mathbb{R}^n} \chi(x)u_0(x)\varphi(x,0)dx + \int_{\mathbb{R}^n\times\mathbb{R}} \chi(x)H(t)\frac{\tilde{\partial}u}{\partial t}dxdt .$$
(1.243)

But we have

$$\int_{\mathbb{R}^n} \chi(x) u_0(x) \varphi(x, 0) \mathrm{d}x = \left((\chi(x) u_0(x)) \times \delta(t), \varphi(x, t) \right), \qquad (1.244)$$

because

$$\begin{aligned} ((\chi(x)u_0(x)) \times \delta(t), \varphi(x, t)) &= (u_0(x)\chi(x), (\delta(t), \varphi(x, t))) \\ &= (u_0(x)\chi(x), \varphi(x, 0)) = \int_{\mathbb{R}^n} u_0(x)\chi(x)\varphi(x, 0) \mathrm{d}x \;. \end{aligned}$$

$$(1.245)$$

Consequently, we obtain

$$\begin{pmatrix} \frac{\partial u^*}{\partial t}, \varphi \end{pmatrix} = ((\chi(x)u_0(x)) \times \delta(t), \varphi(x, t)) + \left(\chi(x)H(t)\frac{\tilde{\partial}u}{\partial t}, \varphi\right)$$
$$= \left(u_0^*(x) \times \delta(t) + \frac{\tilde{\partial}u^*(x, t)}{\partial t}, \varphi\right),$$
(1.246)

from which results the first formula of the sentence, because $\chi(x)H(t)(\tilde{\partial}u/\partial t) = (\tilde{\partial}u^*/\partial t)$ and $u_0(x)\chi(x) = u_0^*(x)$.

To get the second relation, we can write

$$\frac{\partial^2 u^*}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\tilde{\partial} u^*}{\partial t} + u_0^*(x) \times \delta(t) \right] = \frac{\partial}{\partial t} \left[\frac{\tilde{\partial} u^*}{\partial t} \right] + u_0^*(x) \times \delta'(t)$$
$$= \frac{\tilde{\partial}}{\partial t} \left(\frac{\tilde{\partial} u^*}{\partial t} \right) + \frac{\tilde{\partial}}{\partial t u^*} \bigg|_{t=+0} \times \delta(t) + u_0^*(x) \times \delta'(t) , \qquad (1.247)$$

hence

$$\frac{\partial^2}{u^*}\partial t^2 = \frac{\tilde{\partial}^2}{u^*}\partial t^2 + \dot{u}_0^*(x) \times \delta(t) + u_0^*(x) \times \delta'(t) , \qquad (1.248)$$

and thus the proposition is demonstrated.

We note that this proposition generalizes the formula (1.227) and has important applications in mechanics [14–18], because the variable may be interpreted as a time variable.

Remark 1.1 For the functions $u^*(x, t) = \chi(x)H(t)u(x, t)$ and

$$\frac{\tilde{\partial}u^*}{\partial t} = \chi(x)H(t)\frac{\tilde{\partial}u(x,t)}{\partial t}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} ,$$

the hyperplane $t = 0, (x, t) \in \mathbb{R}^n \times \mathbb{R}$ of $\mathbb{R}^n \times \mathbb{R}$ represents a discontinuity hyperplane. When crossing the hyperplane in the direction of the increasing of the variable $t \in \mathbb{R}$, the jumps of the two functions are:

$$u_0^*(x) = u^*(x, 0+0) - u^*(x, 0-0) = u_0^*(x, 0+0) = \chi(x)u_0(x) , \quad (1.249)$$

$$\dot{u}_{0}^{*}(x) = \frac{\tilde{\partial}u^{*}}{\partial t}(x, 0+0) - \frac{\tilde{\partial}u^{*}}{\partial t}(x, 0-0) = \frac{\tilde{\partial}u^{*}}{\partial t}(x, 0+0) = \chi(x)\dot{u}_{0}(x) ,$$
(1.250)

where $x \in \mathbb{R}^n$.

Particularly, if $\Omega = \mathbb{R}^n$, the obtained formulae are simplified, because $\chi(x) = 1$; we have $u_0^*(x) = u_0(x)$ and $\dot{u}_0^*(x) = \dot{u}_0(x)$, where $u_0, \dot{u}_0 \in C^0(\mathbb{R}^n)$.

In connection with the distributions supports, we have the following properties:

1.
$$\operatorname{supp}(\alpha f + \beta g) \subset \operatorname{supp}(f) \cup \operatorname{supp}(g), \quad f, g \in \mathcal{D}'(\mathbb{R}^n), \quad \alpha, \beta \in \mathbb{R};$$

(1.251)

2.
$$\operatorname{supp}\left(\sum_{i=1}^{m} \alpha_{i} f_{i}\right) \subset \bigcup_{i=1}^{m} \operatorname{supp}(f_{i}), \quad f_{i} \in \mathcal{D}'(\mathbb{R}^{n}), \quad \alpha_{i} \in \mathbb{R}; \quad (1.252)$$

3.
$$\operatorname{supp}(D^{\alpha} f) \subset \operatorname{supp}(f);$$
 (1.253)

4.
$$\operatorname{supp}(a(x) \mathbb{D}^{\alpha} f) \subset \operatorname{supp}(f), \quad a(x) \in C^{\infty}(\mathbb{R}^{n});$$
 (1.254)

5. Let

$$P(x, \mathrm{D}) = \sum_{|\alpha|=0}^{m} a_{\alpha}(x) \mathrm{D}^{\alpha}, \quad a_{\alpha}(x) \in C^{\infty}(\mathbb{R}^{n})$$

a linear differential operator with variable coefficients. Then, we have

$$\operatorname{supp}(P(x, \mathbb{D})) \subset \operatorname{supp}(f)$$
. (1.255)

Proposition 1.14 Let *f* be a function of the class C^1 on \mathbb{R}^n , except for a piecewise smooth hypersurface *S*, where it has a first discontinuity; we have

$$\frac{\partial f}{\partial x_i} = \frac{\tilde{\partial} f}{\partial x_i} + \sigma_i \cos \alpha_i \delta_S , \qquad (1.256)$$

where δ_s is the Dirac delta distribution concentrated on the hypersurface *S*, σ_i is the jump function across the hypersurface in the positive direction of the Ox_i axis, and α_i is the angle between the Ox_i -axis and the normal to the hypersurface oriented in the direction of its crossing.

To establish the formula, we see that f is locally integrable and thus we have

$$\begin{pmatrix} \frac{\partial f}{\partial x_i}, \varphi \end{pmatrix} = -\left(f, \frac{\partial \varphi}{\partial x_i}\right) = -\int_{\mathbb{R}^n} f(x) \frac{\partial \varphi}{\partial x_i} dx$$
$$= (-1)^i \int_{\mathbb{R}^{n-1}} dx_1 \dots dx_{i-1} dx_{n+1} \dots dx_n \int_{-\infty}^{\infty} f(x) \frac{\partial \varphi}{\partial x_i} dx_i . \quad (1.257)$$

But, we can write

$$\int_{-\infty}^{\infty} f(x) \frac{\partial \varphi}{\partial x_i} dx_i = \lim_{\epsilon \to +0} \left[\int_{-\infty}^{x_i^* - \epsilon} f(x) \frac{\partial \varphi}{\partial x_i} dx_i + \int_{x_i^* + \epsilon}^{\infty} f(x) \frac{\partial \varphi}{\partial x_i} dx_i \right]$$
$$= \lim_{\epsilon \to +0} \left[(f(x)\varphi(x))|_{-\infty}^{x_i^* - \epsilon} + (f(x)\varphi(x))|_{x_i^* + \epsilon}^{\infty} - \int_{-\infty}^{x_i^* - \epsilon} \varphi(x) \frac{\partial}{\partial x_i} f(x) dx_i - \int_{x_i^* + \epsilon}^{\infty} \varphi(x) \frac{\partial}{\partial x_i} f(x) dx_i \right] = -\sigma_i \varphi(x^*) - \int_{-\infty}^{\infty} \varphi(x) \frac{\partial}{\partial x_i} dx_i , \quad (1.258)$$

where $\sigma_i = f(x_1^*, ..., x_{i-1}^*, x_i^* + 0, x_{i+1}^*, ..., x_n^*) - f(x_1^*, ..., x_{i-1}^*, x_i^* - 0, x_{i+1}^*, ..., x_n^*)$ is the jump of the function *f* at the point $x^* \in S$, when crossing the hypersurface *S* in the positive direction of the Ox_i -axis.

Thus, we get

$$\left(\frac{\partial f}{\partial x_i},\varphi\right) = \int\limits_{\mathbb{R}^n} \frac{\tilde{\partial} f(x)}{\partial x_i} \varphi(x) \mathrm{d}x + (-1)^{i-1} \int\limits_{S} \sigma_i \varphi(x) \mathrm{d}x_1 \dots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \dots \mathrm{d}x_n \,.$$
(1.259)

If we note with α_i the angle formed by the Ox_i -axis with the normal to the hypersurface directed for increasing x_i , then

$$(-1)^{i-1}\int\limits_{S}\sigma_{i}\varphi(x)\mathrm{d}x_{1}\ldots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\ldots\mathrm{d}x_{n}=\int\limits_{S}\sigma_{i}\varphi(x)\cos\alpha_{i}\mathrm{d}S,\quad(1.260)$$

where dS is the area element.

Taking into account the definition of the Dirac delta distribution concentrated on the hypersurface *S*, the previous formula becomes

$$\int_{S} \sigma_{i}\varphi(x)\cos\alpha_{i}\mathrm{d}S = (\sigma_{i}\cos\alpha_{i}\delta_{S},\varphi) .$$
(1.261)

Substituting this in (1.259), we obtain

$$\left(\frac{\partial f}{\partial x_i},\varphi\right) = \left(\frac{\tilde{\partial} f}{\partial x_i},\varphi\right) + \left(\sigma_i \cos \alpha_i \delta_S,\varphi\right), \qquad (1.262)$$

from which we obtain formula (1.256).

We notice that this formula is a generalization of the formula (1.227) established for the case n = 1.

In particular, for the real function f of class $C^1(\mathbb{R}^3)$, with the exception of the piecewise smooth surface $S \subset \mathbb{R}^3$, where it has discontinuities of the first order (Figure 1.2), we have the formula

$$\frac{\partial f(x, y, z)}{\partial z} = \frac{\tilde{\partial} f(x, y, z)}{\partial z} + \sigma_3 \cos \alpha_3 \delta_s , \qquad (1.263)$$

where α_3 is the angle between *n* and *k*, $\sigma_3 = f(x, y, z + 0) - f(x, y, z - 0)$ and

$$(\sigma_3 \cos \alpha_3 \delta_S, \varphi) = \int_S \sigma_3 \cos \alpha_3 \varphi(x, y, z) \mathrm{d}S , \quad \varphi \in \mathcal{D}(\mathbb{R}^3) . \tag{1.264}$$

Regarding formula (1.256), an important case in mechanics is when the hypersurface $S \subset \mathbb{R}^n$ is a cylindrical hypersurface.

Let $\Gamma \subset \mathbb{R}^2$ be a piecewise smooth curve in the $Ox\gamma$ -plane and let us denote by $S = \Gamma \times \mathbb{R} \subset \mathbb{R}^3$ the cylindrical surface with generators parallel to Oz with respect to the orthogonal reference system $Ox\gamma z$ (Figure 1.3).

The curve Γ of the $Ox \gamma$ -plane is the director curve of the cylindrical surface $S = \Gamma \times \mathbb{R} \subset \mathbb{R}^3$ and the normal unit vector \mathbf{n} at the point $M(x, \gamma, z) \in S$ is equal to the normal unit vector \mathbf{n}^* at the point $P(x, \gamma) \in \Gamma$. Hence, between the differential element of area d*S* of the cylindrical surface $S = \Gamma \times \mathbb{R}$ and the differential element d*s* of the curve arc Γ leads to the relation dS = dzds.



Figure 1.2



Figure 1.3

Proposition 1.15 Let the function $f : \mathbb{R}^3 \to \mathbb{R}$ of the class $C^1(\mathbb{R}^3)$, except for cylindrical surface $S = \Gamma \times \mathbb{R} \subset \mathbb{R}^3$, where it has a first-order discontinuity. Then, the following formula results

$$\frac{\partial f(x, y, z)}{\partial y} = \frac{\partial f(x, y, z)}{\partial y} + \sigma_y \cos(\mathbf{n}, y) \delta_s , \qquad (1.265)$$

where $\partial/\partial \gamma$, $\tilde{\partial}/\partial \gamma$ are the derivatives in the sense of distributions and in the ordinary sense, respectively, δ_S is the Dirac delta distribution concentrated on the cylindrical surface $S = \Gamma \times \mathbb{R}$, $\sigma_{\gamma} = f(x, \gamma + 0, z) - f(x, \gamma - 0, z) = \lim_{\epsilon \to +0} f(x, \gamma + \epsilon, z) - \lim_{\epsilon \to +0} f(x, \gamma - \epsilon, z)$, $(x, \gamma, z) \in S$, is the jump of the function f at the crossing of the cylindrical surface in the positive direction of the *O* γ -axis, and $\cos(n, \gamma)$ is the cosine of the angle between the *O* γ -axis with the normal to the surface *S*, oriented in the direction of its crossing.

The Dirac delta distribution $\delta_S = \delta_{\Gamma \times \mathbb{R}} \in \mathcal{D}'(\mathbb{R}^3)$ acts according to the formula

$$(\delta_{S},\varphi) = \int_{S} \varphi(x,y,z) dS = \int_{\mathbb{R}} dz \int_{\Gamma} \varphi(x,y,z) ds , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^{3}) . \quad (1.266)$$

Hence, $\forall \varphi \in \mathcal{D}(\mathbb{R}^3)$ we have

$$(\sigma_{\gamma}\cos(n,\gamma)\delta_{S},\varphi) = (\delta_{S},\cos(n,\gamma)\varphi) = \int_{\mathbb{R}} dz \int_{\Gamma} \cos(n,\gamma)\varphi(x,\gamma,z)ds . \quad (1.267)$$

Proof: For any $\varphi(x, y, z) \in \mathcal{D}(\mathbb{R}^3)$ we get

$$\left(\frac{\partial f}{\partial \gamma},\varphi\right) = -\left(f,\frac{\tilde{\partial}\varphi}{\partial \gamma}\right) = -\int_{\mathbb{R}^3} f\varphi'_{\gamma} dx dy dz = -\int_{\mathbb{R}} dz \int_{\mathbb{R}^2} f(x,y,z)\varphi'_{\gamma} dx dy .$$
(1.268)

On the other hand, we can write

$$\int_{\mathbb{R}^2} f\varphi'_{\gamma} dx dy = \int_{\mathbb{R}} dx \int_{\mathbb{R}} f\varphi'_{\gamma} dy = \int_{\mathbb{R}} dx \left\{ \lim_{\epsilon \to +0} \left[\int_{-\infty}^{\gamma'-\epsilon} f\varphi'_{\gamma} dy + \int_{\gamma'+\epsilon}^{\infty} f\varphi'_{\gamma} dy \right] \right\}.$$
(1.269)

There results

$$\lim_{\varepsilon \to +0} \left[\int_{-\infty}^{\gamma'-\varepsilon} f\varphi'_{\gamma} dy + \int_{\gamma'+\varepsilon}^{\infty} f\varphi'_{\gamma} dy \right] = \lim_{\varepsilon \to +0} \left[f(x, y, z)\varphi(x, y, z) \Big|_{-\infty}^{\gamma'-\varepsilon} + f(x, y, z)\varphi(x, y, z) \Big|_{-\infty}^{\gamma'-\varepsilon} - \int_{-\infty}^{\gamma'-\varepsilon} \varphi \frac{\partial}{\partial y} dy - \int_{\gamma'+\varepsilon}^{\infty} \varphi \frac{\partial}{\partial y} dy \right]$$
$$= -\varphi(x, \gamma', z) [f(x, \gamma'+0, z) - f(x, \gamma'-0, z)] - \int_{\mathbb{R}} \varphi(x, y, z) \frac{\partial}{\partial y} dy$$
$$= -\sigma_{\gamma}\varphi(x, \gamma', z) - \int_{\mathbb{R}} \varphi(x, y, z) \frac{\partial}{\partial \gamma} dy, \qquad (1.270)$$

where

$$\sigma_{\gamma} = \sigma_{\gamma}(x, \gamma', z) = f(x, \gamma' + 0, z) - f(x, \gamma' - 0, z)$$

=
$$\lim_{\varepsilon \to +0} f(x, \gamma' + \varepsilon, z) - \lim_{\varepsilon \to +0} f(x, \gamma' - \varepsilon, z) , \qquad (1.271)$$

represents the jump of the function *f* at the crossing of the cylindrical surface $S = \Gamma \times \mathbb{R} \subset \mathbb{R}^3$ in the positive direction of the *Oy*-axis. Thus, we get

$$\int_{\mathbb{R}^2} f\varphi'_{\gamma} dx dy = -\int_{\mathbb{R}^2} \varphi \frac{\tilde{\partial} f}{\partial \gamma} dx dy - \int_{\mathbb{R}} \sigma_{\gamma} \varphi(x, \gamma', z) dx , \qquad (1.272)$$

and the following relation occurs

$$\left(\frac{\partial f}{\partial \gamma},\varphi\right) = \int_{\mathbb{R}} \mathrm{d}z \int_{\mathbb{R}} \sigma_{\gamma}\varphi(x,\gamma',z)\mathrm{d}x + \int_{\mathbb{R}^3} \varphi(x,\gamma,z)\frac{\tilde{\partial}f}{\partial \gamma}\mathrm{d}x\mathrm{d}\gamma\mathrm{d}z \,. \quad (1.273)$$

If we denote with ds the element of arc on the curve Γ , then d $x = ds \cos(n, \gamma)$, where (n, γ) is the angle between the *O* γ -axis and the normal to the curve Γ at the point (x, γ') .

Consequently, we have

$$\int_{\mathbb{R}} dz \int_{\mathbb{R}} \sigma_{\gamma} \varphi(x, \gamma', z) dx = \int_{\mathbb{R}} dz \int_{\Gamma} \sigma_{\gamma} \varphi(x, \gamma', z) \cos(n, \gamma) ds , \qquad (1.274)$$

which allows us to define the Dirac delta distribution $\delta_{\Gamma \times \mathbb{R}} \in \mathcal{D}'(\mathbb{R}^3)$, concentrated on the cylindrical surface $S = \Gamma \times \mathbb{R} \subset \mathbb{R}^3$, by the formula

$$(\delta_{\Gamma \times \mathbb{R}}, \varphi(x, y, z)) = \int_{\mathbb{R}} dz \int_{\Gamma} \varphi(x, y, z) ds , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3) .$$
(1.275)

Consequently, we can write

$$(\sigma_{\gamma}\cos(n, \gamma)\delta_{\Gamma \times \mathbb{R}}, \varphi) = \int_{S} \sigma_{\gamma}\cos(n, \gamma)\varphi(x, \gamma, z)dS$$
$$= \int_{\mathbb{R}} dz \int_{\Gamma} \sigma_{\gamma}\cos(n, \gamma)\varphi(x, \gamma, z)ds. \qquad (1.276)$$

With these results, one gets

$$\left(\frac{\partial f}{\partial \gamma},\varphi\right) = \left(\frac{\tilde{\partial}f}{\partial \gamma} + \sigma_{\gamma}\cos(n,\gamma)\delta_{S},\varphi\right), \qquad (1.277)$$

giving the requested formula.

The obtained derivation formula can be generalized as

$$\frac{\partial f(x,z)}{\partial x_i} = \frac{\tilde{\partial} f(x,z)}{\partial x_i} + \sigma_{x_i} \cos(n, x_i) \delta_{\Sigma} , \qquad (1.278)$$

where $\Sigma = S \times \mathbb{R} \subset \mathbb{R}^{n+1}$ is the cylindrical surface with generators parallel to the *Oz*-axis with respect to the orthogonal reference system $Ox_1x_2...x_nz$ and $S \subset \mathbb{R}^n$ is a piecewise smooth surface.

The function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is considered to be of class $C^1(\mathbb{R}^{n+1})$ except for the cylindrical surface $\Sigma = S \times \mathbb{R}$, where it has a first-order discontinuity.

Obviously, σ_i is the jump of the function f at the crossing of the cylindrical surface Σ in the positive direction of the Ox_i -axis and (n, x_i) the angle between the Ox_i -axis and the normal to the surface Σ , oriented in the direction of its crossing.

The Dirac delta distribution $\delta_{\Sigma} = \delta_{S \times \mathbb{R}}$, concentrated on the cylindrical surface Σ , acts according to the formula

$$(\delta_{\Sigma},\varphi) = \int_{\Sigma} \varphi dS_{\Sigma} = \int_{\mathbb{R}} dz \int_{S} \varphi dS , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^{n+1}) .$$
(1.279)

Example 1.17 Let the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = -\frac{a}{2}H(y)H(ay - |x|) = \begin{cases} -a/2 & \text{for } |x| \le ay, y > 0, \quad a > 0, \\ 0 & \text{otherwise}. \end{cases}$$
(1.280)

We shall demonstrate that the function type distribution $T_f = f \in D'(\mathbb{R}^2)$ is the fundamental solution of the operator

$$P(\mathsf{D}) = \frac{\partial^2}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2}{\partial \gamma^2} , \quad a > 0 , \qquad (1.281)$$

that is, $P(D) f(x, y) = \delta(x, y)$, by using formula (1.256).

We observe that the function f has the value -a/2 inside the cone Γ^+ (Figure 1.4) and is zero outside it. The frontier of the cone Γ^+ is the curve Γ which consists of the branches Γ_1 and Γ_2 , $\Gamma = \Gamma_1 \cup \Gamma_2$, defined by the parametric equations

$$\Gamma_1 : x = at, \quad y = -t, \quad t \in (-\infty, 0],
 \Gamma_2 : x = at, \quad y = t, \quad t \in [0, \infty).$$
(1.282)

The curve $\Gamma = \Gamma_1 \cup \Gamma_2$ represents the discontinuity curve at the crossing of which the function *f* has a first-order discontinuity. The derivatives in the ordinary sense will be

$$\frac{\tilde{\partial}f}{\partial x} = 0$$
, $\frac{\tilde{\partial}f}{\partial y} = 0$. (1.283)

Applying the formula (1.256), we obtain

$$\frac{\partial f}{\partial x} = \sigma_x |_{\Gamma_1} \cos(\mathbf{n}_1, x)|_{\Gamma_1} \delta_{\Gamma_1} + \sigma_x |_{\Gamma_2} \cos(\mathbf{n}_2, x)|_{\Gamma_2} \delta_{\Gamma_2} ,$$

$$\frac{\partial f}{\partial y} = \sigma_y |_{\Gamma_1} \cos(\mathbf{n}_1, x)|_{\Gamma_1} \delta_{\Gamma_1} + \sigma_y |_{\Gamma_2} \cos(\mathbf{n}_2^*, x)|_{\Gamma_2} \delta_{\Gamma_2} .$$
(1.284)



Figure 1.4

In Figure 1.4, n_1 and n_2 are the normals to Γ_1 and Γ_2 , respectively, oriented in the rising sense of the variable *x*, and n_1 , n_2^* are the normal to Γ_1 and Γ_2 , respectively, oriented in the rising sense of the variable y. Taking this into account we obtain for the function jumps and the directors cosines the values

$$\sigma_{x}|_{\Gamma_{1}} = -\frac{a}{2}, \ \sigma_{x}|_{\Gamma_{2}} = \frac{a}{2}, \ \cos(n_{1}, x)|_{\Gamma_{1}} = \frac{1}{\sqrt{1 + a^{2}}}, \ \cos(n_{2}, x)|_{\Gamma_{2}} = \frac{1}{\sqrt{1 + a^{2}}}, \\ \sigma_{y}|_{\Gamma_{1}} = \sigma_{y}|_{\Gamma_{2}} = -\frac{a}{2}, \ \cos(n_{1}, y)|_{\Gamma_{1}} = \frac{a}{\sqrt{1 + a^{2}}}, \ \cos(n_{2}^{*}, y)|_{\Gamma_{2}} = \frac{a}{\sqrt{1 + a^{2}}},$$

$$(1.285)$$

hence

$$\frac{\partial f}{\partial x} = \frac{-a}{2\sqrt{1+a^2}}\delta_{\Gamma_1} + \frac{a}{2\sqrt{1+a^2}}\delta_{\Gamma_2}, \quad \frac{\partial f}{\partial \gamma} = \frac{-a^2}{2\sqrt{1+a^2}}\delta_{\Gamma_1} - \frac{a^2}{2\sqrt{1+a^2}}\delta_{\Gamma_2}.$$
(1.286)

For the second derivative we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{-a}{2\sqrt{1+a^2}} \frac{\partial}{\partial x} \delta_{\Gamma_1} + \frac{a}{2\sqrt{1+a^2}} \frac{\partial}{\partial x} \delta_{\Gamma_2} , \qquad (1.287)$$

wherefrom, $\forall \varphi \in \mathcal{D}(\mathbb{R}^2)$ it results

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}, \varphi \end{pmatrix} = \frac{a}{2\sqrt{1+a^2}} \left(\delta_{\Gamma_1}, \frac{\partial \varphi}{\partial x} \right) - \frac{a}{2\sqrt{1+a^2}} \left(\delta_{\Gamma_2}, \frac{\partial \varphi}{\partial x} \right)$$
$$= \frac{a}{2\sqrt{1+a^2}} \int_{\Gamma_1} \frac{\partial \varphi}{\partial x} ds_1 - \frac{a}{2\sqrt{1+a^2}} \int_{\Gamma_2} \frac{\partial \varphi}{\partial x} ds_2 .$$
(1.288)

Taking into account the parametric representations (1.282) of the curves Γ_1 and Γ_2 , we obtain

$$ds_1 = \sqrt{1 + a^2} dt$$
, $ds_2 = \sqrt{1 + a^2} dt$. (1.289)

Therefore, the expression (1.288) becomes

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}, \varphi \end{pmatrix} = \frac{a}{2} \int_{-\infty}^{0} \frac{\partial \varphi(at, -t)}{\partial x} dt - \frac{a}{2} \int_{0}^{\infty} \frac{\partial \varphi(at, t)}{\partial x} dt$$

$$= \frac{a}{2} \int_{0}^{\infty} \left[\frac{\partial \varphi(-at, t)}{\partial x} - \frac{\partial \varphi(at, t)}{\partial x} \right] dt .$$
(1.290)

Proceeding similarly, we have

$$\frac{\partial^2 f}{\partial \gamma^2} = \frac{-a^2}{2\sqrt{1+a^2}} \frac{\partial}{\partial \gamma} \delta_{\Gamma_1} - \frac{a^2}{2\sqrt{1+a^2}} \frac{\partial}{\partial \gamma} \delta_{\Gamma_2} , \qquad (1.291)$$

hence

$$\begin{pmatrix} \frac{\partial^2 f}{\partial \gamma^2}, \varphi \end{pmatrix} = \frac{a^2}{2} \int_{-\infty}^{0} \frac{\partial}{\partial \gamma} \varphi(at, -t) dt + \frac{a^2}{2} \int_{0}^{\infty} \frac{\partial}{\partial \gamma} \varphi(at, t) dt$$

$$= \frac{a^2}{2} \int_{0}^{\infty} \left[\frac{\partial}{\partial \gamma} \varphi(-at, t) + \frac{\partial}{\partial \gamma} \varphi(at, t) \right] dt .$$
(1.292)

Because

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(at,t) = a\frac{\partial\varphi(at,t)}{\partial x} + \frac{\partial\varphi(at,t)}{\partial y},$$

$$\frac{\mathrm{d}\varphi(-at,t)}{\mathrm{d}t} = -a\frac{\partial\varphi(-at,t)}{\partial x} + \frac{\partial\varphi(-at,t)}{\partial y},$$
 (1.293)

from (1.288) and (1.292) we get

$$\left(\frac{\partial^2 f}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 f}{\partial y^2}, \varphi\right) = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}[\varphi(-at, t) + \varphi(at, t)]\mathrm{d}t$$
$$= \varphi(0, 0) = (\delta(x, y), \varphi(x, y)), \qquad (1.294)$$

that is,

$$P(\mathbf{D})f = \frac{\partial^2 f}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 f}{\partial y^2} = \delta(x, y).$$
(1.295)

1.3.4 The Fundamental Solution of a Linear Differential Operator

Let the linear differential operator with constant coefficients be $P(D) : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$, having the expression

$$P(\mathsf{D}) = \sum_{|\alpha| \le \ell} a_{\alpha} \, \mathsf{D}^{\alpha} \,, \quad \alpha \in \mathbb{N}_{0}^{n} \,, \quad x \in \mathbb{R}^{n} \,, \tag{1.296}$$

where the scalars $a_{\alpha} \in \Gamma$ represent the operator coefficients.

Definition 1.27 We say that the distribution $E(x) \in \mathcal{D}'(\mathbb{R}^n)$ is the fundamental solution for the operator P(D) if it satisfies the following relation:

$$P(\mathsf{D})E(x) = \delta(x) . \tag{1.297}$$

Based on this definition, we can say that the distribution of function type given by (1.231) is the fundamental solution for the operator $P(D) = (d^2/dx^2) + (d/dx) - (d/dx)$

2. It is verified that the distribution of function type $f_1 = H(x)(e^x - e^{-2x})/3$, $x \in \mathbb{R}$ is the fundamental solution for the same operator.

It follows that the fundamental solution of an operator is generally not unique. Thus, if $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfies the equation P(D) f = 0 and E is a fundamental solution for P(D), then $E_1 = f + E$ is the fundamental solution, because on the basis of linearity of P(D) we can write

$$P(D)E_1 = P(D)(f + E) = P(D)f + P(D)E = \delta.$$
(1.298)

Proposition 1.16 Let there be a linear differential operator with constant coefficients P(D) having the expression

$$P(D) = a_0 \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1} \frac{d}{dx} + a_n , \quad a_0 \neq 0 .$$
 (1.299)

Then, the distribution of function type $E \in \mathcal{D}'(\mathbb{R}^n)$, that is,

$$E(x) = H(x)Y(x)$$
, (1.300)

is the fundamental solution for P(D), where H is the Heaviside function and Y the solution of the homogeneous equation P(D) Y = 0, verifying the initial conditions

$$Y(0) = 0, Y'(0) = 0, \dots, Y^{(n-2)}(0) = 0, Y^{(n-1)}(0) = \frac{1}{a_0}, \quad a_0 \neq 0.$$
 (1.301)

Proof: We note that the function Y is infinitely differentiable and *H*Y is also infinitely differentiable, except at the origin where it has a first-order discontinuity with the jumps

$$s_0(HY^{(p)}) = Y^{(p)}(0), \quad p = 0, 1, 2, ..., n-2, \quad s_0(HY^{(n-1)}) = Y^{(n-1)}(0) = \frac{1}{a_0}.$$

(1.302)

We can write

$$(HY)^{(p)} = HY^{(p)} + \sum_{i=0}^{p-1} s_0(HY^{(i)})\delta^{(p-i-1)}, \quad p = 1, 2, \dots, n.$$
 (1.303)

Consequently, we have

$$(HY)^{(p)} = HY^{(p)}, \quad p = 1, 2, ..., n-1, \quad (HY)^{(n)} = HY^{(n)} + \frac{1}{a_0}\delta.$$

(1.304)

Because P(D) Y = 0, we have $P(D)(HY) = HP(D)Y + \delta = \delta$.

The fundamental solution E = HY is a function of class $C^{\infty}[0,\infty)$ and is unique because of the uniqueness of the solution Y of the Cauchy problem for the equation P(D) Y = 0. Thus, the proposition is proved.

For example, for the operator

$$P(\mathsf{D}) = \frac{\mathsf{d}^2}{\mathsf{d}x^2} + \omega^2 , \quad \omega \in \mathbb{R} \setminus \{0\}$$
(1.305)

we have $Y = (\sin \omega x)/x$, because P(D)Y = 0 and Y(0) = 0, Y'(0) = 1 and thus the fundamental solution is

$$E = H(x)Y(x) = \begin{cases} 0, & x < 0, \\ \frac{\sin(\omega x)}{\omega}, & x \ge 0. \end{cases}$$
 (1.306)

Example 1.18 Let there be the linear differential operator

$$P(\mathsf{D}) = -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\frac{\partial}{\partial y} + 1, \quad (x, y) \in \mathbb{R}^2,$$
(1.307)

and the distribution of function type $E(x, y) \in \mathcal{D}'(\mathbb{R}^2)$,

$$E(x, y) = \begin{cases} 0, & y < 0, & x \in \mathbb{R}, \\ \frac{1}{2}e^{-y}[H(x+y) - H(x-y)], & y \ge 0, & x \in \mathbb{R}, \end{cases}$$
(1.308)

where $H \in \mathcal{D}'(\mathbb{R})$ is the Heaviside distribution.

We will show that the distribution $E \in \mathcal{D}'(\mathbb{R}^2)$ is the fundamental solution of the operator P(D), namely $P(D)E(x, y) = \delta(x, y)$.

We notice that E(x, y) has the value $e^{-y}/2$ inside the cone Γ^+ and is zero outside it (Figure 1.5).

For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we have

$$(P(D) E, \varphi) = -\left(E, \frac{\partial^2 \varphi}{\partial x^2}\right) + \left(E, \frac{\partial^2 \varphi}{\partial y^2}\right) - 2\left(E, \frac{\partial \varphi}{\partial y}\right) + (E, \varphi)$$
$$= -\int_{\Gamma^+} E \frac{\partial^2 \varphi}{\partial x^2} dx dy + \int_{\Gamma^+} E \frac{\partial^2 \varphi}{\partial y^2} dx dy - 2\int_{\Gamma^+} E \frac{\partial \varphi}{\partial y} dx dy + \int_{\Gamma^+} E \varphi dx dy$$



Figure 1.5

By calculation of the four integrals, we obtain:

$$\begin{split} \left(-E, \frac{\partial^2 \varphi}{\partial x^2}\right) &= -\int\limits_{\Gamma^+} E \frac{\partial^2 \varphi}{\partial x^2} \mathrm{d}x \mathrm{d}y = -\frac{1}{2} \int\limits_{0}^{\infty} \mathrm{e}^{-\gamma} \mathrm{d}y \int\limits_{-\gamma}^{\gamma} \frac{\partial^2 \varphi}{\partial x^2} \mathrm{d}x \\ &\quad -\frac{1}{2} \int\limits_{0}^{\infty} \mathrm{e}^{-\gamma} \left[\frac{\partial \varphi(y, \gamma)}{\partial x} - \frac{\partial \varphi(-\gamma, \gamma)}{\partial x} \right] \mathrm{d}y \\ &= -\frac{1}{2} \int\limits_{0}^{\infty} \mathrm{e}^{-t} \left[\frac{\partial \varphi(t, t)}{\partial x} - \frac{\partial \varphi(-t, t)}{\partial x} \right] \mathrm{d}t \;, \\ \left(E, \frac{\partial^2 \varphi}{\partial y^2}\right) &= \int\limits_{\Gamma^+} E \frac{\partial^2 \varphi}{\partial y^2} \mathrm{d}x \mathrm{d}y = \int\limits_{-\infty}^{0} \mathrm{d}x \int\limits_{-x}^{\infty} \frac{1}{2} \mathrm{e}^{-\gamma} \frac{\partial^2 \varphi}{\partial y^2} \mathrm{d}y + \int\limits_{0}^{\infty} \mathrm{d}x \int\limits_{x}^{\infty} \frac{1}{2} \mathrm{e}^{-\gamma} \frac{\partial^2 \varphi}{\partial \gamma^2} \mathrm{d}y \\ &= \int\limits_{-\infty}^{0} \left[-\frac{1}{2} \mathrm{e}^x \frac{\partial \varphi(x, -x)}{\partial y} - \frac{1}{2} \mathrm{e}^x \varphi(x, -x) + \int\limits_{-x}^{\infty} \frac{1}{2} \mathrm{e}^{-\gamma} \varphi \mathrm{d}y \right] \mathrm{d}x \\ &\quad + \int\limits_{0}^{\infty} \left[-\frac{1}{2} \mathrm{e}^{-x} \frac{\partial \varphi(x, x)}{\partial y} - \frac{1}{2} \mathrm{e}^{-x} \varphi(x, x) + \int\limits_{x}^{\infty} \frac{1}{2} \mathrm{e}^{-\gamma} \varphi \mathrm{d}y \right] \mathrm{d}x \\ &= \int\limits_{0}^{\infty} -\frac{1}{2} \mathrm{e}^{-t} \left[\frac{\partial \varphi(-t, t)}{\partial y} + \frac{\partial \varphi(t, t)}{\partial y} + \varphi(-t, t) + \varphi(t, t) \right] \mathrm{d}t \\ &\quad + \int\limits_{\Gamma^+} \frac{1}{2} \mathrm{e}^{-\gamma} \varphi(x, y) \mathrm{d}x \mathrm{d}y \;, \\ \left(-2E, \frac{\partial \varphi}{\partial y}\right) &= -2 \int\limits_{\Gamma^+} \frac{1}{2} \mathrm{e}^{-\gamma} \frac{\partial \varphi}{\partial y} \mathrm{d}x \mathrm{d}y = -\int\limits_{0}^{0} \mathrm{d}x \int\limits_{-\infty}^{\infty} \mathrm{e}^{-y} \frac{\partial \varphi}{\partial y} \mathrm{d}y \\ &\quad -\int\limits_{0}^{\infty} \mathrm{d}x \int\limits_{x}^{\infty} \mathrm{e}^{-\gamma} \frac{\partial \varphi}{\partial y} \mathrm{d}y = \int\limits_{0}^{\infty} \mathrm{e}^{-t} [\varphi(-t, t) + \varphi(t, t)] \mathrm{d}t \end{split}$$

$$-\int_{\Gamma^+} e^{-\gamma} \varphi(x, \gamma) dx d\gamma ,$$

$$(E, \varphi) = \int_{\Gamma^+} \frac{1}{2} e^{-\gamma} \varphi(x, \gamma) dx d\gamma .$$

Consequently, we obtain

$$(P(D)E,\varphi) = \int_{0}^{\infty} \frac{1}{2} e^{-t} \left[\varphi(t,t) + \varphi(-t,t) - \frac{\partial \varphi(t,t)}{\partial x} - \frac{\partial \varphi(t,t)}{\partial y} + \frac{\partial \varphi(-t,t)}{\partial x} - \frac{\partial \varphi(-t,t)}{\partial y} \right] dt .$$
(1.309)

Taking into account the relations

$$\frac{\mathrm{d}\varphi(-t,t)}{\mathrm{d}t} = -\frac{\partial\varphi(-t,t)}{\partial x} + \frac{\partial\varphi(-t,t)}{\partial y}, \quad \frac{\mathrm{d}\varphi(t,t)}{\mathrm{d}t} = \frac{\partial\varphi(t,t)}{\partial x} + \frac{\partial\varphi(t,t)}{\partial y}, \quad (1.310)$$

the previous one becomes

$$(P(D),\varphi) = -\frac{1}{2} \int_{0}^{\infty} \frac{d}{dt} [e^{-t}(\phi(t,t) + \phi(-t,t))] dt = \varphi(0,0) = (\delta(x,y),\varphi(x,y)),$$
(1.311)

namely $P(D)E = \delta(x, y)$.

1.3.5

The Derivation of the Homogeneous Distributions

The distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous and of degree λ if for $\alpha > 0$ it satisfies the relation

$$f(\alpha x) = \alpha^{\lambda} f(x) , \quad x \in \mathbb{R}^{n} , \tag{1.312}$$

or, equivalently

$$\alpha^{\lambda+n}(f(x),\varphi(x)) = \left(f(x),\varphi\left(\frac{x}{\alpha}\right)\right), \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$
(1.313)

It is known that the homogeneous functions of degrees λ and of class $C^1(\mathbb{R}^n)$ satisfy the Euler equation

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = \lambda f .$$
(1.314)

In fact, (1.314) fully characterizes the homogeneous functions of degree λ , since this equation represents the necessary and sufficient condition for a function to be homogeneous of degree λ .

This result is valid for functions which translate to the homogeneous distributions. Suppose that $f \in \mathcal{D}'(\mathbb{R}^n)$ is a homogeneous distribution of degree λ .

Then, we can derive the equality (1.313) with respect to $\alpha > 0$, on the basis of Proposition 1.9, and we obtain

$$(n+\lambda)\alpha^{n+\lambda-1}(f(x),\varphi(x)) = -\frac{1}{\alpha^2} \left(f(x_1,\ldots,x_n), \sum_{i=1}^n x_i \frac{\partial\varphi}{\partial x_i} \left(\frac{x_1}{\alpha},\ldots,\frac{x_n}{\alpha} \right) \right).$$
(1.315)

Considering $\alpha = 1$, the relation (1.315) becomes

$$(n+\lambda)(f,\varphi) = \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(x_i f), \varphi\right), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n), \qquad (1.316)$$

wherefrom

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (x_i f) = (n+\lambda) f , \qquad (1.317)$$

hence

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = \lambda f , \quad f \in \mathcal{D}'(\mathbb{R}^n) .$$
(1.318)

Let us now show the converse. We acknowledge that the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfies (1.316). Differentiating with respect to $\alpha > 0$ the fraction

$$\frac{1}{\alpha^{\lambda+n}}\left(f(x),\varphi\left(\frac{x_1}{\alpha},\ldots,\frac{x_n}{\alpha}\right)\right)$$
(1.319)

and taking into account (1.316), it results that the derivative is zero. This means that the fraction is reduced to a constant.

Hence, we can write

$$\left(f(x),\varphi\left(\frac{x_1}{\alpha},\ldots,\frac{x_n}{\alpha}\right)\right) = c\alpha^{\lambda+n}$$
 (1.320)

To determine the value of the constant, we take $\alpha = 1$ and we obtain $c = (f(x), \varphi(x))$, hence

$$\left(f(x),\varphi\left(\frac{x}{\alpha}\right)\right) = \alpha^{\lambda+n}(f(x),\varphi(x)), \quad \forall \varphi \in \mathcal{D},$$
 (1.321)

which shows that the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous and of degree λ .

The homogeneous distributions with the singularities generated by homogeneous locally integrable functions are of interest in applications.

Let there be the homogeneous function $f : \mathbb{R}^n \setminus \{0\} \to \Gamma$ of degree λ , with a singularity (discontinuity) at the point x = 0, to which we assign the functional $T_f : \mathcal{D}(\mathbb{R}^n) \to \Gamma$ by the formula

$$(T_f,\varphi(x)) = \lim_{\varepsilon \to 0} \int_{\|x\| \ge \varepsilon} f(x)\varphi(x) dx , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n) .$$
(1.322)

Obviously, in the case of the convergence of the integral (1.322), the functional T_f is linear and continuous, hence $T_f \in \mathcal{D}'(\mathbb{R}^n)$.

We note by $F_{\varepsilon}(\varphi)$ the integral

$$F_{\varepsilon}(\varphi) = \int_{\varepsilon \le ||x|| \le a} f(x)\varphi(x)dx , \quad \varphi \in \mathcal{D}(\mathbb{R}^n) , \quad \operatorname{supp}(\varphi) \subset B_a , \qquad (1.323)$$

where B_a is the open sphere of radius *a*, centered at the origin of the coordinates.

We pass to spherical coordinates, expressed by the relations

$$x_{1} = r \sin \theta_{1} \sin \theta_{2} \dots \sin \theta_{n-2} \sin \theta_{n-1} ,$$

$$x_{2} = r \sin \theta_{1} \sin \theta_{2} \dots \sin \theta_{n-2} \cos \theta_{n-1} ,$$

$$x_{3} = r \sin \theta_{1} \sin \theta_{2} \dots \sin \theta_{n-3} \cos \theta_{n-2} ,$$

$$\vdots$$

$$x_{n-2} = r \sin \theta_{1} \sin \theta_{2} \cos \theta_{3} ,$$

$$x_{n-1} = r \sin \theta_{1} \cos \theta_{2} ,$$

$$x_{n} = r \cos \theta_{1} ,$$
(1.324)

where

$$r \ge 0, \, \theta_i \in [0, \, \pi] \,, \quad i = \overline{1, n-2} \,, \quad \theta_{n-1} \in [0, 2\pi) \,,$$
 (1.325)

and where the Jacobian of the transformation is

$$J(r, \theta_1, \theta_2, \dots, \theta_{n-1}) = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(r, \theta_1, \theta_2, \dots, \theta_{n-1})} = r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} ; \qquad (1.326)$$

the expression (1.323) becomes

$$F_{\varepsilon}(\varphi) = \int_{\varepsilon}^{a} \underbrace{\int_{0}^{\pi} \cdots \int_{0}^{\pi} \int_{0}^{2\pi} f^{*}(r, \theta_{1}, \dots, \theta_{n-1}) \varphi^{*}(r, \theta_{1}, \dots, \theta_{n-1}) r^{n-1} \mathrm{d}r \mathrm{d}s_{1}, \quad (1.327)$$

where ds_1 is the element of area of the unit sphere, and f^* , φ^* are the expressions in polar coordinates of the functions f and φ .

Because *f* is a homogeneous function of degree λ , the formula (1.327) becomes

$$F_{\varepsilon}(\varphi) = \int_{\varepsilon}^{a} \underbrace{\int_{0}^{\pi} \cdots \int_{0}^{\pi} \int_{0}^{2\pi} r^{\lambda+n-1} f^{*}(1, \theta_{1}, \dots, \theta_{n-1}) \varphi^{*}(r, \theta_{1}, \dots, \theta_{n-1}) r^{n-1} dr ds_{1} .$$
(1.328)

Taking into account that the functions $f^*(1, \theta_1, ..., \theta_{n-1})$ and $\varphi^*(r, \theta_1, ..., \theta_{n-1})$ are bounded, it results that the integral (1.322) is convergent together with the integral $\lim_{\varepsilon \to 0} \int_{\varepsilon}^{a} r^{\lambda+n-1} dr$.

Thus, the functional T_f , given by (1.322), does exist if $\lambda + n - 1 > -1$, namely $\lambda > -n$.

In particular, if $\lambda \ge -n + 1$, then the homogeneous distribution of function type T_f exists.

We shall establish the formula for the derivative of the homogeneous distribution T_f of degree $\lambda \ge -n + 1$, taking into account (1.322), we have

$$\begin{pmatrix} \frac{\partial T_f}{\partial x_i}, \varphi \end{pmatrix} = -\left(T_f, \frac{\partial \varphi}{\partial x_i}\right) = -\lim_{\epsilon \to +0} \int_{\|x\| \ge \epsilon} f(x) \frac{\partial \varphi}{\partial x_i} dx$$
$$= -\lim_{\epsilon \to +0} \left[\int_{\|x\| \ge \epsilon} \frac{\tilde{\partial}}{\partial x_i} (f\varphi) dx - \int_{\|x\| \ge \epsilon} \frac{\tilde{\partial} f}{\partial x_i} \varphi dx \right],$$
(1.329)

where $\tilde{\partial}/\partial x_i$ is the derivative in the usual sense.

We consider $\varphi \in D(\mathbb{R}^n)$ with supp $(\varphi) \subset B_a$, where B_a is the open sphere of radius *a* centered at the origin. Because $\varepsilon > 0$ is arbitrary, we will take $\varepsilon < a$. Thus, the integrals of (1.329) are performed on a spherical crown $\varepsilon < r < a$, so that we can apply the Gauss-Ostrogradski formula.

Therefore, we can write

$$\int_{\|x\| \ge \varepsilon} \frac{\tilde{\partial}}{\partial x_i} (f\varphi) \mathrm{d}x = -\int_{S_\varepsilon} f\varphi \cos \alpha_i \mathrm{d}S_\varepsilon , \qquad (1.330)$$

where S_{ε} is the sphere centered at the origin, of radius $\varepsilon > 0$, dS_{ε} is the corresponding area element and α_i is the angle formed by the outer normal to S_{ε} and the Ox_i -axis.

We mention that the functions φ and $\partial \varphi / \partial x_i$ are zero on the sphere B_a and beyond it.

Substituting (1.330) in (1.329), we obtain

$$\begin{pmatrix} \frac{\partial T_f}{\partial x_i}, \varphi \end{pmatrix} = \lim_{\epsilon \to +0} \int_{\|x\| \ge \epsilon} \frac{\bar{\partial} f}{\partial x_i} \varphi dx + \lim_{\epsilon \to +0} \int_{S_{\epsilon}} f\varphi \cos \alpha_i dS_{\epsilon}$$

$$= \left(\frac{\tilde{\partial} f}{\partial x_i}, \varphi\right) + \lim_{\epsilon \to +0} \int_{S_{\epsilon}} f\varphi \cos \alpha_i dS_{\epsilon} .$$
(1.331)

In connection with the evaluation of the second term from the right-hand side of the formula (1.331) we introduce:

Definition 1.28 We call residue of the homogeneous function *f* of degree $\lambda \ge -n + 1$ 1 at the singular point x = 0, corresponding to the Ox_i -axis, the number given by the expression:

$$(\operatorname{res} f)_i(0) = \int_{S_1} f(x) \cos \alpha_i \mathrm{d} S_1 . \tag{1.332}$$

Proposition 1.17 Let the homogeneous function $f : \mathbb{R}^n \setminus \{0\} \to \Gamma$ of degree λ have the origin, x = 0, as a singular point. Also, if $\lambda \ge -n + 1$, then the derivative

of the homogeneous distribution of function type T_f given by (1.322) is calculated according to the formula

$$\frac{\partial T_f}{\partial x_i} = \begin{cases} \frac{\partial f}{\partial x_i}, & \lambda > -n+1, \\ \frac{\partial f}{\partial x_i} + \delta(x)(\operatorname{res} f)_i(0), & \operatorname{if} \lambda = -n+1. \end{cases}$$
(1.333)

Indeed, passing to polar coordinates and taking into account that $dS_{\varepsilon} = \varepsilon^{n-1} dS_1$ and that *f* is homogeneous, we can write

$$\lim_{\varepsilon \to +0} \int_{S_{\varepsilon}} f\varphi \cos \alpha_{i} dS_{\varepsilon}$$

$$= \lim_{\varepsilon \to +0} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \int_{0}^{2\pi} f^{*}(\varepsilon, \theta_{1}, \theta_{2}, \dots, \theta_{n-1}) \varphi^{*}(\varepsilon, \theta_{1}, \theta_{2}, \dots, \theta_{n-1}) \cos \alpha_{i} \varepsilon^{n-1} dS_{1}$$

$$= \lim_{\varepsilon \to +0} \int_{S_{1}} \varepsilon^{\lambda+n-1} f(x) \varphi(\varepsilon x) \cos \alpha_{i} dS_{1}, \qquad (1.334)$$

where $x \in S_1$, S_1 is the unit radius sphere.

Since the integral from the right of (1.334) is taken on the unit sphere and the functions $\cos \alpha_i$, f(x) do not depend on the radius ε of the sphere S_{ε} , from (1.334), we obtain

$$\lim_{\varepsilon \to +0} \int_{S_{\varepsilon}} f(x) \cos \alpha_{i} \mathrm{d}S_{\varepsilon} = \begin{cases} 0, & \text{if } \lambda > -n+1, \\ \varphi(0) \int_{S_{1}} f(x) \cos \alpha_{i} \mathrm{d}S_{1}, & \text{if } \lambda = -n+1, \end{cases}$$
(1.335)

namely

$$\lim_{\varepsilon \to +0} \int_{S_{\varepsilon}} f(x)\varphi(x) \cos \alpha_i dS_{\varepsilon} = \begin{cases} 0, & \text{if } \lambda > -n+1, \\ (\delta(x)(\operatorname{res} f)_i(0), \varphi), & \text{if } \lambda = -n+1. \end{cases}$$
(1.336)

Substituting (1.336) in (1.331), we obtain the formula (1.333) and the proposition is proved.

We note that, using the exterior product $\wedge,$ the Gauss–Ostrogradski formula can be written

$$\int_{\Omega} \sum_{i=1}^{n} \frac{\partial a_i(x)}{\partial x_i} dx_1 \wedge \dots \wedge dx_n$$

=
$$\int_{\partial \Omega} (-1)^{i-1} a_i(x) dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_n , \qquad (1.337)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $\partial \Omega$ is its border.

Using the Gauss-Ostrogradski formula, the residue (1.332) can be written in the form

$$(\operatorname{res} f)_i(0) = (-1)^{i-1} \int_{\overline{U}_1} f(x) \mathrm{d} x_1 \wedge \cdots \wedge \mathrm{d} x_{i-1} \wedge \mathrm{d} x_{i+1} \wedge \cdots \wedge \mathrm{d} x_n , \quad (1.338)$$

where \overline{U}_1 is the closed sphere of the unit radius centered at the coordinates origin, because $\cos \alpha_i dS_1 = (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$.

Example 1.19 To illustrate the application of the formula (1.333), we establish the following relations

$$\Delta \frac{1}{r^{n-2}} = -(n-2)\delta(x)S_1 = -(n-2)\delta(x)\frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad x \in \mathbb{R}^n, \quad n \ge 3,$$
(1.339)

$$\Delta \ln r = 2\pi \delta(x) , \quad n = 2 , \qquad (1.340)$$

where r = ||x||, $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, Re z > 0, is the Euler gamma function, z = x + iy, S_1 is the area of the unit radius sphere in \mathbb{R}^n and Δ is the Laplace operator.

We consider the function $f(x) = 1/r^{n-2}$, $x \in \mathbb{R}^n \setminus \{0\}$, $n \ge 3$, which is homogeneous and of the degree $\lambda = -n + 2$.

Taking into account (1.333), the function f is locally integrable and we have

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} = (-n+2)\frac{x_i}{r^n} . \tag{1.341}$$

We observe that the function $g(x) = x_i/r^n$ is also homogeneous and of degree -n + 1.

Consequently, we can apply the formula (1.333), thus we may write

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x_i^2} + (-n+2)\delta(x) \int_{S_1} \frac{x_i}{r^n} \cos \alpha_i dS_1$$

= $(-n+2)\frac{r^2 - nx_i^2}{r^{n+2}} + (-n+2)\delta(x) \int_{S_1} x_i^2 dS_1$. (1.342)

From (1.342) we obtain, by summing,

$$\Delta \frac{1}{r^{n-2}} = \tilde{\Delta} \frac{1}{r^{n-2}} + (-n+2)\delta(x) \int_{S_1} \mathrm{d}S_1 = (-n+2)S_1\delta(x) , \qquad (1.343)$$

because $\tilde{\Delta} = 1/r^{n-2} = 0$, $\tilde{\Delta} = (\tilde{\partial}^2/\partial x_1^2) + \dots + (\tilde{\partial}/\partial x_n^2)$.

Observing that $S_1 = 2\pi^{n/2}/\Gamma(n/2)$, we obtain the formula (1.339).

Particularly, for n = 3 we have $\Delta 1/r = -4\pi \delta(x)$.

As regards the formula (1.340), we consider the locally integrable function $h(x) = \ln r, r = \sqrt{x^2 + y^2}, (x, y) \in \mathbb{R}^2 \setminus \{0\}$, for which we can write

$$\frac{\partial}{\partial x}\ln(x^2+y^2) = \frac{\tilde{\partial}}{\partial x}\ln(x^2+y^2) = \frac{2x}{x^2+y^2}.$$
(1.344)

Because the function $F(x) = \frac{2x}{x^2 + y^2}$ is a homogeneous function of degree $\lambda = -1$, we can apply the formula (1.333) and read

$$\frac{\partial^2}{\partial x^2} \ln(x^2 + y^2) = \frac{\tilde{\partial}^2}{\partial x^2} \ln(x^2 + y^2) + 2\delta(x, y) \int_{S_1} \frac{x}{x^2 + y^2} \cos \alpha dS_1 , \quad (1.345)$$

where S_1 is the unit circle and $\cos \alpha = x/\sqrt{x^2 + \gamma^2}$. Consequently, we get

$$\frac{\partial^2 \ln(x^2 + y^2)}{\partial x^2} = \frac{\tilde{\partial}^2}{\partial x^2} \ln(x^2 + y^2) + 2\delta(x, y) \int_{S_1} x^2 dS_1 .$$
(1.346)

Similarly, we have

$$\frac{\partial^2 \ln(x^2 + y^2)}{\partial y^2} = \frac{\tilde{\partial}^2 \ln(x^2 + y^2)}{\partial y^2} + 2\delta(x, y) \int_{S_1} y^2 dS_1.$$
(1.347)

By summing up, and because $\tilde{\Delta} \ln(x^2 + y^2) = 0$, we obtain the formula (1.340). The function $f(x, y) = y^3/(x^2 + y^2)^2$, $(x, y) \in \mathbb{R}^2 \setminus \{0\}$ has an integrable singularity at the origin of coordinates and is a homogeneous function of degree $\lambda = -1$.

Consequently, we can apply the formula (1.333) and get

$$\frac{\partial f}{\partial x} = -\frac{4x\gamma^3}{(x^2 + \gamma^2)^3} , \qquad (1.348)$$

$$\frac{\partial f}{\partial y} = \frac{y^2 (3x^2 - y^2)}{(x^2 + y^2)^3} + \frac{3\pi}{4} \delta(x, y) , \qquad (1.349)$$

because $(\text{res } f)_x(0) = 0$ and $(\text{res } f)_y(0) = 3\pi/4$.

The formula (1.339) can be derived using the Green second formula. In this case, because the function $f(x) = 1/r^{n-2}$, $x \in \mathbb{R}^n \setminus \{0\}$, $n \ge 3$, r = ||x|| is locally integrable, for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\left(\Delta \frac{1}{r^{n-2}},\varphi\right) = \left(\frac{1}{r^{n-2}},\Delta\varphi\right) = \int_{\mathbb{R}^n} \frac{1}{r^{n-2}} \Delta\varphi dx = \lim_{\varepsilon \to +0} \int_{r \ge \varepsilon} \frac{1}{r^{n-2}} \Delta\varphi dx .$$
(1.350)

Further, we shall apply the Green second formula

$$\int_{\Omega} (f\Delta\varphi - \varphi\Delta f) dx = \int_{S} \left(f \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial f}{\partial n} \right) dS , \qquad (1.351)$$

where *n* is the outer normal to the surface *S*, bordering the bounded domain $\Omega \subset \mathbb{R}^n$, *S* being the spherical crown $\varepsilon < r < a$, with supp $(\varphi) \subset B_a$; the support of φ included in the sphere centered at O and of radius *a*.

We have

$$\int_{r \ge \varepsilon} \frac{1}{r^{n-2}} \Delta \varphi dx = \int_{\varepsilon < r < a} \frac{1}{r^{n-2}} \Delta \varphi dx = -\int_{S_{\varepsilon}} \left[\frac{1}{r^{n-2}} \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial}{\partial r} \left(\frac{1}{r^{n-2}} \right) \right] dS ,$$
(1.352)

where S_{ε} is the sphere centered at the origin and of radius ε .

This occurs because $\Delta(1/r^{n-2}) = 0$ for $r > \varepsilon$, and $\varphi, \partial \varphi / \partial r$ are zero on the sphere U_a and beyond. We notice that $(\partial/\partial n)\varphi = \operatorname{grad} \varphi \cdot n = (\partial/\partial r)\varphi$ and $(\partial/\partial n)(1/r^{n-2}) = (\partial/\partial r)(1/r^{n-2})$.

Consequently, we get

$$\int_{r\geq\varepsilon} \frac{1}{r^{n-2}} \Delta \varphi \, \mathrm{d}x = -\left[\frac{1}{\varepsilon^{n-2}} \frac{\partial \varphi(\xi)}{\partial r} + \frac{n-2}{\varepsilon^{n-1}} \varphi(\xi)\right] S_{\varepsilon} , \quad \xi \in U_{\varepsilon} , \quad (1.353)$$

namely

$$\int_{r\geq\varepsilon} \frac{1}{r^{n-2}} \Delta \varphi dx = -\frac{\varepsilon \pi^{n/2}}{\Gamma(n/2)} \frac{\partial \varphi(\xi)}{\partial r} - \frac{(n-2)\pi^{n/2}}{\Gamma(n/2)} \varphi(\xi) , \qquad (1.354)$$

because $S_{\varepsilon} = \varepsilon^{n-1} \pi^{n/2} / \Gamma(n/2)$ is the area of the sphere of radius ε from \mathbb{R}^n .

For $\varepsilon \to 0$ we have $\xi \to 0$, hence $(\Delta(1/r^{n-2}), \varphi) = -((n-2)\pi^{n/2}/\Gamma(n/2))\varphi(0) = (-(n-2)(\pi^{n/2}/\Gamma(n/2))\delta(x), \varphi)$ giving the formula (1.339).

We note that this result is correct because the distribution $\Delta(1/r^{n-2})$ has as support the origin.

1.3.6

Dirac Representative Sequences: Criteria for the Representative Dirac Sequences

The Dirac delta distribution $\delta(x) \in \mathcal{D}'(\mathbb{R}^n)$ plays an important role in operational calculus, in the theory of electrical systems and the construction of fundamental solutions of linear differential operators with constant coefficients.

In many theoretical or practical problems (Fourier series, Fourier integral, elasticity problems, and so on) occur sequences of locally integrable functions that are convergent in the sense of convergence in the distribution space, having as a limit the Dirac delta distribution δ . Such sequences of the locally integrable functions are called the representative sequences. The functions which form the representative sequences δ are also called the impulsive functions.

We can say that any term of such a sequence represents a certain approximation of the Dirac delta distribution δ ; this is very important from the practical point of view. Indeed, assuming that we want to obtain numerical values in a problem in which the results are expressed as distributions, we can substitute – for the calculations – a singular distribution by a term of a corresponding representative sequence. Thus, the obtained formulae can be used in calculations by the computer, obtaining a desired approximation, depending on the chosen term of the representative sequence.

Definition 1.29 Let $f_i : \mathbb{R}^n \to \mathbb{C}, i \in \mathbb{N}$, be a sequence of locally integrable functions. We say that $(f_i)_{i \ge 1}$ is a Dirac representative sequence if on the space $\mathcal{D}'(\mathbb{R}^n)$ we have $\lim_{i\to\infty} f_i(x) = \delta(x)$, that is,

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^n) \Rightarrow \lim_{i \to \infty} (f_i(x), \varphi(x)) = (\delta(x), \varphi(x)) = \varphi(0) .$$
(1.355)

We will show that continuous functions and locally integrable functions with certain properties allow for the construction of the representative Dirac sequences. Thus, we mention the following proposition.

Proposition 1.18 Let there be $f \in C^0(\mathbb{R}^n)$, $f : \mathbb{R}^n \to \mathbb{C}$ with the property $\int_{\mathbb{R}^n} f(x) dx = 1$; then the family of functions $f_{\varepsilon}, \varepsilon > 0$, having the expression

$$f_{\varepsilon}(x) = \frac{1}{\varepsilon^n} f\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^n, \quad \varepsilon > 0,$$
 (1.356)

forms a representative Dirac family; hence

$$\lim_{\varepsilon \to +0} f_{\varepsilon}(x) = \delta(x) .$$
(1.357)

Proof: For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$(f_{\varepsilon}(x),\varphi(x)) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f\left(\frac{x}{\varepsilon}\right) \varphi(x) \mathrm{d}x , \quad \varepsilon > 0 .$$
(1.358)

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Performing the change of variable $x = \varepsilon u$, $x_k = \varepsilon u_k$, $k = \overline{1, n}$, the Jacobian of the transformations is

$$J(u) = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \dots & \frac{\partial x_1}{\partial u_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \dots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$
$$= \begin{vmatrix} \varepsilon & 0 & \dots & 0 & 0 \\ 0 & \varepsilon & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \varepsilon & 0 \\ 0 & 0 & \dots & 0 & \varepsilon \end{vmatrix} = \varepsilon^n ;$$
(1.359)

thus we can write

$$(f_{\varepsilon}(x),\varphi(x)) = \int_{\mathbb{R}^n} f(u)\varphi(\varepsilon u) du .$$
(1.360)

Hence, it follows that

$$\lim_{\varepsilon \to 0} (f_{\varepsilon}(x), \varphi(x)) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f(u)\varphi(\varepsilon u) du$$
$$= \int_{\mathbb{R}^n} f(u)\varphi(0) du = \varphi(0) = (\delta(x), \varphi(x)) .$$
(1.361)

Example 1.20 Let there be the function

$$f(x) = \frac{n}{S_1} \frac{1}{(\|x\|^2 + 1)^{(n+2)/2}}, \quad x \in \mathbb{R}^n, \quad n \ge 2,$$
(1.362)

where S_1 represents the area of the unit radius sphere from \mathbb{R}^n . Obviously, f is a continuous function, hence $f \in C^0(\mathbb{R}^n)$.

Using the polar coordinates $(r, \theta_1, \theta_2, ..., \theta_{n-1}) \in \mathbb{R}^n$, whose connection with the Cartesian coordinates $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is expressed by the relations

$$x_{1} = r \sin \theta_{1} \sin \theta_{2} \dots \sin \theta_{n-2} \sin \theta_{n-1}$$

$$x_{2} = r \sin \theta_{1} \sin \theta_{2} \dots \sin \theta_{n-2} \cos \theta_{n-1}$$

$$x_{3} = r \sin \theta_{1} \sin \theta_{2} \dots \sin \theta_{n-3} \cos \theta_{n-2}$$

$$\vdots$$

$$x_{n-2} = r \sin \theta_{1} \sin \theta_{2} \cos \theta_{3}$$

$$x_{n-1} = r \sin \theta_{1} \cos \theta_{2}$$

$$x_{n} = r \cos \theta_{1}, \qquad (1.363)$$

where

$$r \ge 0$$
, $\theta_i \in [0, \pi]$, $i = \overline{1, n-2}$, $\theta_{n-1} \in [0, 2\pi)$, (1.364)

we have

$$\int_{\mathbb{R}^n} f(x) \mathrm{d}x = \frac{n}{S_1} \int_0^\infty \frac{r^{n-1}}{(r^2+1)^{(n+2)/2}} \left\{ \int_\Omega \mathrm{d}\Omega \right\} \mathrm{d}r = n \int_0^\infty \frac{r^{n-1}}{(r^2+1)^{(n+2)/2}} \mathrm{d}r ,$$
(1.365)

where d Ω represents the area element of the unit radius sphere from \mathbb{R}^n , centered at the origin.

We observe that the Jacobian of the transformation (1.363) is

$$J(r, \theta_1, \theta_2, \dots, \theta_{n-1}) = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(r, \theta_1, \theta_2, \dots, \theta_{n-1})} = r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} , \qquad (1.366)$$

so that $\mathrm{d} \varOmega$ has the expression

$$\mathrm{d}\Omega = \frac{J}{r^{n-1}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \dots \mathrm{d}\theta_{n-1} \,. \tag{1.367}$$

Regarding the volume element $dx = dx_1 \dots dx_n$ in polar coordinates (1.363) it has the expression

$$dx = dv = J(r, \theta_1, \dots, \theta_{n-1}) dr d\theta_1 \dots d\theta_{n-1} = r^{n-1} dr d\Omega .$$
(1.368)

Making the change of variable $t = 1/r^2$, $t \in \mathbb{R}$, we have

$$I = \int_{0}^{\infty} \frac{r^{n-1}}{(r^2+1)^{(n+2)/2}} dr = \int_{0}^{\infty} \frac{r^{n-1}}{r^{n+2} (1+(1/r^2))^{(n+2)/2}} dr$$
$$= \int_{0}^{\infty} \frac{dr}{r^3 (1+(1/r^2))^{(n+2)/2}} = -\int_{\infty}^{0} \frac{dt}{2(1+t)^{(n+2)/2}}$$
$$= \frac{1}{2} \int_{0}^{\infty} (1+t)^{-(n+2)/2} dt = \frac{1}{2} \cdot \frac{1}{(-n)/2} (1+t)^{-n/2} \Big|_{0}^{\infty} = \frac{1}{n} .$$
(1.369)

Taking into account (1.365) we can write $\int_{\mathbb{R}^n} f(x) dx = 1$ and as $f \in C^0(\mathbb{R}^n)$ follows that the two conditions of Proposition 1.18 are fulfilled. With this on the basis of (1.356) we have

$$f_{\varepsilon}(x) = \frac{1}{\varepsilon^{n}} \cdot \frac{n}{S_{1}} f\left(\frac{\varepsilon}{n}\right) = \frac{1}{\varepsilon^{n}} \cdot \frac{n}{S_{1}} \cdot \varepsilon^{n+2} \frac{1}{\left(\|x\|^{2} + \varepsilon^{2}\right)^{(n+2)/2}}$$
$$= \frac{n}{S_{1}} \frac{\varepsilon^{2}}{\left(\|x\|^{2} + \varepsilon^{2}\right)^{(n+2)/2}}, \quad \varepsilon > 0, \qquad (1.370)$$

and thus $\lim_{\epsilon \to +0} f_{\epsilon}(x) = \delta(x)$, namely $f_{\epsilon} \xrightarrow{\mathcal{D}'(\mathbb{R}^n)}_{\epsilon \to +0} \delta(x)$.

Thus, the family of functions $f_{\varepsilon}(x), \varepsilon > 0, x \in \mathbb{R}^n$ is a representative Dirac family.

Particularly, for n = 2 we obtain

$$f_{\varepsilon}(x, \gamma) = \frac{2}{2\pi} \frac{\varepsilon^2}{\left(x^2 + \gamma^2 + \varepsilon^2\right)^2} , \quad \varepsilon > 0 , \qquad (1.371)$$

hence

$$\lim_{\varepsilon \to +0} \frac{1}{\pi} \frac{\varepsilon^2}{\left(x^2 + \gamma^2 + \varepsilon^2\right)^2} = \delta(x, \gamma) .$$
(1.372)

Example 1.21 Let there be the function

$$f(x) = \frac{1}{(\sqrt{\pi})^n} e^{-\|x\|^2}, \quad x \in \mathbb{R}^n.$$
(1.373)
Obviously, $f \in C^0(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} f(x) \mathrm{d}x = \frac{1}{\left(\sqrt{\pi}\right)^n} \prod_{k=1}^n \int_{\mathbb{R}} \exp\left(-x_k^2\right) \mathrm{d}x_k = 1.$$

According to Proposition 1.18, we have

$$f_{\varepsilon}(x) = \frac{1}{\varepsilon^n} \frac{1}{(\sqrt{\pi})^n} \exp\left(-\frac{\|x\|^2}{\varepsilon^2}\right), \quad x \in \mathbb{R}^n,$$
(1.374)

and $\lim_{\varepsilon \to +0} f_{\varepsilon}(x) = \delta(x)$.

In the Fourier integral theory, the Dirichlet function is used in the form

$$\lim_{n\to\infty}\frac{\sin nx}{\pi x}=\delta(x).$$

Particularly, for n = 1 we obtain

$$\lim_{\varepsilon \to +0} \frac{1}{\varepsilon} \frac{1}{\sqrt{\pi}} \exp\left(\frac{-x^2}{\varepsilon^2}\right) = \delta(x) .$$
 (1.375)

Particular forms of Dirac sequences of one variable have been used in connection with Fourier integrals, heat propagation, wave theory of light, representation of concentrated loads, and so on.

Thus, G.R. Kirchhoff, formulating Huygens principle in the wave theory of light, mentions the function

$$f_n(x) = \sqrt{\frac{n}{2\pi}} \exp\left(-n\frac{x^2}{2}\right) , \qquad (1.376)$$

which is obtained from (1.375), thus $\lim_{n\to\infty} f_n(x) = \delta(x)$.

Lord Kelvin used this function to represent the point heat sources in the form

$$q_t(x) = \frac{1}{2\sqrt{\pi kt}} \exp\left(-\frac{x^2}{4kt}\right), \quad k > 0, \quad t > 0, \quad \lim_{t \to 0} q_t(x) = \delta(x).$$
(1.377)

We also mention the impulsive function of Stieltjes

$$f_n(x) = \frac{2}{\pi} \frac{n}{\cosh nx} , \quad n \in \mathbb{N}$$
(1.378)

and the Cauchy impulsive function

$$q_{\varepsilon}(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} , \quad \varepsilon > 0 .$$
(1.379)

For these functions we have

$$\lim_{n \to \infty} f_n(x) = \delta(x) , \quad \lim_{\epsilon \to 0} q_\epsilon(x) = \delta(x) .$$
(1.380)

Example 1.22 Let there be the function

$$f: \mathbb{R}^2 \to \mathbb{R}$$
, $f(x, y) = \frac{p-2}{2\pi} \cdot \frac{1}{(x^2 + y^2 + 1)^{p/2}}$, $p > 2$. (1.381)

This function is obviously continuous, thus $f \in C^0(\mathbb{R}^2)$ and we obtain

$$\int_{\mathbb{R}^2} f(x, y) \mathrm{d}x \mathrm{d}y = 1 . \tag{1.382}$$

Indeed, passing to polar coordinates $x = \rho \cos \theta$, $y = \rho \sin \theta$, $\theta \in [0, 2\pi]$, $\rho \ge 0$ we can write

$$\int_{\mathbb{R}^{2}} f(x, y) dx dy = \frac{p-2}{2\pi} \int_{\mathbb{R}^{2}} \frac{dx dy}{(x^{2} + y^{2} + 1)^{p/2}}$$
$$= \frac{p-2}{2\pi} \int_{0}^{2\pi} d\theta \int_{0}^{\infty} \frac{\rho d\rho}{(\rho^{2} + 1)^{p/2}} = \frac{(p-2)}{2} \int_{0}^{\infty} (\rho^{2} + 1)^{-p/2} (\rho^{2} + 1)' d\rho$$
$$= \frac{p-2}{2} \frac{(\rho^{2} + 1)^{-(p/2)+1}}{1 - p/2} \Big|_{0}^{\infty} = \frac{p-2}{2} \cdot \frac{1}{-1 + p/2} = 1.$$
(1.383)

Thus, the conditions of Proposition 1.18 are fulfilled and we can build the function

$$f_{\varepsilon}(x, \gamma) = \frac{1}{\varepsilon^2} f\left(\frac{x}{\varepsilon}, \frac{\gamma}{\varepsilon}\right) = \frac{p-2}{2\pi \cdot \varepsilon^2} \cdot \frac{\varepsilon^p}{\left(x^2 + \gamma^2 + \varepsilon^2\right)^{p/2}}$$
$$= \frac{p-2}{2\pi} \frac{\varepsilon^{p-2}}{\left(x^2 + \gamma^2 + \varepsilon^2\right)^{p/2}}, \quad \varepsilon > 0, \quad p > 2.$$
(1.384)

Consequently, the relation follows

$$\lim_{\epsilon \to +0} \frac{p-2}{2\pi} \frac{\epsilon^{p-2}}{\left(x^2 + \gamma^2 + \epsilon^2\right)^{p/2}} = \delta(x, \gamma) , \quad p > 2 .$$
 (1.385)

We note that the family of Dirac representative functions (1.384) plays an important role in the construction of the fundamental solution of the elastic half-space problem [19].

Particularly, for n = 3 and n = 5 we obtain

$$\lim_{\varepsilon \to +0} \frac{\varepsilon}{2\pi} \cdot \frac{1}{\left(x^2 + \gamma^2 + \varepsilon^2\right)^{3/2}} = \lim_{\varepsilon \to +0} \frac{3\varepsilon^3}{2\pi} \cdot \frac{1}{\left(x^2 + \gamma^2 + \varepsilon^2\right)^{5/2}} = \delta(x, \gamma) .$$
(1.386)

Also, taking into account improper integrals values

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_{0}^{\infty} \frac{\sin x}{x} dx = \pi , \quad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = 2 \int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi ,$$
(1.387)

based on Proposition 1.18, we get

1.
$$\lim_{\varepsilon \to +0} \frac{1}{\pi x} \sin \frac{x}{\varepsilon} = \delta(x) , \qquad 2. \quad \lim_{\varepsilon \to +0} \frac{\varepsilon}{\pi x^2} \sin^2 \frac{x}{\varepsilon} = \delta(x) . \quad (1.388)$$

Thus, for 1., the continuous function $f(x) = (\sin x)/(\pi x), x \in \mathbb{R}$ is considered, resulting in $f(x/\varepsilon)/\varepsilon = (\sin /\varepsilon x)/(\pi x)$.

For 2., the continuous function $g(x) = \sin^2 x / (\pi x^2), x \in \mathbb{R}$ is considered. It follows

$$g_{\varepsilon}(x) = \frac{1}{\varepsilon}g\left(\frac{x}{\varepsilon}\right) = \frac{\varepsilon}{\pi x^2}\sin^2\frac{x}{\varepsilon}, \quad \varepsilon > 0.$$

Another criterion for the Dirac representative sequences is given by the following.

Proposition 1.19 Let there be $f_{\varepsilon} \in L^1_{loc}(\mathbb{R}^n)$, $\varepsilon > 0$, a family of locally integrable functions with the properties:

1. $f_{\varepsilon}(x) \ge 0$, $\forall x \in \mathbb{R}^{n}, \forall \varepsilon > 0$, 2. $\int_{\mathbb{R}^{n}} f_{\varepsilon}(x) dx = 1$, 3. $\forall R > 0$ we have $\lim_{\varepsilon \to +0} \int_{\|x\| \ge R} f_{\varepsilon}(x) dx = 0$;

then

$$\lim_{\varepsilon \to +0} f_{\varepsilon}(x) = \delta(x) \Leftrightarrow f_{\varepsilon}(x) \xrightarrow{\mathcal{D}'(\mathbb{R}^n)} \delta(x) .$$
(1.389)

Proof: Taking into account 1 and 2 for $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\begin{split} |(f_{\varepsilon},\varphi)-\varphi(0)| &= \left| \int_{\mathbb{R}^{n}} f_{\varepsilon}(x)\varphi(x)dx - \varphi(0)\int_{\mathbb{R}^{n}} f_{\varepsilon}dx \right| \\ &= \left| \int_{\mathbb{R}^{n}} f_{\varepsilon}(x)[\varphi(x)-\varphi(0)]dx \right| \leq \int_{\mathbb{R}^{n}} |\varphi(x)-\varphi(0)| f_{\varepsilon}(x)dx \;. \end{split}$$
(1.390)

Based on the continuity of the function $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$ in the origin, we can write $|\varphi(x) - \varphi(0)| < \varepsilon_0/2$, $||x|| < \eta_{\varepsilon_0}$.

Consequently, for the integral on the right-hand side of the relation (1.390) we obtain

$$\int_{\mathbb{R}^{n}} |\varphi(x) - \varphi(0)| f_{\varepsilon}(x) dx = \int_{\|x\| < \eta_{\varepsilon_{0}}} |\varphi(x) - \varphi(0)| f_{\varepsilon}(x) dx$$
$$+ \int_{\|x\| \ge \eta_{\varepsilon_{0}}} |\varphi(x) - \varphi(0)| f_{\varepsilon}(x) dx \le \frac{\varepsilon_{0}}{2} + \int_{\|x\| \ge \eta_{\varepsilon_{0}}} |\varphi(x) - \varphi(0)| f_{\varepsilon}(x) dx .$$
(1.391)

But

$$\int_{\|x\| \ge \eta_{\varepsilon_0}} |\varphi(x) - \varphi(0)| f_{\varepsilon}(x) dx$$

$$\leq \sup_{x \in \mathbb{R}^n} |\varphi(x) - \varphi(0)| \int_{\|x\| \ge \eta_{\varepsilon_0}} f_{\varepsilon}(x) dx = M^* \int_{\|x\| \ge \eta_{\varepsilon_0}} f_{\varepsilon}(x) dx , \quad (1.392)$$

where $M^* = \sup_{x \in \mathbb{R}^n} |\varphi(x) - \varphi(0)| > 0$ (which exists because it is continuous and has compact support).

Substituting (1.392) in (1.391), we obtain

$$\int_{\mathbb{R}^n} |\varphi(x) - \varphi(0)| f_{\varepsilon}(x) \mathrm{d}x < \frac{\varepsilon_0}{2} + M^* \int_{\|x\| \ge \eta_{\varepsilon_0}} f_{\varepsilon}(x) \mathrm{d}x .$$
(1.393)

On the other hand, condition 3 of the proposition means the following: $\forall \varepsilon_0 > 0$, $\exists M_{\varepsilon_0} > 0$ so that for $\varepsilon \leq M_{\varepsilon_0}$ we have

$$\int_{\|x\| \ge \mathbb{R}} f_{\varepsilon}(x) \mathrm{d}x < \frac{\varepsilon_0}{2M^*} .$$
(1.394)

The relation (1.393) becomes

$$\int_{\mathbb{R}^n} |\varphi(x) - \varphi(0)| f_{\varepsilon}(x) \mathrm{d}x < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0 , \quad ||x|| < \eta_{\varepsilon_0} , \quad \text{and } 0 < \varepsilon \le M_{\varepsilon_0} .$$
(1.395)

Taking into account (1.395) and (1.390), it follows that $\forall \varepsilon_0 > 0, \exists \eta_{\varepsilon_0} > 0$ and $M_{\varepsilon_0} > 0$ so that

$$|(f_{\varepsilon}, \varphi) - \varphi(0)| < \varepsilon_0 , \quad \text{for } ||x|| < \eta_{\varepsilon_0} , \quad \text{and } 0 < \varepsilon \le M_{\varepsilon_0} . \tag{1.396}$$

The last relation is equivalent to $\lim_{\varepsilon \to +0} |(f_{\varepsilon}, \varphi) - \varphi(0)| = 0$, thus $\lim_{\varepsilon \to +0} (f_{\varepsilon}, \varphi) = \varphi(0) = (\delta(x), \varphi(x))$, namely $\lim_{\varepsilon \to +0} f_{\varepsilon}(x) = \delta(x)$, and thus the proposition is proved.

We consider the family of functions (1.384), namely

$$f_{\varepsilon}(x, y) = \frac{p-2}{2\pi} \cdot \frac{\varepsilon^{p-2}}{(x^2 + y^2 + \varepsilon^2)^{p/2}}, \quad \varepsilon > 0, p > 2.$$
 (1.397)

We note that $f_{\varepsilon}(x, y) \ge 0$ and, according to (1.383), we have $\int_{\mathbb{R}^2} f_{\varepsilon}(x, y) dx dy = 1$, so that the conditions 1. and 2. of Proposition 1.19 are satisfied. We will show that condition 3. is satisfied, namely $\forall R > 0$ we have

$$\lim_{\varepsilon \to +0} \iint_{r=\sqrt{x^2+y^2} \ge R} f_{\varepsilon}(x,y) \mathrm{d}x \mathrm{d}y = 0.$$
(1.398)

Indeed, we have

$$\iint_{r=\sqrt{x^2+y^2} \ge R} f_{\varepsilon}(x,y) dx dy = \lim_{r_1 \to \infty} \iint_{r_1 \ge r \ge R} f_{\varepsilon}(x,y) dx dy$$
$$= \frac{p-2}{2\pi} \cdot \varepsilon^{p-2} \lim_{r_1 \to \infty} \iint_{R \le r \le r_1} \frac{dx dy}{(x^2+y^2+\varepsilon^2)^{p/2}}.$$
(1.399)

Using the polar coordinates $x = \rho \cos \theta$, $y = \rho \sin \theta$, we obtain

$$\iint_{R < r \le r_1} \frac{\mathrm{d}x \mathrm{d}y}{\left(x^2 + y^2 + \varepsilon^2\right)^{p/2}} = \int_{R}^{r_1} \int_{0}^{2\pi} \frac{\rho \mathrm{d}\rho \mathrm{d}\theta}{\left(\rho^2 + \varepsilon^2\right)^{p/2}}$$
$$= \pi \int_{R}^{r_1} \left(\rho^2 + \varepsilon^2\right)^{-p/2} \left(\rho^2 + \varepsilon^2\right)' \mathrm{d}\rho = \frac{2\pi}{2-p} \left[\left(r_1^2 + \varepsilon^2\right)^{(2-p)/2} - \left(R^2 + \varepsilon^2\right)^{(2-p)/2} \right].$$
(1.400)

Substituting in (1.399) we have

$$\iint_{\substack{r=\sqrt{x^2+y^2} \ge R}} f_{\varepsilon}(x,y) dx dy = \frac{p-2}{2\pi} \varepsilon^{p-2} \cdot \frac{2\pi}{2-p} \left[-(R^2 + \varepsilon^2)^{(2-p)/2} \right]$$
$$= \varepsilon^{p-2} (R^2 + \varepsilon^2)^{(2-p)/2} < \varepsilon^{p-2} (R^2)^{(2-p)/2} = \left(\frac{\varepsilon}{R}\right)^{p-2}, \quad p > 2, \quad (1.401)$$

giving $\lim_{\epsilon \to +0} \iint_{r \ge R} f_{\epsilon}(x, \gamma) dx d\gamma \le 0$, hence $\lim_{\epsilon \to +0} \iint_{r \ge R} f_{\epsilon}(x, \gamma) dx d\gamma = 0$ because $f_{\varepsilon} \geq 0$.

We showed that the conditions of Proposition 1.19 are fulfilled, hence $\lim_{\epsilon \to +0}$ $f_{\varepsilon}(x, y) = \delta(x, y)$, a result that was obtained using Proposition 1.18.

Example 1.23 Let there be the sequence $(f_n(x))_{n\geq 1}$ (Figure 1.6) where

$$f_n(x) = \begin{cases} n - n^2 x/2, & x \in [0, 2/n], \\ 0, & x \notin [0, 2/n]. \end{cases}$$
(1.402)

We note that the three conditions of Proposition 1.19 are satisfied.

Indeed, $f_n(x) \ge 0$, $\int_{\mathbb{R}} f_n(x) dx = 1$ and $\lim_{n \to \infty} \int_{|x| \ge r>0} f_n(x) dx = 0$, $\forall r > 0$ 0, because

$$\int_{|x|\ge r>0} f_n(x)\mathrm{d}x = \int_{-\infty}^r f_n(x)\mathrm{d}x + \int_r^\infty f_n(x)\mathrm{d}x = \int_r^\infty f_n(x)\mathrm{d}x , \qquad (1.403)$$

and for 2/n < r it results $\int_r^{\infty} f_n(x) dx = 0$, hence $\lim_{n \to \infty} \int_{|x| \ge r>0} f_n(x) dx = 0$. Consequently, we have $\lim_{n\to\infty} f_n(x) = \delta(x)$.





Example 1.24 Let there be $f_{\varepsilon} \in L^{1}_{loc}(\mathbb{R}), \varepsilon > 0$ a family of locally integrable functions, where $f_{\varepsilon}(x) = (2H(x)/\pi) \cdot (\varepsilon/(x^{2} + \varepsilon^{2})), \varepsilon > 0$ (Figure 1.7), and *H* is the Heaviside function.

We shall show that $f_{\varepsilon} \xrightarrow{\mathcal{D}'(\mathbb{R})} {\varepsilon \to +0} \delta(x)$.

Indeed, we note that the first two conditions of Proposition 1.19 are fulfilled because:

1.
$$f_{\varepsilon} \ge 0$$
, $\forall \varepsilon > 0$, $\forall x \in \mathbb{R}$, $f_{\varepsilon} \in L^{1}_{loc}(\mathbb{R})$;

2.
$$\int_{\mathbb{R}} f_{\varepsilon}(x) dx = \int_{0}^{\infty} \frac{2\varepsilon}{\pi (x^{2} + \varepsilon^{2})} dx = \frac{2\varepsilon}{\pi} \cdot \frac{1}{\varepsilon} \arctan \left. \frac{x}{\varepsilon} \right|_{0}^{\infty} = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1.$$

Regarding the third condition, we have

$$\lim_{\substack{\varepsilon \to +0 \\ \forall R > 0}} \int_{|x| \ge R} f_{\varepsilon}(x) dx = \lim_{\varepsilon \to +0} \int_{R}^{\infty} \frac{2\varepsilon}{\pi (x^2 + \varepsilon^2)} dx$$
$$= \lim_{\varepsilon \to +0} \frac{2\varepsilon}{\pi} \cdot \frac{1}{\varepsilon} \arctan \frac{x}{\varepsilon} \Big|_{R}^{\infty} = \lim_{\varepsilon \to +0} \frac{2}{\pi} \left(\frac{\pi}{2} - \arctan \frac{R}{\varepsilon} \right) = 0. \quad (1.404)$$

Thus, the conditions of Proposition 1.19 being fulfilled, we have $\lim_{\epsilon \to +0} f_{\epsilon}(x) = \delta(x)$.

Example 1.25 Let there be the sequence of locally integrable functions $f_n(x) = nH(x)e^{-nx}$, $x \in \mathbb{R}$ (Figure 1.8). We shall show that $\lim_{n \to +\infty} f_n(x) = \delta(x)$.



Figure 1.7



Figure 1.8

We will use Proposition 1.19, because the function f_n is discontinuous at the origin. We note that $f_n(x) \ge 0$, $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$. Also

$$\int_{\mathbb{R}} f_n(x) dx = \int_{0}^{\infty} n e^{-nx} dx = \left. \frac{n e^{-nx}}{-n} \right|_{0}^{\infty} = -e^{-nx} |_{0}^{\infty} = 1.$$
(1.405)

Thus, the first two conditions of the proposition are met. We will check the third condition, namely $\lim_{\substack{n\to\infty\\\forall R>0}} \int_{|x|\geq R} f_n(x) dx = 0$. We can write

$$\lim_{n \to \infty} \int_{|x| \ge R} f_n(x) dx = \lim_{n \to \infty} \int_R^\infty n e^{-nx} dx = \lim_{n \to \infty} \left. \frac{n e^{-nx}}{-n} \right|_R^\infty$$
$$= \lim_{n \to \infty} \left[-e^{-nx} \right]_R^\infty = \lim_{n \to \infty} e^{-nR} = 0. \quad (1.406)$$

Consequently, the conditions of the Proposition 1.19 are fulfilled, so we have $\lim_{n\to\infty} f_n(x) = \delta(x)$.

Example 1.26 We shall show that the sequence $(g_n(x))_{n\geq 1}, g_n(x) = xn^2 H(x)e^{-nx}$, $x \in \mathbb{R}$ is a Dirac representative sequence, hence $\lim_{n\to\infty} g_n(x) = \delta(x)$.

Indeed, $f_n(x) \ge 0, \forall x \in \mathbb{R}$, and we have

$$\int_{\mathbb{R}} f_n(x) dx = n^2 \int_0^\infty x e^{-nx} dx = \frac{n^2}{-n} \int_0^\infty x (e^{-nx})' dx$$
$$= -n \left[x e^{-nx} \Big|_0^\infty - \int_0^\infty e^{-nx} dx \right] = n \int_0^\infty e^{-nx} dx = \left. \frac{n e^{-nx}}{-n} \right|_0^\infty = 1 \,. (1.407)$$

As regards the third condition of the Proposition 1.19 we have

$$\lim_{\substack{n \to \infty \\ \forall R > 0}} \int_{|x| \ge R} g_n(x) dx = \lim_{n \to \infty} n^2 \int_R^\infty x e^{-nx} dx$$
$$= \lim_{n \to \infty} (-n) \left[x e^{-nx} |_R^\infty - \int_R^\infty e^{-nx} dx \right]$$
$$= \lim_{n \to \infty} (-n) \left[-R \exp(-Rn) - \frac{\exp(-nx)}{-n} \Big|_R^\infty \right]$$
$$= \lim_{n \to \infty} \left[e^{-Rn} + Rn e^{-Rn} \right] = 0.$$
(1.408)

With this, the conditions of the Proposition 1.19 are fulfilled, so we have $\lim_{n\to\infty} g_n(x) = \delta(x)$.

Below, we will give another criterion for Dirac representative sequences that complement the Proposition 1.19.

Proposition 1.20 Let there be the sequence of functions $(f_i(x))_{i \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$, which satisfies the conditions:

1.
$$f_i(x) \ge 0$$
, $\forall i \in \mathbb{N}$, $\forall x \in \mathbb{R}^n$;

2.
$$\int_{\mathbb{R}^n} f_i(x) \mathrm{d}x = 1 , \quad \forall i \in \mathbb{N} ;$$

3.
$$\forall i \in \mathbb{N}, \operatorname{supp}(f_i) \subset \overline{U}_{\varepsilon_i} = \{x | x \in \mathbb{R}^n, \|x\| \le \varepsilon_i\} \text{ and } \lim_{i \to \infty} \varepsilon_i = 0$$

Then $\lim_{i\to\infty} f_i(x) = \delta(x)$.

Proof: For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$|(f_i,\varphi)-\varphi(0)| \leq \int_{\overline{U}_{\varepsilon_i}} f_i(x)|\varphi(x)-\varphi(0)|dx \leq \sup_{\|x\| \leq \varepsilon_i} |\varphi(x)-\varphi(0)|. \quad (1.409)$$

On the basis of the uniform continuity of the function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and because if $i \to \infty$, then $\varepsilon_i \to 0$, we have $\sup_{\|x\| \le \varepsilon_i} |\varphi(x) - \varphi(0)| \to 0$, hence $\lim_i (f_i, \varphi) = \varphi(0) = (\delta(x), \varphi(x))$, namely $\lim_i f_i(x) = \delta(x)$.

Example 1.27 Let there be the family of sequences $\rho_{\varepsilon}(x)$, $\varepsilon > 0$, $x \in \mathbb{R}^{n}$, namely

$$\rho_{\varepsilon}(x) = \begin{cases} c_{\varepsilon} \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - \|x\|^2}\right), & \|x\| < \varepsilon, \\ 0, & \|x\| \ge \varepsilon, \end{cases}$$
(1.410)

where the constant c_{ε} has the expression

$$c_{\varepsilon} = \varepsilon^{-n} \left[\int_{\|x\| \le 1} \exp\left(-\frac{1}{1 - \|x\|^2}\right) \mathrm{d}x \right]^{-1}.$$

We observe that the function $\rho_{\varepsilon} \geq 0$, $\varepsilon > 0$, is a test function of Schwartz's space, hence $\rho_{\varepsilon} \in \mathcal{D}(\mathbb{R}^n)$. The support of the function ρ_{ε} is the closed ball $\overline{B}_{\varepsilon} = \{x | x \in \mathbb{R}^n, \|x\| \le \varepsilon\}$, hence $\sup(\rho_{\varepsilon}) = \overline{B}_{\varepsilon}$.

Due to the value chosen for the constant c_{ε} , the function $\rho_{\varepsilon}(x)$ has the property $\int_{\mathbb{R}^n} \rho_{\varepsilon}(x) dx = 1$. Consequently, the family of functions $\rho_{\varepsilon}, \varepsilon > 0$, has the properties $\rho_{\varepsilon} \ge 0, \varepsilon > 0, \int_{\mathbb{R}^n} \rho_{\varepsilon}(x) dx = 1$ and $\operatorname{supp}(\rho_{\varepsilon}) \subset \overline{B}_{\varepsilon}$, hence $\varepsilon \to +0$ involves $\operatorname{supp}(\rho_{\varepsilon}) \to 0$.

Thus, the conditions of the Proposition 1.20 are fulfilled and we can write $\lim_{\varepsilon \to +0} \rho_{\varepsilon}(x) = \delta(x)$; namely $\rho_{\varepsilon}, \varepsilon > 0$, is a family of Dirac representative sequences, but from the space of test functions $\mathcal{D}(\mathbb{R}^n)$.

Example 1.28 Let there be the family of sequences of polynomials $L_{\varepsilon}(x), \varepsilon > 0, x \in$ \mathbb{R}^n , where

$$L_{\varepsilon}(x) = \begin{cases} \frac{1}{c_{\varepsilon}} \left(\varepsilon^{2} - \|x\|^{2}\right)^{p}, & \|x\| \leq \varepsilon, \quad p \in \mathbb{N} \text{ fixed }, \\ 0, & \|x\| > \varepsilon, \end{cases}$$
(1.411)

and $c_{\varepsilon} = \int_{\|x\| < \varepsilon} (\varepsilon^2 - \|x\|^2)^p \mathrm{d}x.$

We shall show that $\lim_{\epsilon \to +0} L_{\epsilon}(x) = \delta(x)$.

Indeed, $L_{\varepsilon} \geq 0, \forall \varepsilon > 0, \int_{\mathbb{R}^n} L_{\varepsilon}(x) dx = 1$, $\operatorname{supp}(L_{\varepsilon}(x)) = \overline{U}_{\varepsilon} = \{x | x \in \mathbb{R}^n, t \in \mathbb{R}^n\}$ $||x|| \leq \varepsilon$ and $\lim_{\varepsilon \to +0} \operatorname{supp}(L_{\varepsilon}(x)) = 0$.

The conditions of the proposition are fulfilled, hence $\lim_{\varepsilon \to +0} L_{\varepsilon}(x) = \delta(x)$.

We note that $L_{\varepsilon}(x)$ are polynomials of degree 2*p*, but with compact support, which tends to zero when $\varepsilon \rightarrow 0$ [20, p. 51].

1.3.7

Distributions Depending on a Parameter

1.3.7.1 Differentiation of Distributions Depending on a Parameter

The different quantities encountered in the mathematical-physical problems are generally functions of space variable $x \in \mathbb{R}^n$, but they may also depend on certain parameters of real or complex variable, such as the temporal variable $t \in I \subset \mathbb{R}$.

This requires considerations on distributions depending on a real or complex parameter $t \in \Omega \subset \mathbb{C}^m$.

In the following, we consider the real parameter t, hence $t = (t_1, t_2, \ldots, t_m) \in$ $\Omega \subset \mathbb{R}^m$. For example, the Dirac delta distribution $\delta(x-t) = \delta(x_1 - t_1, x_2 - t_2)$ $t_2, x_3 - t_3 \in \mathcal{D}'(\mathbb{R}^3)$ depends on the real parameter $t = (t_1, t_2, t_3) \in \mathbb{R}^3$.

If for any $t \in \Omega \subset \mathbb{R}^m$ we can associate, after a certain rule, a single distribution $f_t(x) \in \mathcal{D}'(\mathbb{R}^n)$, we say that this distribution depends on the real parameter $t \in \mathcal{D}'(\mathbb{R}^n)$ $\Omega \subset \mathbb{R}^m$.

Definition 1.30 We say that the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is the limit of the distribution $f_t \in \mathcal{D}'(\mathbb{R}^n)$, $t \in \Omega \subset \mathbb{R}^m$, when $t \to t_0$, t_0 being the accumulation point for $\Omega \subset \mathbb{R}^m$, and we write $\lim_{t\to t_0} f_t(x) = f(x)$, if $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ the function $(f_t(x), \varphi(x)), t \in \Omega \subset \mathbb{R}^m$, converges to $(f(x), \varphi(x))$, that is, we have

$$\lim_{t \to t_0} (f_t(x), \varphi(x)) = (f(x), \varphi(x)) .$$
(1.412)

Proposition 1.21 Let there be the distributions $f_t, g_t \in \mathcal{D}'(\mathbb{R}^n)$ depending on the parameter $t \in \Omega \subset \mathbb{R}^m$. If $\lim_{t \to t_0} f_t(x)$, $\lim_{t \to t_0} g_t(x) \in \mathcal{D}'(\mathbb{R}^n)$ exist, then we have

$$\lim_{t \to t_0} (\alpha f_t + \beta g_t) = \alpha \lim_{t \to t_0} f_t + \beta \lim_{t \to t_0} g_t , \quad \forall \alpha, \beta \in \mathbb{R} .$$
(1.413)

Indeed, noting $\lim_{t\to t_0} f_t(x) = a(x)$, $\lim_{t\to t_0} g_t(x) = b(x)$, $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, we can write

$$\lim_{t \to t_0} (\alpha f_t(x) + \beta g_t(x), \varphi(x)) = \lim_{t \to t_0} [\alpha (f_t, \varphi) + \beta (g_t, \varphi)]$$

= $\alpha \lim_{t \to t_0} (f_t, \varphi) + \beta \lim_{t \to t_0} (g_t, \varphi)$
= $\alpha (a(x), \varphi) + \beta (b(x), \varphi) = (\alpha a(x) + \beta b(x), \varphi)$, (1.414)

which leads to (1.413).

Definition 1.31 The distribution $f_t \in \mathcal{D}'(\mathbb{R}^n)$, $t \in \Omega \subset \mathbb{R}^m$, is continuous with respect to the parameter *t* on the set $\Omega \subset \mathbb{R}^m$ if $\forall t_0 \in \Omega$, we have $\lim_{t \to t_0} f_t(x) = f_{t_0}(x)$, hence

$$\lim_{t \to t_0} (f_t(x), \varphi(x)) = (f_{t_0}(x), \varphi(x)), \forall \varphi \in \mathcal{D}(\mathbb{R}^n) .$$
(1.415)

From the definition of continuity and the limit property with respect to the parameter $t \in \Omega \subset \mathbb{R}^m$, it follows that $\forall \alpha, \beta \in \mathbb{R}$ the distribution $\alpha f_t(x) + \beta g_t(x)$ is continuous on Ω if the distributions $f_t(x), g_t(x) \in \mathcal{D}'(\mathbb{R}^n)$ are continuous on Ω .

Definition 1.32 Let $f_t \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution depending on the parameter $t \in \Omega \subset \mathbb{R}^m$. We call derivative of the distribution f_t with respect to $t_j \in \mathbb{R}$, $j = \overline{1, m}, t = (t_1, t_2, \dots, t_m)$, the distribution $\partial f_t(x)/\partial t_j \in \mathcal{D}'(\mathbb{R}^n)$, defined by

$$\frac{\partial}{\partial t_j} f_t(x) = \lim_{\Delta t_j \to 0} \frac{f_{(t_1, t_2, \dots, t_j + \Delta t_j, \dots, t_m)}(x) - f_{(t_1, t_2, \dots, t_j, \dots, t_m)}(x)}{\Delta t_j} , \qquad (1.416)$$

if the limit exists and is unique.

This means that $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\begin{pmatrix} \frac{\partial}{\partial t_j} f_t(x), \varphi(x) \end{pmatrix}$$

$$= \lim_{\Delta t_j \to 0} \frac{(f_{(t_1, \dots, t_j + \Delta t_j, \dots, t_m)}(x), \varphi(x)) - (f_{(t_1, \dots, t_j, \dots, t_m)}(x), \varphi(x))}{\Delta t_j}$$

$$= \frac{\partial}{\partial t_j} \psi(t_1, \dots, t_j, \dots, t_m) = \frac{\partial}{\partial t_j} (f_t(x), \varphi(x)) ,$$

$$(1.417)$$

where

$$\psi(t_1,\ldots,t_j,\ldots,t_m) = (f_t(x),\varphi(x)), \quad t \in \Omega \subset \mathbb{R}^m.$$
(1.418)

Proposition 1.22 The necessary and sufficient condition that the derivative $\partial f_t(x)/\partial t_j \in \mathcal{D}'(\mathbb{R}^n)$ does exist is that the function $\psi(t) = (f_t(x), \varphi(x))$ be differentiable with respect to the variable t_j , $j = \overline{1, m}$.

We note that the existence of the limit (1.416) implies the existence of the limit

$$\lim_{\Delta t_{j} \to 0} \left(\frac{f_{(t_{1},\dots,t_{j}+\Delta t_{j},\dots,t_{m})}(x) - f_{(t_{1},\dots,t_{j},\dots,t_{m})}(x)}{\Delta t_{j}},\varphi(x) \right)$$
(1.419)

and, on the basis of the completeness theorem of the distribution space $\mathcal{D}'(\mathbb{R}^n)$, it defines a distribution depending on the parameter $t \in \Omega \subset \mathbb{R}^m$.

Consequently, $\partial f_t(x)/\partial t_i$ is a distribution from $\mathcal{D}'(\mathbb{R}^n)$ depending on the parameter $t \in \Omega$.

Proposition 1.23 If the derivative $\partial f_t(x)/\partial t_i \in \mathcal{D}'(\mathbb{R}^n)$, $t \in \Omega \subset \mathbb{R}^m$, then $(\partial/\partial t_i)$ $(\partial f_t(x)/\partial x_i)$ exists and the following formula occurs

$$\frac{\partial}{\partial t_j} \left(\frac{\partial f_t(x)}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial t_j} f_t(x) \right) , \quad i = \overline{1, n} .$$
(1.420)

Proof: For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\left(\frac{\partial f_t(x)}{\partial x_i},\varphi(x)\right) = \left(f_t(x), -\frac{\partial\varphi(x)}{\partial x_i}\right), \quad t \in \Omega \subset \mathbb{R}^m.$$
(1.421)

From the existence of the derivative $\partial f_t(x)/\partial t_i$ and taking into account (1.417) it results that the function defined by (1.418) is differentiable with respect to t_i ; hence we get

$$\frac{\partial}{\partial t_j} \left(f_t(x), -\frac{\partial \varphi}{\partial x_i} \right) = \left(\frac{\partial}{\partial t_j} f_t(x), -\frac{\partial \varphi}{\partial x_i} \right) = \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial t_j} f_t(x), \varphi(x) \right)$$
$$= \frac{\partial}{\partial t_j} \left(\frac{\partial}{\partial x_i} f_t, \varphi \right) = \left(\frac{\partial}{\partial t_j} \frac{\partial}{\partial x_i} g_t, \varphi \right), \qquad (1.422)$$

wherefrom we obtain the relation (1.420).

Proposition 1.24 Let there be the distribution $f_t \in \mathcal{D}'(\mathbb{R}^n), t \in \Omega \subset \mathbb{R}^m$. If $\lim_{t\to t_0} f_t(x)$ exists, then

$$\lim_{t \to t_0} \left(\frac{\partial}{\partial x_i} f_t(x) \right) = \frac{\partial}{\partial x_i} \left[\lim_{t \to t_0} f_t(x) \right].$$
(1.423)

Indeed, if we note $\lim_{t\to t_0} f_t(x) = a(x) \in \mathcal{D}'(\mathbb{R}^n)$, then $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ we have $\lim_{t\to t_0} (f_t(x), \varphi(x)) = (a(x), \varphi(x))$, and, consequently, we obtain

$$\lim_{t \to t_0} \left(\frac{\partial}{\partial x_i} f_t(x), \varphi(x) \right) = \lim_{t \to t_0} \left(f_t(x), -\frac{\partial \varphi(x)}{\partial x_i} \right)$$
$$= \left(a(x), -\frac{\partial \varphi(x)}{\partial x_i} \right) = \left(\frac{\partial a(x)}{\partial x_i}, \varphi(x) \right) , \qquad (1.424)$$

namely

$$\lim_{t \to t_0} \frac{\partial}{\partial x_i} f_t(x) = \frac{\partial a(x)}{\partial x_i} = \frac{\partial}{\partial x_i} \left[\lim_{t \to t_0} f_t(x) \right].$$
(1.425)

Proposition 1.25 Let there be the distribution $f_t(x) = F(u) \in \mathcal{D}'(\mathbb{R})$, $u = ax + \alpha(t)$, $x \in \mathbb{R}$, where $a \in \mathbb{R} \setminus \{0\}$ and $\alpha \in C^1(I)$, $I \subset \mathbb{R}$. We have

$$\frac{\partial}{\partial t}F(ax+\alpha(t)) = \alpha'(t)F'(ax+\alpha(t)), \qquad (1.426)$$

$$\frac{\partial^2}{\partial t \partial x} F(ax + \alpha(t)) = a\alpha'(t)F''(ax + \alpha(t)).$$
(1.427)

Example 1.29 Let there be the distributions $f_t(x) = \delta(x - at) \in \mathcal{D}'(\mathbb{R})$, $g_t(x) = H(x - bt) \in \mathcal{D}'(\mathbb{R})$ depending on the parameter $t \in \mathbb{R}$, where $a, b \in \mathbb{R}$, and H(u) is the Heaviside distribution. We have

$$\frac{\partial}{\partial t}\delta(x-at) = -a\delta'(x-at), \quad \frac{\partial}{\partial t}H(x-bt) = -b\delta(x-bt). \quad (1.428)$$

Indeed, the relations (1.428) are obtained directly by applying the formula (1.426), because $H'(u) = \delta(u)$.

1.3.7.2 Integration of Distributions Depending on a Parameter

For the distributions depending on a real parameter $t \in I \subset \mathbb{R}$ we can define the integral with respect to the corresponding parameter.

Let there be $f_t \in \mathcal{D}'(\mathbb{R}), t \in I \subset \mathbb{R}$, a distribution depending on the real parameter *t*. If the distribution f_t is continuous on $I \subset \mathbb{R}$ with respect to the parameter *t*, then, according to the continuity definition, $\forall \varphi \in \mathcal{D}(\mathbb{R})$, the function $\psi : I \subset \mathbb{R} \to \mathbb{R}$, having the expression,

$$\psi(t) = (f_t(x), \varphi(x)),$$
 (1.429)

is continuous on *I*.

Consequently, the functional $F : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$, defined by

$$(F(x),\varphi(x)) = \int_a^b \psi(t) \mathrm{d}t = \int_a^b (f_t(x),\varphi(x)) \mathrm{d}t , \quad t \in [a,b] \subset I , \qquad (1.430)$$

exists $\forall \varphi \in \mathcal{D}(\mathbb{R})$ and it represents a distribution from $\mathcal{D}'(\mathbb{R})$.

Indeed, according to the definition integral (1.430) we can write

$$(F,\varphi) = \int_{a}^{b} \psi(t) dt = \lim_{\nu(\pi) \to 0} \sum_{i=1}^{n} \psi(\tau_{i}) \Delta t_{i} = \lim_{\nu(\pi) \to 0} \sum_{i=1}^{n} (f_{\tau_{i}}(x), \varphi(x)) \Delta t_{i} ,$$
(1.431)

where $\pi = \{t_0 = a, t_1, \dots, t_i, \dots, t_n = b\}$ is a partition of the interval $[a, b] \subset I$, with the norm $\nu(\pi) = \max_{1 \le i \le n} \Delta t_i, \Delta t_i = t_i - t_{i-1}$, and $\tau_i \in [t_{i-1}, t_i], i = \overline{1, n}$, are the intermediary points of the partition. Denoting $\sigma_{\pi}(f_t) \in \mathcal{D}'(\mathbb{R})$ the distribution depending on the parameter $t \in [a, b]$, namely $\sigma_{\pi}(f_t) = \sum_{i=1}^{n} f_{\tau_i}(x) \Delta t_i$, then we have

$$(\sigma_{\pi}(f_t),\varphi(x)) = \sum_{i=1}^n (f_{\tau_i}(x)\Delta t_i,\varphi(x)) = \sum_{i=1}^n \psi(\tau_i)\Delta t_i , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}) .$$
(1.432)

Thus, (1.431) becomes

$$(F,\varphi) = \lim_{\nu(\pi)\to 0} (\sigma_{\pi}(f_t),\varphi(x)) = \int_{a}^{b} \psi(t)dt = \int_{a}^{b} (f_t(x),\varphi(x))dt .$$
(1.433)

Because, the limit of (1.433) exists, according to the theorem of completeness of the distribution space $\mathcal{D}'(\mathbb{R})$, we obtain

$$\lim_{\nu(\pi) \to 0} \sigma_{\pi}(f_t(x)) = F(x) , \qquad (1.434)$$

hence the functional F(x) is a distribution from $\mathcal{D}'(\mathbb{R})$.

The distribution $F \in \mathcal{D}'(\mathbb{R})$ is denoted by

$$F(x) = \int_{a}^{b} f_{t}(x) dt , \qquad (1.435)$$

and will be called the integral of the distribution $f_t \in \mathcal{D}'(\mathbb{R})$ depending on the parameter $t \in [a, b] \subset I \subset \mathbb{R}$.

Obviously, the distribution (1.435) exists if $f_t \in \mathcal{D}'(\mathbb{R})$ is continuous for $t \in [a, b]$, and its mode of action on the test functions space $\mathcal{D}(\mathbb{R})$ is given by the formula (1.430), that is,

$$\left(\int_{a}^{b} f_{t}(x) \mathrm{d}t, \varphi(x)\right) = \int_{a}^{b} (f_{t}(x), \varphi(x)) \mathrm{d}t , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}) .$$
(1.436)

We note that the distribution (1.435) exists even if the distribution $f_t \in \mathcal{D}'(\mathbb{R})$ is not continuous, but the function ψ defined by (1.429) is integrable on $[a, b] \subset I$.

Proposition 1.26 Let there be the distribution $f_t \in \mathcal{D}'_+$ continuous for $t \in [a, b] \subset \mathbb{R}$. Then, we have $F(x) = \int_a^b f_t(x) dt \in \mathcal{D}'_+$.

Proposition 1.27 If the distribution $f_t \in \mathcal{D}'(\mathbb{R})$ is continuous on $[a, b] \subset \mathbb{R}$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{b}f_{t}(x)\mathrm{d}t = \int_{a}^{b}\frac{\partial}{\partial x}f_{t}(x)\mathrm{d}t . \qquad (1.437)$$

Proposition 1.28 Let there be the distribution $f \in \mathcal{D}'(\mathbb{R})$ and the integrable function $g : [a, b] \subset \mathbb{R} \to \mathbb{R}$. Then, for the distribution $f_t(x) = f(x)g(t) \in \mathcal{D}'(\mathbb{R})$ depending on the parameter $t \in [a, b]$, the following results:

$$\int_{a}^{b} f_{t}(x) dt = \int_{a}^{b} f(x)g(t) dt = f(x) \int_{a}^{b} g(t) dt .$$
 (1.438)

Example 1.30 Let there be the distributions $\delta(x - t)$, $H(x - t) \in \mathcal{D}'(\mathbb{R})$, depending on the parameter $t \in \mathbb{R}$, where *H* is the Heaviside distribution. The following relations take place

$$F_{1}(x) = \int_{a}^{b} H(x-t) dt = \begin{cases} 0, & x \le a < b, \\ b-a, & a < b < x, \\ x-a, & a \le x \le b, \end{cases}$$
(1.439)

$$F_2(x) = \int_a^b \delta(x-t) dt = H(x-a) - H(x-b) = \begin{cases} 1, & x \in [a, b], \\ 0, & x \notin [a, b]. \end{cases}$$
(1.440)

Obviously, $H(x - t) \in L^1_{loc}(\mathbb{R})$, and, using the definition of the Heaviside function, we obtain

$$F_1(x) = \int_a^b H(x-t) dt = \begin{cases} 0, & x \le a < b, \\ b-a, & a < b < x, \\ x-a, & a \le x \le b. \end{cases}$$

As regards the formula (1.440), this is obtained by applying the definition of the integral of a distribution depending on a parameter.

Thus, $\forall \varphi \in \mathcal{D}(\mathbb{R})$, we can write

$$\left(\int_{a}^{b} \delta(x-t) \mathrm{d}t, \varphi(x)\right) = \int_{a}^{b} \left(\delta(x-t), \varphi(x)\right) \mathrm{d}t = \int_{a}^{b} \varphi(t) \mathrm{d}t \;. \tag{1.441}$$

Because φ has compact support, the expression (1.441) may be written as

$$\int_{a}^{b} \varphi(t)dt = \int_{a}^{\infty} \varphi(t)dt - \int_{b}^{\infty} \varphi(t)dt = \int_{0}^{\infty} \varphi(x+a)dx - \int_{0}^{\infty} \varphi(x+b)dx$$
$$= \int_{0}^{\infty} [\varphi(x+a) - \varphi(x+b)]dx = (H(x), \varphi(x+a) - \varphi(x+b)). \quad (1.442)$$

Consequently, (1.441) becomes

$$\left(\int_{a}^{b} \delta(x-t) dt, \varphi(x)\right) = (H(x), \varphi(x+a) - \varphi(x+b))$$
$$= (H(x-a) - H(x-b), \varphi(x)), \qquad (1.443)$$

wherefrom results (1.440).

We note that the main properties of the defined integral are maintained for distributions depending on a parameter.

Proposition 1.29 Let there be the distributions $f_t, g_t \in \mathcal{D}'(\mathbb{R})$ depending on the real parameter $t \in [a, b] \subset \mathbb{R}$.

1. If $\partial f_t / \partial t \in \mathcal{D}'(\mathbb{R})$ exists, then we have

$$\int_{a}^{b} \frac{\partial}{\partial t} f_t(x) \mathrm{d}t = f_t(x)|_a^b = f_b(x) - f_a(x) , \qquad (1.444)$$

which is an analogue of the Leibniz-Newton formula.

2. If $f_t \in \mathcal{D}'(\mathbb{R})$ is continuous on $[a, b] \subset \mathbb{R}$, $c \in [a, b]$, we have

$$\int_{a}^{b} f_{t}(x) dt = \int_{a}^{c} f_{t}(x) dt + \int_{c}^{b} f_{t}(x) dt , \qquad (1.445)$$

$$\int_{a}^{b} f_{t}(x) dt = -\int_{b}^{a} f_{t}(x) dt .$$
(1.446)

Particularly, $\int_{a}^{a} f_{t}(x) dt = 0$ and

$$\frac{\partial}{\partial t} \int_{a}^{t} f_{u}(x) \mathrm{d}u = f_{t}(x) , \quad t \in [a, b] .$$
(1.447)

Example 1.31 The relation (1.440) is obtained by applying the formula (1.444). Indeed, because $\partial H(x - t)/\partial t = -H'(x - t) = -\delta(x - t)$, we have

$$F_2(x) = \int_a^b \delta(x-t) dt = -\int_a^b \frac{\partial}{\partial t} H(x-t) dt$$
$$= -H(x-t) \Big|_a^b = H(x-a) - H(x-b).$$
(1.448)

1.3.8

Direct Product and Convolution Product of Functions and Distributions

The direct (or tensor) product of two distributions is a new operation with distributions which extends the usual product of two functions.

Let there be \mathbb{R}^n and \mathbb{R}^m two Euclidean spaces with the dimensions n, m, respectively, and let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, ..., y_m) \in \mathbb{R}^m$ be points of these spaces. Then the Cartesian product of these spaces is $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ and represents a new n + m-dimensional Euclidean space with generic point $(x, y) \in \mathbb{R}^{n+m}$.

Let *f* and *g* be two complex functions defined on \mathbb{R}^n and \mathbb{R}^m , respectively, with the generic points $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.

Definition 1.33 The function $f \times g : \mathbb{R}^{n+m} \to \Gamma$, defined by the relation $(f \times g)(x, y) = f(x)g(y)$ is called direct or tensor product of the function f by the function g.

So, the direct product of two numerical functions coincides with their usual product.

Proposition 1.30 Let there be the functions $f \in C^k(\mathbb{R}^n)$, $g \in C^k(\mathbb{R}^m)$. Then, the following properties occur

- 1. $f \times g \in C^k(\mathbb{R}^{n+m});$
- 2. $D_x^p D_y^q (f(x) \times g(x)) = D_x^p f \times D_y^q g$, $|p| \le k$, $|q| \le k$, where D_x^p, D_y^q are derivation operators of orders |p|, |q|, respectively, in relation to the variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$;
- 3. $\operatorname{supp}(f \times g) = \operatorname{supp}(f) \times \operatorname{supp}(g)$.

Proof: Indeed, we have $D_x^p D_y^q (f \times g)(x, y) = D_x^p f(x) D_y^q g(y) = (D_x^p f \times D_y^q g)(x, y)$, wherefrom results the properties 1. and 2.

Let there be $(x_0, y_0) \in \text{supp}(f \times g)$. Then $\forall U_r(x_0, y_0), \exists (x, y) \in U_r(x_0, y_0)$, so that $(f \times g)(x, y) \neq 0$ which implies $f(x) \neq 0$ and $g(y) \neq 0$, hence $x_0 \in \text{supp}(f)$ and $y_0 \in \text{supp}(g)$, wherefrom results 3.

We note with $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^m)$, $\mathcal{D}(\mathbb{R}^n \times \mathbb{R}^m)$ the indefinitely derivable test functions spaces with compact support on \mathbb{R}^n , \mathbb{R}^m , \mathbb{R}^{n+m} , and with $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^m)$, $\mathcal{D}'(\mathbb{R}^{n+m})$ (the space \mathcal{D}' of corresponding distributions).

We note that $\mathcal{D}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^m)$ is a vector subspace of $\mathcal{D}(\mathbb{R}^{n+m})$, generated by functions of the form $u \times v$, $u \in \mathcal{D}(\mathbb{R}^n)$, $v \in \mathcal{D}(\mathbb{R}^m)$.

Proposition 1.31 The space $\mathcal{D}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^m)$ is dense in $\mathcal{D}(\mathbb{R}^{n+m})$.

This means that $\forall \varphi(x, y) \in \mathcal{D}(\mathbb{R}^{n+m})$, there exists the sequence of functions $(\varphi_i(x, y))_{i \in \mathbb{N}}$ of the form $\varphi_i(x, y) = \sum_{k=1}^p u_{ik}(x)v_{ik}(y)$, with $u_{ik} \in \mathcal{D}(\mathbb{R}^n)$, $v_{ik} \in \mathcal{D}(\mathbb{R}^m)$ so that $\varphi_i \xrightarrow{\mathcal{D}(\mathbb{R}^{n+m})} \varphi$.

This result can be generalized, so that $\mathcal{D}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^m) \times \mathcal{D}(\mathbb{R}^\ell)$ is a vector subspace of $\mathcal{D}(\mathbb{R}^{n+m+\ell})$ and is dense in it.

Let *f* and *g* be locally integrable functions on \mathbb{R}^n and \mathbb{R}^m , respectively. Then, their direct product $h(x, y) = f(x)g(y), (x, y) \in \mathbb{R}^{n+m}$ is a locally integrable function, which generates a regular distribution on the test functions space $\mathcal{D}(\mathbb{R}^{n+m})$. Then, $\forall \varphi \in \mathcal{D}(\mathbb{R}^{n+m})$, on the basis of Fubini's theorem of interchange of the order of integration, we can write

$$(f(x) \times g(y), \varphi(x, y)) = \int_{\mathbb{R}^{n+m}} f(x)g(y)\varphi(x, y)dxdy$$

=
$$\int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^m} g(y)\varphi(x, y)dy = (f(x), (g(y), \varphi(x, y))), \qquad (1.449)$$

namely

$$(f(x) \times g(y), \varphi(x, y)) = (f(x), (g(y), \varphi(x, y))), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^{n+m}).$$
(1.450)

This relation will be adopted as the definition of the direct product of two distributions.

Definition 1.34 Let there be the distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in \mathcal{D}'(\mathbb{R}^m)$. We call direct or tensor product of the distribution f with g, the functional $f \times g : \mathcal{D}(\mathbb{R}^{n+m}) \to \Gamma$ defined by the relation

$$(f(x) \times g(y), \varphi(x, y)) = (f(x), (g(y), \varphi(x, y))), \quad \varphi \in \mathcal{D}(\mathbb{R}^{n+m}).$$
(1.451)

Proposition 1.32 The direct product $f \times g$ of the distributions $f \in \mathcal{D}'(\mathbb{R}^n), g \in \mathcal{D}'(\mathbb{R}^m)$, defined through the relation (1.451), exists and is a distribution from $\mathcal{D}'(\mathbb{R}^{n+m})$, namely $f \times g \in \mathcal{D}'(\mathbb{R}^{n+m})$.

In particular, if $\varphi(x, y) \in \mathcal{D}(\mathbb{R}^{n+m})$ is of the form $\varphi(x, y) = \varphi_1(x)\varphi_2(y)$, where $\varphi_1 \in \mathcal{D}(\mathbb{R}^n), \varphi_2 \in \mathcal{D}(\mathbb{R}^m)$, then the formula (1.451) becomes

$$(f(x) \times g(y), \varphi_1(x)\varphi_2(y)) = (f(x), \varphi_1(x)) \cdot (g(y), \varphi_2(y)) .$$
(1.452)

From the formula (1.452) we obtain the following.

Proposition 1.33 The necessary and sufficient condition for $f \times g = 0, f \in \mathcal{D}'(\mathbb{R}^n), g \in \mathcal{D}'(\mathbb{R}^m)$ is that one of the factors be null.

Indeed, if f(x) = 0, then $\forall \varphi_1 \in \mathcal{D}(\mathbb{R}^n)$ we have $(f(x), \varphi_1(x)) = 0$ and, from (1.452), it results $f \times g = 0$.

Conversely, if $f \times g = 0$, then $\forall \varphi_1(x) \in \mathcal{D}(\mathbb{R}^n)$ and $\forall \varphi_2(y) \in \mathcal{D}(\mathbb{R}^m)$ we have

$$(f(x) \times g(y), \varphi_1(x)\varphi_2(y)) = (f(x), \varphi_1(x)) \cdot (g(y), \varphi_2(y)) = 0, \qquad (1.453)$$

wherefrom it results that one of the factors is zero. Thus, if $(f(x), \varphi_1(x)) = 0$ ($\varphi_1(x)$ is arbitrary), then we get f(x) = 0.

1.3.8.1 Properties of the Direct Product

Proposition 1.34 The direct product is commutative, associative and distributive with respect to the addition of distributions from the same space. Namely, we have

- 1. $f \times g = g \times f$, $\forall f \in \mathcal{D}'(\mathbb{R}^n)$, $g \in \mathcal{D}'(\mathbb{R}^m)$;
- 2. $f \times (g \times h) = (f \times g) \times h, \forall f \in \mathcal{D}'(\mathbb{R}^n), g \in \mathcal{D}'(\mathbb{R}^m), h \in \mathcal{D}'(\mathbb{R}^\ell);$
- 3. $f \times (\alpha g \times \beta h) = \alpha (f \times g) + \beta (f \times h), \forall f \in \mathcal{D}'(\mathbb{R}^n), g, h \in \mathcal{D}'(\mathbb{R}^m), \alpha, \beta \in \Gamma.$

Proposition 1.35 Let there be the distributions $f \in \mathcal{D}'(\mathbb{R}^n)$, $g \in \mathcal{D}'(\mathbb{R}^m)$, ν the symmetry operator and D_x^{α} , D_y^{β} the derivation operators. The following relations take place:

$$supp(f \times g) = supp(f) \times supp(g) ,$$

$$D_x^{\alpha} D_y^{\beta}(f(x) \times g(y)) = D_x^{\alpha} f(x) \times D_y^{\beta} f(y)$$

$$(f \times g)^{\nu} = f^{\nu} \times g^{\nu} ,$$

$$a(x)b(y)(f(x) \times g(y)) = a(x) f(x) \times b(y)g(y) ,$$

$$a(x) \in C^{\infty}(\mathbb{R}^n), b(y) \in C^{\infty}(\mathbb{R}^m) .$$

Example 1.32 Let $H(x), x \in \mathbb{R}^n$ be the Heaviside distribution of *n* variables. By means of the direct product, it can be written as

$$H(x_1,\ldots,x_n) = H(x_1) \times H(x_2) \times \cdots \times H(x_n) .$$
(1.454)

Because $dH(x_i)/dx_i = \delta(x_i)$, we obtain

$$\frac{\partial^n H(x_1, \dots, x_n)}{\partial x_1 \partial x_2 \dots \partial x_n} = \delta(x_1, x_2, \dots, x_n)$$

= $\frac{dH(x_1)}{dx_1} \times \frac{dH(x_2)}{dx_2} \times \dots \times \frac{dH(x_n)}{dx_n} = \delta(x_1) \times \dots \times \delta(x_n)$, (1.455)

hence

$$\delta(x_1,\ldots,x_n) = \delta(x_1) \times \delta(x_2) \times \cdots \times \delta(x_n) . \tag{1.456}$$

Definition 1.35 We say that the distribution $g(x, y) \in \mathcal{D}'(\mathbb{R}^{n+m})$ does not depend on the variable $y \in \mathbb{R}^m$ if it is of the form

$$g(x, y) = f(x) \times 1(y), f \in \mathcal{D}'(\mathbb{R}^n) .$$

$$(1.457)$$

This distribution will be denoted by $f(x) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and should not be confused with $f(x) \in \mathcal{D}'(\mathbb{R}^n)$, which is defined on the test functions space $\mathcal{D}(\mathbb{R}^n)$.

For $\forall f \in \mathcal{D}'(\mathbb{R}^n)$, the relation follows

$$\int_{\mathbb{R}^m} (f(x), \varphi(x, y)) dy = \left(f(x), \int_{\mathbb{R}^m} \varphi(x, y) dy \right), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^{n+m}). \quad (1.458)$$

Indeed, on the basis of the direct product, $\forall \varphi(x, y) \in \mathcal{D}(\mathbb{R}^{n+m})$ we can write

$$(f(x) \times 1(y), \varphi(x, y)) = (f(x), (1(y), \varphi(x, y))) = \left(f(x), \int_{\mathbb{R}^m} \varphi(x, y) dy\right)$$
$$= (1(y), (f(x), \varphi(x, y))) = \int_{\mathbb{R}^m} (f(x), \varphi(x, y)) dy, \quad (1.459)$$

giving the formula (1.458).

If the distribution $g(x, y) \in \mathcal{D}'(\mathbb{R}^{n+m})$ does not depend on the variable $y \in \mathbb{R}^m$, then we have

$$D_i g(x, y) = 0, D_i = \frac{\partial}{\partial y_i} .$$
(1.460)

Indeed, $D_i g(x, y) = D_i (f(x) \times 1(y)) = f(x) \times D_i 1(y) = f(x) \times 0 = 0.$

In general we have the following: The necessary and sufficient condition for the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ should not depend on the variable x_i is $\partial f / \partial x_i = 0$.

Proposition 1.36 The necessary and sufficient condition for the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ to be a constant is

$$\frac{\partial f}{\partial x_i} = 0 , \quad i = \overline{1, n} . \tag{1.461}$$

Proposition 1.37 Let there be the distribution $f(x) \in \mathcal{D}'(\mathbb{R})$ and the Dirac representative sequence $g_{\ell}(t) \xrightarrow{\mathcal{D}'} \delta(t)$, where $g_{\ell} \in L^1_{loc}(\mathbb{R})$. Then, we have

$$\lim_{\ell \to \infty} (f(x) \times g_{\ell}(t)) = f(x) \times \delta(t) .$$
(1.462)

Indeed, $\forall \varphi(x, t) \in \mathcal{D}(\mathbb{R}^2)$ we can write

$$(f(x) \times g_{\ell}(t), \varphi(x, t)) = (g_{\ell}(t), (f(x), \varphi(x, t))), \qquad (1.463)$$

where $(f(x), \varphi(x, t)) = \psi(t) \in \mathcal{D}(\mathbb{R})$.

Consequently, we obtain

$$\lim_{\ell \to \infty} (f(x) \times g_{\ell}(t), \varphi(x, t)) = \lim_{\ell \to \infty} (g_{\ell}(t), \psi(t)) = (\delta(t), \psi(t))$$
$$= (\delta(x), (f(x), \varphi(x, t))) = (f(x) \times \delta(t), \varphi(x, t)) ,$$
(1.464)

giving the formula (1.462).

In general, we can write the formulae

$$\lim_{\ell \to \infty} (f(x) \times g_{\ell}(t - t_k)) = f(x) \times \delta(t - t_k) ,$$

$$\lim_{\ell \to \infty} (f(x) \times g'_{\ell}(t - t_k)) = f(x) \times \delta'(t - t_k) , \qquad (1.465)$$

because

$$\lim_{\ell \to \infty} g_{\ell}(t - t_k) = \delta(t - t_k) \text{ , and } \lim_{\ell \to \infty} g'_{\ell}(t - t_k) = \delta'(t - t_k) \text{ .}$$
(1.466)

1.3.8.2 The Convolution Product of Distributions

In order to extend the convolution product to distribution, we will consider the functions $f, g \in L^1(\mathbb{R}^n)$. Then, $f * g \in L^1(\mathbb{R}^n)$, hence it is a regular distribution and, $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$((f * g)(x), \varphi(x)) = \int_{\mathbb{R}^n} (f * g)(x)\varphi(x)dx = \int_{\mathbb{R}^n} \varphi(x) \left[\int_{\mathbb{R}^n} f(t)g(x-t)dt \right] dx .$$
(1.467)

Making the change of variables u = x - t, v = t, the previous relation becomes

$$((f * g)(x), \varphi(x)) = \int_{\mathbb{R}^{2n}} f(t)g(x - t)\varphi(x)dxdt$$
$$= \int_{\mathbb{R}^{2n}} f(v)g(u)\varphi(u + v)dudv .$$
(1.468)

Taking into account that $f(v)g(u) = f(v) \times g(u)$, formula (1.468) can be written in the form

$$((f * g)(x), \varphi(x)) = (f(x) \times g(y), \varphi(x + y)).$$
(1.469)

This relation is considered to be a definition formula of the convolution product of two distributions.

Definition 1.36 If $f, g \in \mathcal{D}'(\mathbb{R}^n)$, then their convolution product f * g represents a new distribution from $\mathcal{D}'(\mathbb{R}^n)$, defined by the formula

$$(f * g, \varphi) = (f(x) \times g(y), \varphi(x + y)), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$
(1.470)

We note that the distribution $f * g \in \mathcal{D}'(\mathbb{R}^n)$ does not exist for any distributions $f, g \in \mathcal{D}'(\mathbb{R}^n)$.

Indeed, when $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$ the function $\varphi(x + y)$ is indefinitely differentiable on \mathbb{R}^{2n} , but does not have compact support, hence $\varphi(x + y) \notin \mathcal{D}(\mathbb{R}^{2n})$.

The convolution product f * g exists if the sets $supp(f(x) \times g(y))$ and $supp(\varphi(x + y))$ have a compact intersection.

Proposition 1.38 Let there be $f, g \in \mathcal{D}'(\mathbb{R}^n)$. The convolution product $f * g \in \mathcal{D}'(\mathbb{R}^n)$ exists if one of the distributions f, g has compact support.

Proof: Let us assume that the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ has compact support, hence $\operatorname{supp}(f) = \Omega = \operatorname{compact}$. We notice that $\forall \alpha \in \mathcal{D}'(\mathbb{R}^n)$ the function $\psi(\gamma) = (\alpha(x), \varphi(x + \gamma))$ is indefinitely differentiable. In particular, the function $(f(x), \varphi(x + \gamma)) = \psi_1(\gamma)$ has compact support, because $x \in \Omega =$ $\operatorname{supp}(f)$ is bounded; it means that, for $|\gamma|$ large enough, $\varphi(x + \gamma) = 0$. Hence, $(f(x), \varphi(x + \gamma)) \in \mathcal{D}(\mathbb{R}^n)$.

Consequently, formula (1.469) makes sense and we can write

$$(f * g, \varphi) = (f(x) \times g(y), \varphi(x + y)) = (g(y), (f(x), \varphi(x + y)))$$

= $(f(x), (g(y), \varphi(x + y))), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$ (1.471)

Proposition 1.39 Let there be the distributions $\delta(x)$, $f(x) \in \mathcal{D}'(\mathbb{R}^n)$ and D^{α} the derivation operator. Then, we have

$$D^{\alpha}\delta * f = D^{\alpha}f, \qquad (1.472)$$

$$\delta(x-a) * f(x) = f(x-a) .$$
(1.473)

Proof: Because $D^{\alpha}\delta(x)$ has as support the origin, it means that the product $D^{\alpha}\delta(x) * f(x) \in \mathcal{D}'(\mathbb{R}^n)$ exists and for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\begin{aligned} (\mathsf{D}^{\alpha}\delta * f,\varphi) &= (\mathsf{D}^{\alpha}\delta(x) \times f(\gamma),\varphi(x+\gamma)) \\ &= (-1)^{|\alpha|}(f(\gamma),(\delta(x),\mathsf{D}^{\alpha}\varphi(x+\gamma))) = (-1)^{|\alpha|}(f(\gamma),\mathsf{D}^{\alpha}\varphi(\gamma)) \\ &= (\mathsf{D}^{\alpha}f(\gamma),\varphi(\gamma)) = (\mathsf{D}^{\alpha}f(x),\varphi(x)) , \end{aligned}$$
(1.474)

giving the formula (1.472).

Also, we have

$$\begin{aligned} (\delta(x-a) * f(x), \varphi(x)) &= (\delta(x-a) \times f(y), \varphi(x+y)) \\ &= (f(y), (\delta(x-a), \varphi(x+y))) = (f(y), (\delta(x), \varphi(x+a+y))) \\ &= (f(y), \varphi(a+y)) = (f(y-a), \varphi(y)), \end{aligned}$$
(1.475)

hence

$$(\delta(x-a) * f(x), \varphi(x)) = (f(x-a), \varphi(x)); \qquad (1.476)$$

thus, the relation (1.473) is proved.

Particularly, for a = 0 we have

$$\delta(x) * f(x) = f(x) , \quad \forall f \in \mathcal{D}'(\mathbb{R}^n) .$$
(1.477)

This property shows that Dirac's delta distribution $\delta(x) \in \mathcal{D}'(\mathbb{R}^n)$ is the unit element with respect to the convolution product.

Corollary 1.3 Let there be $P(D) = \sum_{|\alpha| < l} a_{\alpha} D^{\alpha}, \alpha \in \mathbb{N}_{0}^{n}, a_{\alpha} \in \mathbb{C}$ a linear differential operator with constant coefficients. Then, $\forall f \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$P(D)\delta(x) * f(x) = P(D)f(x).$$
(1.478)

This result is obtained from property (1.472) and the linearity of the operator P(D).

Thus, for $P(D) = \Delta = \sum_{k=1}^{n} (\partial^2 / \partial x_k^2)$ representing the Laplace operator in \mathbb{R}^n and $\forall f \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$\Delta\delta(x) * f = \Delta f . \tag{1.479}$$

For the operational calculus, of special importance are the distributions from $\mathcal{D}'(\mathbb{R})$, having bounded supports to the left, hence the supports on $[a, \infty)$.

Proposition 1.40 If $f, g \in \mathcal{D}'(\mathbb{R})$ and have supports bounded to the left, then the product f * g exists.

Proof: Let there be supp(f) = Ω_1 , supp(g) = Ω_2 and Ω_1 , $\Omega_2 \subset [a, \infty)$. Let there be supp $(\varphi) = \Omega, \varphi \in \mathcal{D}(\mathbb{R})$. Then, for $x \in \Omega_1, y \in \Omega_2$, because $x + y \in \Omega$ and Ω is compact, it results $x \ge a$, $y \ge a$, $b_1 \le x + y \le b_2$. Hence, $x \le b_2 - a$, $y \le b_2 - a$ and therefore the set $(x, y) \in \mathbb{R}^2$ with $x \in \Omega_1, y \in \Omega_2, x + y \in \Omega$ is bounded and thus the product f * g exists (Figure 1.9).

We note that the hatched trapezoid ABCD is the intersection of the sets $\sup(f(x) \times g(y))$ and $\sup(\varphi(x + y))$, which is a bounded set for which there exists the convolution product $f * g \in \mathcal{D}'(\mathbb{R})$. Similarly, it is shows the existence of the product f * g if the distributions $f, g \in \mathcal{D}'(\mathbb{R})$ have bounded supports to the right, hence supp(f), supp(g) $\subset (-\infty, b]$.



Figure 1.9

We note by \mathcal{D}'_+ all the distributions of $\mathcal{D}'(\mathbb{R})$ with the support on the half-axis $[0, \infty)$.

Proposition 1.41 If $f, g \in \mathcal{D}'(\mathbb{R}^n)$ and f * g exists, then we have

$$\operatorname{supp}(f * g) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)} . \tag{1.480}$$

Proof: Let there be $A = \operatorname{supp}(f)$, $B = \operatorname{supp}(g)$. To justify the sentence, it is enough to show that $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, such that $\operatorname{supp}(\varphi) \subset C_{\mathbb{R}^n}(\overline{A+B})$, we have $(f * g, \varphi) = 0$.

Because $\operatorname{supp}(f(x) \times g(y)) = A \times B$ and $x + y \in \operatorname{supp}(\varphi) \subset C_{\mathbb{R}^n}(\overline{A + B})$, from $x \in A$ and $y \in B$ it results $x + y \in A + B$; but as $(A + B) \cap \operatorname{supp}(\varphi(x + y)) = \emptyset$, we have $(f * g, \varphi) = (f(x) \times g(y), \varphi(x + y)) = 0$ for $\operatorname{supp}(\varphi) \subset C_{\mathbb{R}^n}(\overline{A + B})$.

Taking into account that the closure of a set is a closed set, we deduce that $\underline{C}_{\mathbb{R}^n}(\overline{A+B})$ is an open set, hence f * g = 0 in $C_{\mathbb{R}^n}(\overline{A+B})$, that is, $\operatorname{supp}(f * g) \subset \overline{A+B}$.

Corollary 1.4 Let there be $f, g \in \mathcal{D}'(\mathbb{R})$, having $\operatorname{supp}(f) \subset [a, \infty)$ and $\operatorname{supp}(g) \subset [b, \infty)$. Then f * g exists and we have

$$\operatorname{supp}(f * g) \subset [a + b, \infty) . \tag{1.481}$$

Indeed, on the basis of the Proposition 1.40, the product f * g exists and, according to the Proposition 1.41, we have

$$\operatorname{supp}(f * g) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)} \subset [a, \infty) + [b, \infty) = [a + b, \infty) . (1.482)$$

Particularly, if $f, g \in \mathcal{D}'_+$, that is, $\operatorname{supp}(f)$, $\operatorname{supp}(g) \subset [0, \infty)$, then $\operatorname{supp}(f \times g) \subset [0, \infty)$, wherefrom it results $f * g \in \mathcal{D}'_+$.

Thus, for example, we have

$$H(x) * H(x) = x H(x) = \begin{cases} 0, & x < 0, \\ x, & x \ge 0, \end{cases}$$
(1.483)

where $H \in \mathcal{D}'_+$ is the Heaviside distribution.

Corollary 1.5 If one of the distributions $f, g \in \mathcal{D}'(\mathbb{R}^n)$ has compact support, then $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$.

Proposition 1.42 Let there be the distributions $f, g \in \mathcal{D}'(\mathbb{R}^n)$. If f * g exists, then we have f * g = g * f, that is, the convolution product is commutative.

Indeed, because $f(x) \times g(y) = g(y) \times f(x)$, $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$(f * g, \varphi) = (f(x) \times g(y), \varphi(x + y))$$
$$= (g(y) \times f(x), \varphi(x + y)) = (g * f, \varphi) , \qquad (1.484)$$

namely f * g = g * f.

Proposition 1.43 If two of the distributions $f, g, h \in \mathcal{D}'(\mathbb{R}^n)$ have compact support, then the convolution product is associative, that is,

$$f * (g * h) = (f * g) * h$$
. (1.485)

Proof: Suppose that the distributions *f* and *g* have compact supports. Then, the distributions f * (g * h) and (f * g) * h exist and $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$(f * (g * h), \varphi) = (f(x) \times (g \times h)(y), \varphi(x + y)) = ((g * h)(y), (f_x, \varphi(x + y))) = (g(y) \times h(z), (f(x), \varphi(x + y + z))) = (f(x) \times g(y) \times h(z), \varphi(x + y + z)).$$
(1.486)

Proceeding analogously, we obtain

$$((f * g) * h, \varphi) = ((f * g)(y) \times h(z), \varphi(y + z)) = (f(x) \times g(y), (h(z), \varphi(x + y + z))) = (f(x) \times g(y) \times h(z), \varphi(x + y + z)) .$$
 (1.487)

Comparing the two expressions, we obtain the associativity of the convolution product. $\hfill \Box$

In connection with the property of associativity of distributions bounded to the left, we can state the following.

Proposition 1.44 Let there be $f, g, h \in \mathcal{D}'(\mathbb{R})$ and supp $(f, g, h) \subset [a, \infty)$. Then, the convolution product of these distributions is associative.

Remark 1.2 Apart from the cases presented by associativity, we note that the convolution product f * g * h, $f, g, h \in \mathcal{D}'(\mathbb{R}^n)$ is associative if the following conditions are fulfilled: $f \in \mathcal{E}'(\mathbb{R}^n)$, hence it is a distribution with compact support and there exists the product $g * h \in \mathcal{D}'(\mathbb{R}^n)$.

Thus, for example, we have

$$(\delta(x) * H(x)) * H(x) = \delta(x) * (H * H) = x H(x).$$
(1.488)

Instead, the product $1 * \delta'(x) * H(x)$, $x \in \mathbb{R}$ is not associative because the associativity conditions are not respected and we have

$$(1 * \delta'(x)) * H(x) = 0 * H = 0, \qquad (1.489)$$

$$1 * (\delta' * H) = 1 * \delta = 1.$$
(1.490)

In connection with the distributivity of the convolution product with respect to the addition operation, we state the following.

Proposition 1.45 Let there be the distributions $f, g, h \in \mathcal{D}'(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{C}$. If two of the products $f * (ag + \beta h)$, f * g, f * h exist, then the third one exists as well, and we have

$$f * (\alpha g + \beta h) = \alpha (f * g) + \beta (f * h).$$
 (1.491)

Corollary 1.6 If $f, g, h \in \mathcal{D}'(\mathbb{R}^n)$ and f has compact support, hence $f \in \mathcal{E}'(\mathbb{R}^n)$, then the convolution product is distributive, namely

$$f * (\alpha g + \beta h) = \alpha (f * g) + \beta (f * h), \quad \alpha, \beta \in \mathbb{C} .$$

$$(1.492)$$

Indeed, because *f* has compact support, the three products exist and, on the basis of the Proposition 1.45, the distributivity property occurs.

Proposition 1.46 Let there be $f, g \in \mathcal{D}'(\mathbb{R}^n)$ and $P(D) = \sum_{|\alpha| < l} a_{\alpha} D^{\alpha}$ a linear differential operator with constant coefficients. If f * g exists, then the distributions P(D) f * g, f * P(D)g exist as well, and we have

$$P(D)(f * g) = P(D)f * g = f * P(D)g.$$
(1.493)

Proof: Let there be $\delta(x) \in \mathcal{D}'(\mathbb{R}^n)$ the Dirac delta distribution. Then, the distribution $P(D)\delta \in \mathcal{E}'(\mathbb{R}^n)$ hence, it has compact support and therefore the product $P(D)\delta * (f * g)$ is commutative and associative. Based on the associative property and on the Corollary 1.3, we have

$$P(D)\delta * (f * g) = P(D)(f * g) = (P(D)\delta * f) * g = P(D)f * g. \quad (1.494)$$

Similarly, using the commutativity of the convolution product and the previous formula, we have

$$P(D)\delta * (f * g) = P(D)\delta * (g * f) = P(D)g * f , \qquad (1.495)$$

wherefrom the required result is obtained.

Example 1.33 Let $H(x) \in \mathcal{D}'_+$ be the Heaviside distribution. Then, we can write

$$(H * H)' = H' * H = \delta * H = H, \qquad (1.496)$$

because H * H exists and the obtained result is verified directly. Thus, as H * H =x H(x), by the derivation of this product, which is allowed because the function $\mu(x) = x, x \in \mathbb{R}$, is of class $C^{\infty}(\mathbb{R})$, hence it is the multiplier of the space $\mathcal{D}(\mathbb{R})$, we obtain

$$(H * H)' = (xH)' = H + xH' = H + x\delta(x) = H, \qquad (1.497)$$

because $x\delta(x) = 0$.

Let there be the distribution $f\in \mathcal{D}'_+$. Then, the distribution $F\in \mathcal{D}'_+$ given by the expression

$$F(x) = f(x) * H^{*m} = f(x) * \underbrace{(H * H * \dots * H)}_{m}, \qquad (1.498)$$

is a primitive of order *m* for the distribution *f*.

Indeed, $F \in \mathcal{D}'_+$ and we have

$$F^{(m)}(x) = f(x) * \underbrace{(H' * H' * \dots * H')}_{m} = f * \underbrace{(\delta * \delta * \dots * \delta)}_{m} = f * \delta = f .$$
(1.499)

It follows that the Heaviside distribution $H \in \mathcal{D}'_+$ has the role of integration operator for the distributions from \mathcal{D}'_+ , which is of particular importance in the operational calculus.

In particular, if $f \in C^0(\mathbb{R}) \cap \mathcal{D}'_+$, then $F(x) = f(x) * H(x) \in C^1(\mathbb{R}) \cap \mathcal{D}'_+$ and $F'(x) = H'(x) * f(x) = \delta(x) * f(x) = f(x)$.

Example 1.34 If $f, g \in \mathcal{D}'(\mathbb{R})$ and f * g exist, then for $P(D) = d^n/dx^n$ we have

$$\delta^{(n)}(x) * (f * g) = (f * g)^{(n)} = f^{(n)} * g = f * g^{(n)} .$$
(1.500)

Hence, for the derivative of order *n* of a distribution from $\mathcal{D}'(\mathbb{R})$, the convolution of the respective distribution with the distribution $\delta^{(n)}(x)$ is performed.

We can say that the distribution $\delta^{(n)}(x)$ acts as an operator of derivation of order *n* with respect to the convolution products. This property plays an essential role in the operational calculus.

Proposition 1.47 Let there be τ_a , $a \in \mathbb{R}^n$, and ν symmetry operator with respect to the origin of the coordinates. If $f, g \in \mathcal{D}'(\mathbb{R}^n)$ and f * g exists, then the following formulae take place

$$\tau_a(f * g) = \tau_a f * g = f * \tau_a g , \qquad (1.501)$$

$$(f * g)^{\nu} = f^{\nu} * g^{\nu} . \tag{1.502}$$

Below, we will state the continuity property of the convolution product.

Proposition 1.48 Let there be the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ and the sequence of distributions $(f_i)_{i \in \mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ with the property $f_i \xrightarrow{\mathcal{D}'} f$ and $\operatorname{supp}(f_i) \subset \Omega$ bounded. Then, $\forall g \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$f_i * g \xrightarrow{\mathcal{D}'} f * g . \tag{1.503}$$

Proof: Because f_i has compact support $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$(f_i * g, \varphi) = (f_i(\gamma), (g(x), \varphi(x + \gamma))) = (f_i(\gamma), h(\gamma)(g(x), \varphi(x + \gamma))), \quad (1.504)$$

where $h(\gamma) \in \mathcal{D}(\mathbb{R}^n)$ and has the value 1 in a compact neighborhood of the bounded set in which are contained the supports of the distributions f_i .

Consequently, $h(y)(g(x), \varphi(x + y)) \in \mathcal{D}(\mathbb{R}^n)$ and, on the basis of the convergence of the distributions, we have

$$(f_i * g, \varphi) \to (f, h(\gamma)(g(x), \varphi(x+\gamma))) = (f * g, \varphi), \qquad (1.505)$$

namely $f_i * g \xrightarrow{\mathcal{D}'} f * g$.

The property of continuity of the convolution product occurs in the following cases:

1.
$$f_i \xrightarrow{\mathcal{D}'} f_i, f \in \mathcal{D}'(\mathbb{R}^n), g \in \mathcal{E}'(\mathbb{R}^n) \text{ involves } f_i * g \xrightarrow{\mathcal{D}'} f * g$$

2.

$$f_i \xrightarrow{\mathcal{D}'} f, g, f, f_i \in \mathcal{D}'(\mathbb{R}), \operatorname{supp}(f_i), \operatorname{supp}(g) \subset (a, \infty) \text{ or } (-\infty, b)$$

involves $f_i * g \xrightarrow{\mathcal{D}'} f * g$

In particular, if $g \in \mathcal{D}'(\mathbb{R}^n)$ and $f_i \to \delta(x)$ with supp $(f_i) \subset \Omega$ bounded, then $f_i * g \to \delta(x) * g = g$.

3.

$$f_i \xrightarrow{\mathcal{D}'} f_i g_i \xrightarrow{\mathcal{D}'} g_i f_i, g_i, f_i g \in \mathcal{D}'(\mathbb{R})$$
 and

 $\operatorname{supp}(f_i), \operatorname{supp}(g_i), \operatorname{supp}(f), \operatorname{supp}(g) \subset (a, \infty) \text{ involve } f_i * g_i \xrightarrow{\mathcal{D}'} f * g$.

Proposition 1.49 Let there be $f \in \mathcal{D}'(\mathbb{R}^n)$, $\alpha \in \mathcal{D}(\mathbb{R}^n)$ and D^p the derivation operator. Then, $f * \alpha \in C^{\infty}(\mathbb{R}^n)$, $(f * \alpha)(x) = (f_t, \alpha(x - t))$ and

$$D^{p}(f * \alpha)(x) = (D^{p} f_{t}, \alpha(x - t)) = (f_{t}, D^{p}_{x}\alpha(x - t)).$$
(1.506)

Proof: Because α has compact support, then the convolution $f * \alpha$ exists and for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$(f * \alpha, \varphi) = (f(t) \times \alpha(u), \varphi(t+u)) = (f(t), (\alpha(u), \varphi(t+u)))$$

$$= \left(f(t), \int_{\mathbb{R}^n} \alpha(u)\varphi(t+u)du \right) = \left(f(t), \int_{\mathbb{R}^n} \alpha(x-t)\varphi(x)dx \right)$$

$$= (f(t), (\varphi(x), \alpha(x-t))) = (f(t) \times \varphi(x), \alpha(x-t))$$

$$= (\varphi(x), (f(t), \alpha(x-t)))$$

$$= \int_{\mathbb{R}^n} (f(t), \alpha(x-t))\varphi(x)dx = ((f(t), \alpha(x-t)), \varphi), \quad (1.507)$$

hence $f * \alpha$ is a function of variable $x \in \mathbb{R}^n$ and $(f * \alpha)(x) = (f(t), \alpha(x - t))$.

To show that the function $\psi(x) = (f * \alpha)(x)$ is indefinitely derivable, for *x* fixed and $h = (0, 0, ..., 0, h_i, 0, 0, ..., 0)$, we consider

$$\frac{\psi(x+h) - \psi(x)}{h_i} = \left(f(t), \frac{\alpha(x+h-t) - \alpha(x-t)}{h_i}\right) \,. \tag{1.508}$$

For $h_i \to 0$, because *f* is a continuous functional on $\mathcal{D}(\mathbb{R}^n)$ and

$$\frac{\alpha(x+h-t)-\alpha(x-t)}{h_i} \rightarrow \frac{\partial \alpha(x-t)}{\partial x_i}$$

in the sense of the convergence in the space $\mathcal{D}(\mathbb{R}^n)$, we obtain

$$\frac{\partial}{\partial x_i}(f*\alpha)(x) = \left(f(t), \frac{\partial \alpha(x-t)}{\partial x_i}\right) \\ = \left(f(t), -\frac{\partial \alpha(x-t)}{\partial t_i}\right) = \left(\frac{\partial}{\partial t_i}f(t), \alpha(x-t)\right), \quad (1.509)$$

giving

$$D^{p}(f * \alpha)(x) = (f(t), D^{p}_{x}\alpha(x-t)) = (D^{p}f(t), \alpha(x-t)).$$
(1.510)

We mention that the function $f * a \in C^{\infty}(\mathbb{R}^n)$ is called the regularized of the distribution *f*.

Particularly, if $f \in \mathcal{E}'(\mathbb{R}^n)$, so that it has compact support, then the function $f * \alpha$ has compact support, because $\operatorname{supp}(f * \alpha) \subset \operatorname{supp}(f) + \operatorname{supp}(\alpha)$ and $\operatorname{supp}(f)$, $\operatorname{supp}(\alpha)$ are compact sets, hence $f * \alpha \in \mathcal{D}(\mathbb{R}^n)$. \Box

Corollary 1.7 If $f \in \mathcal{E}'(\mathbb{R}^n)$ and $\alpha \in \mathcal{E}(\mathbb{R}^n)$ then

$$f * \alpha \in C^{\infty}(\mathbb{R}^n)$$
 and $(f * \alpha)(x) = (f(t), \alpha(x-t))$. (1.511)

Indeed, it follows since we have for $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$(f * \alpha, \varphi) = (f(t) \times \alpha(u), \varphi(t+u)) = (f(t), (\alpha(u), \varphi(t+u))) = (f(t), h(t)(\alpha(u), \varphi(t+u))), \quad (1.512)$$

where $h \in \mathcal{D}(\mathbb{R}^n)$ and it has a value equal to 1 on a compact neighborhood of the distribution support $f \in \mathcal{E}'(\mathbb{R}^n)$.

Thus, if $f \in \mathcal{E}'(\mathbb{R}^n)$ and $\alpha = 1$ on \mathbb{R}^n , then because $\alpha \in \mathcal{E}(\mathbb{R}^n)$ we have

$$f * 1 = (f(t), 1) = (f, 1) = \int_{\mathbb{R}^n} f(x) dx$$
 (1.513)

This convolution is called the integral of the distribution *f*.

If D^p is a derivation operator, then we obtain $D^p f * 1 = D^p (f * 1) = 0$, hence $\int_{\mathbb{R}^n} D^p f dx = 0$.

Definition 1.37 We call the trace at the origin of the continuous function $f : \mathbb{R}^n \to \mathbb{C}$, the number f(0) denoted by Tr f(x) = f(0).

According to the Proposition 1.49, if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $f * \varphi \in C^{\infty}(\mathbb{R}^n)$ and we have

$$(f * \varphi)(x) = (f(t), \varphi(x - t)),$$
 (1.514)

giving

$$\operatorname{Tr}(f * \varphi) = (f * \varphi)(0) = (f(t), \varphi(-t)) = (f, \varphi^{\nu}) = (f^{\nu}, \varphi) .$$
(1.515)

Consequently, we have

$$\operatorname{Tr}(f^{\nu} * \varphi) = \operatorname{Tr}(f * \varphi^{\nu}) = (f^{\nu}, \varphi^{\nu}) = (f, (\varphi^{\nu})^{\nu}) = (f, \varphi) .$$
(1.516)

Proposition 1.50 The necessary and sufficient condition for the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ to be null is that, for $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, we should have $f * \varphi = 0$.

Indeed, if f = 0, then $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ we can write

$$(f, \varphi) = 0 = \operatorname{Tr}(f * \varphi^{\nu}) \Rightarrow f * \varphi = 0$$

Reciprocally, if $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ we have $f * \varphi = 0$, then $f * \varphi^{\nu} = 0$. Applying the formula (1.516), we have $(f, \varphi) = \text{Tr}(f * \varphi^{\nu}) = 0$, hence f = 0.

Proposition 1.51 Let there be $f, g \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. If the convolution product $f * g \in \mathcal{D}'(\mathbb{R}^n)$ exists, then the formula follows:

$$(f * g, \varphi) = (f, g^{\nu} * \varphi) = (g, f^{\nu} * \varphi).$$
 (1.517)

Proof: Because $\varphi \in \mathcal{D}(\mathbb{R}^n)$, the product $(f * g) * \varphi$ exists, is commutative and associative. Consequently, we have

$$(f * g, \varphi) = \text{Tr}[(f * g) * \varphi^{\nu}] = \text{Tr}[f * (g * \varphi^{\nu})] = \text{Tr}[f * (g^{\nu} * \varphi)^{\nu}] = (f, g^{\nu} * \varphi).$$
(1.518)

We obtain the required result on the basis of the commutativity of the convolution product. $\hfill \Box$

From the formula (1.517), it follows that the convolution product f * g is determined if we know one of the functions $f^{\nu} * \varphi$ or $g^{\nu} * \varphi$, $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$.

Let there be the distribution $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$. We define the distribution

$$(f(x,t),\varphi(x)) \in \mathcal{D}'(\mathbb{R}^m) \tag{1.519}$$

by the formula

$$((f(x, t), \varphi(x)), \psi(t)) = (f(x, t), \varphi(x)\psi(t)), \qquad (1.520)$$

where $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\psi \in \mathcal{D}(\mathbb{R}^m)$.

Proposition 1.52 Let there be $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and D_t^{β} the derivation operator with respect to $t \in \mathbb{R}^m$. Then, $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\left(\mathsf{D}_t^{\beta}f(x,t),\varphi(x)\right) = \mathsf{D}_t^{\beta}(f(x,t),\varphi(x)), \quad \beta \in \mathbb{N}_0^m.$$
(1.521)

Indeed, taking into account (1.520) for $\psi(t) \in \mathcal{D}(\mathbb{R}^m)$, we can write

$$\begin{pmatrix} \left(\mathsf{D}_{t}^{\beta} f(x,t),\varphi(x)\right),\psi(t) \right) = \left(\mathsf{D}_{t}^{\beta} f(x,t),\varphi(x)\psi(t) \right) \\ = (-1)^{|\beta|} \left(f(x,t),\varphi\mathsf{D}_{t}^{\beta}\psi(t) \right) = (-1)^{|\beta|} \left((f(x,t),\varphi(x)),\mathsf{D}_{t}^{\beta}\psi \right) \\ = \left(\mathsf{D}_{t}^{\beta} (f(x,t),\varphi(x)),\psi(t) \right) ;$$
 (1.522)

hence, based on the equality of two distributions, we obtain the formula (1.521).

1.3.8.3 The Convolution of Distributions Depending on a Parameter: Properties

Proposition 1.53 Let there be the distribution $f_t, g \in \mathcal{D}'(\mathbb{R}^n), t \in T \subset \mathbb{R}$ being a parameter. If $\partial f_t(x)/\partial t$ and $f_t * g \in \mathcal{D}'(\mathbb{R}^n)$ exist, then we have

$$\frac{\partial}{\partial t}(f_t(x) * g(x)) = \frac{\partial f_t(x)}{\partial t} * g(x) .$$
(1.523)

Remark 1.3 The formula (1.523) remains valid if $\partial f_t(x)/\partial t$, $\forall t \in T \subset \mathbb{R}$, exists and if the distributions f_t , g satisfy one of the conditions:

- 1. $f_t \in \mathcal{E}'(\mathbb{R}^n)$, $\forall t \in T \subset \mathbb{R}$,
- 2. $g \in \mathcal{E}'(\mathbb{R}^n)$,
- 3. $f_t, g \in \mathcal{D}'(\mathbb{R})$ and $\operatorname{supp}(f_t), \operatorname{supp}(g) \subset [a, \infty)$ or $\operatorname{supp}(f_t), \operatorname{supp}(g) \subset (-\infty, b]$.

In particular, if $f_t(x) = \varphi(x, t) \in D(\mathbb{R}^{n+1})$, then $\forall g \in \mathcal{D}'(\mathbb{R}^n)$ on the basis of the formula (1.523), we obtain

$$\frac{\partial}{\partial t} \left(g(x) \mathop{*}_{x} \varphi(x, t) \right) = g(x) \mathop{*}_{x} \frac{\partial}{\partial t} \varphi(x, t) , \qquad (1.524)$$

where $t \in \mathbb{R}$ is considered parameter, and the convolution is performed with respect to the variable $x \in \mathbb{R}^n$.

Indeed, for $t \in \mathbb{R}$ fixed, $\varphi(x, t) \in \mathcal{D}(\mathbb{R}^n)$ and consequently $\forall g \in \mathcal{D}'(\mathbb{R}^n)$, the convolution product $g(x) *_x(\partial/\partial t)\varphi(x, t)$ exists and is a function of class $C^{\infty}(\mathbb{R}^{n+1})$, hence $g(x) *_x(\partial/\partial t)\varphi(x, t) \in C^{\infty}(\mathbb{R}^{n+1})$.

If $g \in \mathcal{E}'(\mathbb{R}^n)$, then $g(x) *_x(\partial/\partial t)\varphi(x, t) \in \mathcal{D}(\mathbb{R}^{n+1})$.

Proposition 1.54 Let there be the distributions $f_t, g_t \in \mathcal{D}'_+$ depending on the parameter $t \in T \subset \mathbb{R}$. If $(\partial/\partial t) f_t(x), (\partial/\partial t)g_t(x) \in \mathcal{D}'_+$, then $(\partial/\partial t)(f_t(x) * g_t(x)) \in \mathcal{D}'_+$ exists and the formula follows:

$$\frac{\partial}{\partial t}(f_t * g_t) = \frac{\partial}{\partial t}f_t * g_t + f_t * \frac{\partial}{\partial t}g_t .$$
(1.525)

Proof: On the basis of the definition of a derivative with respect to the parameter $t \in T \subset \mathbb{R}$ we have

$$\frac{\partial}{\partial t}(f_t * g_t) = \lim_{\Delta t \to 0} \frac{f_{t+\Delta t} * g_{t+\Delta t} - f_t * g_t}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \left(\frac{f_{t+\Delta t} - f_t}{\Delta t} * g_{t+\Delta t} + \frac{g_{t+\Delta t} - g_t}{\Delta t} * f_t \right) . \quad (1.526)$$

Taking into account the Proposition 1.48, we obtain

$$\lim_{\Delta t \to 0} \frac{f_{t+\Delta t} - f_t}{\Delta t} * g_{t+\Delta t} = \frac{\partial}{\partial t} f_t * g_t \text{ and } \lim_{\Delta t \to 0} \frac{g_{t+\Delta t} - g_t}{\Delta t} * f_t = \frac{\partial}{\partial t} g_t * f_t .$$
(1.527)

It follows that the right-hand side of the relation (1.526) exists, which implies the existence of the derivative $(\partial/\partial t)(f_t * g_t)$ and also the formula (1.525).

Example 1.35 Let there be the distributions $f_t(x) = \delta(x - at) \in \mathcal{D}'_+, g_t(x) = H(x - bt) \in \mathcal{D}'_+$ depending on the parameter $t \ge 0$, where $a, b \in \mathbb{R}_+$ and $H(u) \in \mathcal{D}'_+$ is the Heaviside distribution. We have

$$\frac{\partial}{\partial t}(\delta(x-at)*H(x-bt)) = -(a+b)\delta(x-(a+b)t) .$$
(1.528)

Indeed, applying the formula (1.525), we obtain

$$\frac{\partial}{\partial t} (\delta(x-at) * H(x-bt))$$

$$= \frac{\partial}{\partial t} \delta(x-at) * H(x-bt) + \delta(x-at) * \frac{\partial}{\partial t} H(x-bt)$$

$$= -a\delta'(x-at) * H(x-bt) - b\delta(x-at) * \delta(x-bt)$$

$$= -a\delta(x-at) * \delta(x-bt) - b\delta(x-at) * \delta(x-bt)$$

$$= -(a+b)\delta(x-(a+b)t).$$
(1.529)

Proposition 1.55 Let there be the distributions $f_t, g_t \in \mathcal{D}'_+$ depending on the parameter $t \in [a, b]$, continuous on [a, b] and let $F(x), G(x) \in \mathcal{D}'_+$. We have

$$\int_{a}^{b} [f_{t}(x) * F(x) + g_{t}(x) * G(x)]dt$$

= $F(x) * \int_{a}^{b} f_{t}(x)dt + G(x) * \int_{a}^{b} g_{t}(x)dt$. (1.530)

Particularly, for $F(x) = \alpha \delta(x)$, $G(x) = \beta \delta(x)$, $\alpha, \beta \in \mathbb{R}$ we obtain

$$\int_{a}^{b} \left[\alpha f_t(x) + \beta g_t(x)\right] \mathrm{d}t = \alpha \int_{a}^{b} f_t(x) \mathrm{d}t + \beta \int_{a}^{b} g_t(x) \mathrm{d}t \;. \tag{1.531}$$

Proposition 1.56 Let there be the distributions $f_t, g_t \in \mathcal{D}'_+$ depending on the parameter $t \in [a, b]$. If $(\partial/\partial t) f_t(x)$, $(\partial/\partial t)g_t(x) \in \mathcal{D}'_+$ exist, then we have

$$\int_{a}^{b} \left(\frac{\partial}{\partial t} f_{t}(x) * g_{t}(x)\right) dt = \left(f_{t}(x) * g_{t}(x)\right)|_{a}^{b} - \int_{a}^{b} \left(f_{t}(x) * \frac{\partial}{\partial t} g_{t}(x)\right) dt ,$$
(1.532)

where

$$(f_t * g_t)|_a^b = f_b(x) * g_b(x) - f_a(x) * g_a(x) .$$
(1.533)

Proof: Indeed, because $f_t, g_t \in \mathcal{D}'_+$, it follows that $\partial f_t/\partial t, \partial g_t/\partial t \in \mathcal{D}'_+$ and, as the convolution products between the distributions $f_t, g_t, \partial f_t/\partial t, \partial g_t/\partial t$ exist, according to formulae (1.525), (1.531), and (1.444), we obtain

$$\int_{a}^{b} \frac{\partial}{\partial t} (f_t * g_t) dt = (f_t * g_t)|_{a}^{b} = \int_{a}^{b} \left(\frac{\partial}{\partial t} f_t * g_t \right) dt + \int_{a}^{b} \left(f_t * \frac{\partial}{\partial t} g_t \right) dt ,$$
(1.534)

that is, the formula (1.532).

The relation (1.532) represents the analogue of the integration by parts formula. $\hfill \Box$

Example 1.36 Applying the formula (1.530) to calculate the integral

$$I = \int_{a}^{b} \delta'(x) * H(x - t) dt , \qquad (1.535)$$

where 0 < a < b and $t \in [a, b]$ is a parameter.

We have

$$I = \delta'(x) * \int_a^b H(x-t) dt = \frac{d}{dx} \int_a^b H(x-t) dt = \frac{d}{dx} F_1(x) ,$$

where, according to the Example 1.30, we obtain

$$F_1(x) = \begin{cases} 0, & x \le a < b, \\ b-a, & a < b < x, \\ x-a, & a \le x < b. \end{cases}$$
(1.536)

Consequently, applying the differentiation formula of the functions with discontinuities of the first order we obtain

$$I = \frac{d}{dx}F_1(x) = \begin{cases} 0, & x < a, \\ 1, & x \in (a, b) = H(x - a) - H(x - b), \\ 0, & x > b. \end{cases}$$
(1.537)

Remark 1.4 By direct calculation, we have

$$I = \int_{a}^{b} \delta(x-t) dt = -\int_{a}^{b} \frac{\partial}{\partial t} H(x-t) dt$$

= -[H(x-t)]|^b_a = H(x-a) - H(x-b). (1.538)

1.3.8.4 The Partial Convolution Product for Functions and Distributions

Definition 1.38 Let there be the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x) \in \mathcal{D}'(\mathbb{R}^n)$. We call partial convolution product of the distribution *f* with *g*, the distribution denoted $f(x, t) \otimes_x g(x) \in \mathcal{D}'(\mathbb{R}^n)$, defined by the formula

$$f(x, t) \otimes g(x) = f(x, t) * (g(x) \times \delta(t)),$$
 (1.539)

where $\delta(t) \in \mathcal{D}'(\mathbb{R}^m)$ is Dirac's delta distribution.

The symbol \otimes_x for the convolution product denotes that the convolution is performed only with respect to the variable $x \in \mathbb{R}^n$, common to the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x) \in \mathcal{D}'(\mathbb{R}^n)$, considered in different spaces.

On the right-hand side of the formula (1.539), the convolution product denoted by the symbol * obviously refers to the variables $(x, t) \in \mathbb{R}^n \times \mathbb{R}^m$.

In the case of existence of the partial convolution product, the latter is a distribution from $\mathcal{D}'(\mathbb{R}^{n+m})$, hence $f(x, t) \otimes_x g(x) \in \mathcal{D}'(\mathbb{R}^{n+m})$.

Taking into account the definition of the commutativity of the partial convolution product, we will not distinguish between the distributions $f(x, t) \otimes_x g(x)$ and $g(x) \otimes_x f(x, t)$.

Proposition 1.57 Let there be $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x) \in \mathcal{D}'(\mathbb{R}^n)$. The partial convolution product $f(x, t) \otimes_x g(x) \in \mathcal{D}'(\mathbb{R}^{n+m})$ exists if one of the distributions *f*, *g* has compact support.

Indeed, if $g(x) \in \mathcal{E}'(\mathbb{R}^n)$, hence *g* has compact support, then $g(x) \times \delta(t)$ has compact support, hence the right-hand side of expression (1.539) exists.

Also, if $f(x, t) \in \mathcal{E}'(\mathbb{R}^{n+m})$, then the product $f(x, t) * (g(x) \times \delta(t))$ exists, wherefrom the proposition is proved.

From the above considerations, it follows that the partial convolution product denoted by the symbol \otimes_x is a new law of composition for the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x) \in \mathcal{D}'(\mathbb{R}^n)$ with respect to the common variable $x \in \mathbb{R}^n$.

This new introduced convolution product [20] has wide applications in deformable solid mechanics and in particular in viscoelasticity [21–23].

The structure relation of the partial convolution product is shown as follows:

Proposition 1.58 Representation formula Let there be the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x) \in \mathcal{D}'(\mathbb{R}^n)$. If the partial convolution product $f(x, t) \otimes_x g(x) \in \mathcal{D}'(\mathbb{R}^n)$.

 $\mathcal{D}'(\mathbb{R}^{n+m})$, then the relation

$$\left(f(x,t) \underset{x}{\otimes} g(x), \varphi\right) = \left(f(x,t), g^{\nu} \underset{x}{*} \varphi(x,t)\right), \quad \forall \varphi(x,t) \in \mathcal{D}(\mathbb{R}^{n+m}), \quad (1.540)$$

occurs, where g^{ν} is symmetric with respect to the origin of the distribution $g(x) \in \mathcal{D}'(\mathbb{R}^n)$.

Proof: We suppose that $g(x) \in \mathcal{E}'(\mathbb{R}^n)$. Then the convolution product exists and $\forall \varphi(x, t) \in \mathcal{D}(\mathbb{R}^{n+m})$, thus we can write

$$\begin{pmatrix} f(x,t) \bigotimes_{x} g(x), \varphi(x,t) \end{pmatrix} = (f(x,t) * [g(x) \times \delta(t)], \varphi(x,t))$$

= $(f(x,t) \times g(u) \times \delta(v), \varphi(x+u,t+v))$
= $(f(x,t) \times g(u), \varphi(x+u,t)) = (f(x,t), (g(u), \varphi(x+u,t))) .$ (1.541)

On the other hand, on the basis of the Proposition 1.49, because $g(x) \in \mathcal{E}'(\mathbb{R}^n)$, the convolution product $g^v *_x \varphi(x, t) \in \mathcal{D}'(\mathbb{R}^n)$ exists and we have

$$g^{\nu} *_{x} \varphi(x, t) = (g^{\nu}(u), \varphi(x - u, t)) = (g(u), \varphi(x + u, t)) .$$
(1.542)

Taking into account (1.541), we obtain

$$(f(x,t) \underset{x}{\otimes} g(x), \varphi(x,t)) = \left(f(x,t), g^{\nu} \underset{x}{*} \varphi(x,t)\right), \quad \forall \varphi(x,t) \in \mathcal{D}(\mathbb{R}^{n+m}), (1.543)$$

namely the required formula (1.540).

If $f(x, t) \in \mathcal{E}'(\mathbb{R}^{n+m})$, then on the basis of the Proposition 1.49, $\forall g \in \mathcal{D}'(\mathbb{R}^n)$ we have $g^{\nu} *_x \varphi(x, t) = (g(u), \varphi(x + u, t)) \in C^{\infty}(\mathbb{R}^{n+m})$ and the formula (1.541) becomes

$$\left(f(x,t) \underset{x}{\otimes} g(x), \varphi(x,t) \right) = \left(f(x,t), g^{\nu} \underset{x}{*} \varphi(x,t) \right)$$
$$= \left(f(x,t), h(x,t) \left(g^{\nu} \underset{x}{*} \varphi(x,t) \right) \right) ,$$
(1.544)

where $h(x, t) \in \mathcal{D}(\mathbb{R}^{n+m})$ and has the value 1 in a compact neighborhood of the distribution support $f(x, t) \in \mathcal{E}'(\mathbb{R}^{n+m})$.

Obviously, $h(x, t)(g^{\nu} *_x \varphi(x, t)) \in \mathcal{D}(\mathbb{R}^{n+m})$, therefore the right side of the formula (1.544) makes sense, which proves the equality (1.540).

Comparing the formula (1.540) of the partial convolution product with the formula (1.517), that is,

$$(f * g, \varphi) = (f, g^{\nu} * \varphi) = (g, f^{\nu} * \varphi), \quad f, g \in \mathcal{D}'(\mathbb{R}^n), \quad \varphi \in \mathcal{D}(\mathbb{R}^n), \quad (1.545)$$

we see that these two types of convolutions have the same structure, in the sense that they are expressed with respect to the common variable of both distributions which are convoluted. From this point of view, we can say that the partial convolution product is a generalization of the ordinary convolution product.

Below, we give some properties of the partial convolution product.

Proposition 1.59 Let there be the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m}), g(x) \in \mathcal{D}'(\mathbb{R}^n)$. If the product $f(x, t) \otimes_x g(x) \in \mathcal{D}'(\mathbb{R}^{n+m})$ exists and $D_x^{\alpha}, D_t^{\beta}$ are derivation operators with respect to the variables $x \in \mathbb{R}^n, t \in \mathbb{R}^m$, respectively, then the following formulae take place

$$D_x^{\alpha}\left[f(x,t)\bigotimes_x g(x)\right] = D_x^{\alpha}f(x,t)\bigotimes_x g(x) = f(x,t)\bigotimes_x D_x^{\alpha}g(x), \qquad (1.546)$$

$$D_t^{\beta}\left[f(x,t)\underset{x}{\otimes}g(x)\right] = D_t^{\beta}f(x,t)\underset{x}{\otimes}g(x).$$
(1.547)

Proposition 1.60 Let there be the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$, $g(x) \in \mathcal{D}'(\mathbb{R}^n)$ and D_t^{β} the derivation operators with respect to the variable $t \in \mathbb{R}^m$. If the partial convolution product $f(x, t) \otimes_x g(x)$ exists, then the formula follows:

$$\left(\mathsf{D}_{t}^{\beta}f(x,t)\underset{x}{\otimes}g(x),\varphi(x)\right) = \mathsf{D}_{t}^{\beta}(f(x,t)\underset{x}{\otimes}g(x),\varphi(x)), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^{n}).$$
(1.548)

Indeed, on the basis of the Proposition 1.52 and the formula (1.547) follows the required relation.

Remark 1.5 A similar relation occurs for the distributions depending on the parameter $t \in \mathbb{R}^m$. Thus, if $f_t(x), g(x) \in \mathcal{D}'(\mathbb{R}^n)$ and $f_t(x) * g(x) \in \mathcal{D}'(\mathbb{R}^n)$ exists, then $\forall \varphi(x) \in \mathcal{D}(\mathbb{R}^n)$ and we have

$$\left(\mathsf{D}_t^\beta f_t(x) * g(x), \varphi(x)\right) = \mathsf{D}_t^\beta (f_t(x) * g(x), \varphi(x)) . \tag{1.549}$$

Proposition 1.61 Let there be the distributions $f \in \mathcal{D}'(\mathbb{R}^n), h \in \mathcal{E}'(\mathbb{R}^n), g \in \mathcal{D}'(\mathbb{R}^m)$. Then we have

$$(f(x) \times g(t)) \underset{x}{\otimes} h(x) = (f * h)(x) \times g(t) .$$

$$(1.550)$$

Example 1.37 Let there be D_x^{α} the derivation operator with respect to the variable $x \in \mathbb{R}^n$; then we have the relations

$$f(x,t) \underset{x}{\otimes} D_x^{\alpha} \delta(x) = D_x^{\alpha} f(x,t) , \quad \forall f(x,t) \in \mathcal{D}'(\mathbb{R}^{n+m}) , \qquad (1.551)$$

$$\delta(x,t) \bigotimes_{x} D_{x}^{a} g(x) = D_{x}^{a} g(x) \times \delta(t), \forall g(x) \in \mathcal{D}'(\mathbb{R}^{n}) .$$
(1.552)

In particular, for $|\alpha| = 0$ we have

$$f(x,t) \underset{x}{\otimes} \delta(x) = f(x,t), \quad \forall f(x,t) \in \mathcal{D}'(\mathbb{R}^{n+m}),$$
(1.553)

$$\delta(x,t) \underset{x}{\otimes} g(x) = g(x) \times \delta(t) , \quad \forall g(x) \in \mathcal{D}'(\mathbb{R}^n) .$$
(1.554)

Proposition 1.62 Let there be the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m}), g(x) \in \mathcal{D}'(\mathbb{R}^n)$. If the product $f(x, t) \otimes_x g(x) \in \mathcal{D}'(\mathbb{R}^{n+m})$ exists, then we have

$$\left(f(x,t)\bigotimes_{x}g(x)\right)^{\nu} = f^{\nu}\bigotimes_{x}g^{\nu}, \qquad (1.555)$$

$$\tau_a \Big(f(x,t) \underset{x}{\otimes} g(x) \Big) = \tau_a f(x,t) \underset{x}{\otimes} g(x) , \qquad (1.556)$$

where τ_a is the translation operator by the vector $a \in \mathbb{R}^{n+m}$.

The partial convolution product has the property of continuity as the usual convolution product.

Proposition 1.63 Let there be the distribution $g(x) \in \mathcal{D}'(\mathbb{R}^n)$ and the sequence of distributions $(g_i(x))_{i \in \mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ with the properties $g_i \xrightarrow{\mathcal{D}'(\mathbb{R}^n)} g$ and $\operatorname{supp}(g_i) \subset \Omega$ bounded. Then, $\forall f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ we have

$$f(x,t) \underset{x}{\otimes} g_i(x) \xrightarrow{\mathcal{D}'(\mathbb{R}^{n+m})} f(x,t) \underset{x}{\otimes} g(x) .$$
(1.557)

As regards the support of the partial convolution product we can state [24]:

Proposition 1.64 Let there be the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x) \in \mathcal{D}'(\mathbb{R}^n)$. If $f \otimes_x g \in \mathcal{D}'(\mathbb{R}^{n+m})$ exists, $\operatorname{supp}(f) = \Omega \times T$, $\operatorname{supp}(g) = \Omega', \Omega, \Omega' \subset \mathbb{R}^n, T \subset \mathbb{R}^m$, then

$$\operatorname{supp}\left(f\underset{x}{\otimes}g\right)\subset\overline{(\Omega+\Omega')\times T}.$$
(1.558)

In particular, if $f(x, t) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$, $g(x) \in \mathcal{D}'(\mathbb{R})$ and $\operatorname{supp}(f) = [0, \infty) \times T$, $\operatorname{supp}(g) \subset [0, \infty) \times T$, $T \subset \mathbb{R}^m$, then $\Omega + \Omega' \subset [0, \infty)$ and the formula (1.558) becomes

$$\operatorname{supp}\left(f\underset{x}{\otimes}g\right) \subset [0,\infty) \times T . \tag{1.559}$$

We have seen that the partial convolution product exists if one of the factors is a distribution with compact support. Another case of existence of the partial convolution product which has particular importance in mechanics is given by [24]:

Proposition 1.65 If $f(x, t) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$, $g(x) \in \mathcal{D}'(\mathbb{R})$ and $\operatorname{supp}(f) = (a, \infty) \times T$, $\operatorname{supp}(g) \subset (b, \infty)$, $T \subset \mathbb{R}^m$, then $f(x, t) \otimes_x g(x) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$ exists.

Proposition 1.66 Let there be $f(x, t) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$, $g(x) \in \mathcal{D}'(\mathbb{R})$. If $\operatorname{supp}(f) \subset \Omega \times T$, Ω -compact, $T \subset \mathbb{R}^m$ and $\operatorname{supp}(g) = \Omega'$ arbitrary, then the partial convolution product $f(x, t) \otimes_x g(x) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$ exists.
A property that expresses a certain relation between the partial convolution product and the usual one is given by the following.

Proposition 1.67 Let there be the distributions $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g_1(x)$, $g_2(x) \in \mathcal{E}'(\mathbb{R}^n)$. We have

$$f \bigotimes_{x} (g_1 * g_2) = \left(f \bigotimes_{x} g_1 \right) \bigotimes_{x} g_2 = \left(f \bigotimes_{x} g_2 \right) \bigotimes_{x} g_1 .$$
(1.560)

Remark 1.6 The formula (1.560) remains valid if $f(x,t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $\operatorname{supp}(f) \subset [0,\infty) \times T, T \subset \mathbb{R}^m, g_1(x), g_2(x) \in \mathcal{D}'_+$, hence $\operatorname{supp}(g_1), \operatorname{supp}(g_2) \subset [0,\infty)$.

Proposition 1.68 Let there be $E(x) \in \mathcal{D}'(\mathbb{R}^n)$ a fundamental solution of the differential operator with constant coefficients $P(D_x) = \sum_{|\alpha| \le l} a_\alpha D_x^\alpha$, $\alpha \in \mathbb{N}_0^n$, $a_\alpha \in \mathbb{C}$. If $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$; if the partial convolution product $f(x, t) \otimes_x E(x) \in \mathcal{D}'(\mathbb{R}^{n+m})$ exists, then the distribution

$$u(x,t) = f(x,t) \bigotimes_{x} E(x) = f(x,t) * [E(x) \times \delta(t)], \qquad (1.561)$$

is a solution for the equation

$$P(D_x)u(x,t) = f(x,t) . (1.562)$$

Indeed, using the formula (1.546) we obtain

$$P(\mathsf{D}_x)u(x,t) = f(x,t) \otimes P(\mathsf{D}_x)E(x) .$$
(1.563)

Observing that $P(D_x)E(x) = \delta(x)$ and taking into account (1.554), the relation (1.563) becomes $P(D_x)u(x, t) = f(x, t) \otimes_x \delta(x) = f(x, t)$.

Remark 1.7 It follows that a fundamental solution of the operator $P(D_x)$ in $\mathcal{D}'(\mathbb{R}^{n+m})$ is the distribution $E_1(x,t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ with

$$E_1(x,t) = E(x) \times \delta(t), E(x) \in \mathcal{D}'(\mathbb{R}^n), \quad t \in \mathbb{R}^m.$$
(1.564)

Indeed, we have

$$P(D_x)E_1(x,t) = P(D_x)[E(x) \times \delta(t)]$$

= [P(D_x)E(x)] × $\delta(t) = \delta(x) \times \delta(t) = \delta(x,t)$. (1.565)

Example 1.38 Let there be $\delta_{\lambda}(x) = \delta(x - \lambda) \in E'(\mathbb{R}^n)$ and $f(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$. Then, we have $f \otimes_x (\delta_a * \delta_b) = f(x - a - b, t)$. Because $f(x, t) \otimes_x \delta_a(x) = f(x, t) * [\delta_a(x) \times \delta(x)] = f(x - a, t)$, on the basis of the formula (1.560) we obtain

$$f \underset{x}{\otimes} (\delta_a * \delta_b) = \left(f \underset{x}{\otimes} \delta_a \right) \underset{x}{\otimes} \delta_b = f(x - a, t) \underset{x}{\otimes} \delta_b(x) = f(x - a - b, t) .$$
(1.566)

Example 1.39 We consider the Poisson equation

$$\Delta u(x,t) = f(x,t) , \qquad (1.567)$$

where $\Delta = (\partial^2 / \partial x_1^2) + \dots + (\partial^2 / \partial x_n^2)$ is the Laplace operator in \mathbb{R}^n and $f(x, t) \in \mathcal{E}'(\mathbb{R}^{n+m}) \subset \mathcal{D}'(\mathbb{R}^{n+m})$ a distribution with compact support.

Let us show that a solution in $\mathcal{D}'(\mathbb{R}^{n+m})$ of (1.567) is the distribution $u(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ given by

$$u(x,t) = f(x,t) \underset{x}{\otimes} E(x) = f(x,t) * (E(x) \times \delta(t)) , \qquad (1.568)$$

where

$$E(x) = \begin{cases} \frac{-\Gamma[n/2]}{2(n-2)\pi^{n/2}} \cdot \frac{1}{r^{n-2}}, & n \ge 3, \\ \frac{\ln r}{2\pi}, & n = 2, \\ \frac{r}{2}, & n = 1, \end{cases}$$
(1.569)

 $r = ||x||, x \in \mathbb{R}^n$, while $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt, p > 0$ is the Euler gamma function.

Indeed, the linear differential operator with constant coefficients $P(D_x)$ is the Laplace operator Δ , hence $P(D_x) = \Delta$; taking into account the Example 1.19, the distribution $E(x) \in \mathcal{D}'(\mathbb{R}^n)$ given by (1.569) is the fundamental solution of the operator Δ in \mathbb{R}^n , $n \ge 1$.

Consequently, based on the Proposition 1.68, it follows that the distribution $u(x, t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ specified by (1.569) is the solution of (1.567).

Example 1.40 We consider the equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t) , \quad u, f \in \mathcal{D}'(\mathbb{R}^2) .$$
(1.570)

For $f(x, t) \in \mathcal{D}'(\mathbb{R}^2)$ where $\operatorname{supp}(f(x, t)) = [0, \infty) \times T, T \subset \mathbb{R}$, a solution of (1.570) is the distribution

$$u(x,t) = f(x,t) \bigotimes_{x} x H(x) = f(x,t) * [x H(x) \times \delta(t)], \qquad (1.571)$$

 $H(x) \in \mathcal{D}'(\mathbb{R})$ being the Heaviside distribution.

Indeed, a fundamental solution of the operator $P(D_x) = \partial^2 / \partial x^2$ in $\mathcal{D}'(\mathbb{R})$ is the distribution $E(x) = x H(x) \in \mathcal{D}'(\mathbb{R})$.

Consequently, a solution of (1.570) is the distribution

$$u(x,t) = f(x,t) \otimes x H(x) , \qquad (1.572)$$

which exists, taking into account the Proposition 1.65.

1.3.9

Partial Convolution Product of Functions

To prove the consistency of the partial convolution product introduced for distributions of different spaces, we show that in the case of functions, that is, distributions of function type, this operation coincides with the convolution operation with respect to the common variable of the two functions.

We can state the following.

Proposition 1.69 If $f(x,t) \in L^1(\mathbb{R}^{n+m})$ and $g(x) \in L^1(\mathbb{R}^n)$, then the product $f(x,t) \otimes_x g(x)$ exists and we have

$$f(x,t) * g(x) = f(x,t) \otimes_{x} g(x) \in L^{1}(\mathbb{R}^{n+m}), \qquad (1.573)$$

$$\int_{\mathbb{R}^{n+m}} \left(f(x,t) \underset{x}{*} g(x) \right) \mathrm{d}x \mathrm{d}t = \int_{\mathbb{R}^{n+m}} f(x,t) \mathrm{d}x \mathrm{d}t \cdot \int_{\mathbb{R}^n} g(x) \mathrm{d}x , \qquad (1.574)$$

$$\|f(x,t) * g(x)\|_{1} \le \|f\|_{1} \cdot \|g\|_{1}.$$
(1.575)

From above, the symbol $*_x$ means the usual convolution with respect to $x \in \mathbb{R}^n$ considering $t \in \mathbb{R}^m$ fixed, and the symbol \otimes_x represents the partial convolution product with respect to $x \in \mathbb{R}^n$ introduced by the Definition 1.38 for distribution from the spaces $\mathcal{D}'(\mathbb{R}^{n+m})$ and $\mathcal{D}'(\mathbb{R}^n)$.

Proposition 1.70 Let there be the locally integrable functions $f(x, t) \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^m)$ and $g(x) \in L^1_{loc}(\mathbb{R})$. If $\operatorname{supp}(f(x, t)) \subset [0, \infty) \times T$, $\operatorname{supp}(g(x)) \subset [0, \infty)$, $T \subset \mathbb{R}^m$, then $f \otimes_x g$ is a locally integrable function on $\mathbb{R} \times \mathbb{R}^m$ and we have

$$(f \bigotimes_{x} g)(x, t) = (f \bigotimes_{x} g)(x, t) = \begin{cases} 0, & (x, t) \in (-\infty, 0] \times T, \\ \int_{0}^{x} f(\zeta, t)g(x - \zeta)d\zeta, & (x, t) \in [0, \infty) \times T. \end{cases}$$
(1.576)

Example 1.41 For Heaviside distributions $H(x, t) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R})$ and $H(x) \in \mathcal{D}'(\mathbb{R})$ we have

$$H(x, t) *_{x} H(x) = x H(x, t) .$$
(1.577)

Because $H(x, t) \in L^1_{loc}(\mathbb{R}^2)$, $H(x) \in L^1_{loc}(\mathbb{R})$, the product $H(x, t) *_x H(x)$ exists, so that applying the formula (1.576) we obtain

$$H(x,t) *_{x} H(x) = \begin{cases} 0, & (x,t) \in (-\infty,0] \times \mathbb{R}, \\ \int_{0}^{x} H(\zeta,t) H(x-\zeta) d\zeta, & (x,t) \in [0,\infty) \times \mathbb{R}. \end{cases}$$
(1.578)

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By the change of variable $x - \zeta = u$, we have

$$\int_{0}^{x} H(\zeta, t) H(x - \zeta) d\zeta = \begin{cases} 0, & x \ge 0, & t < 0, \\ \int_{0}^{x} H(x - \zeta) d\zeta, & x \ge 0, & t \ge 0, \end{cases}$$
$$= \begin{cases} 0, & (x, t) \in [0, \infty) \times (-\infty, 0), \\ x, & (x, t) \in [0, \infty) \times [0, \infty), \end{cases}$$
(1.579)

and thus the relation (1.578) becomes

$$H(x, t) *_{x} H(x) = \begin{cases} x , & \text{for } (x, t) \in [0, \infty) \times [0, \infty) \\ 0 , & \text{for } (x, t) \in \mathbb{R} \setminus [0, \infty) \times [0, \infty) \end{cases} = x H(x, t) ,$$
(1.580)

that is, the formula (1.577).