1
SDOF Autonomous Systems

1.1
Introduction

In this chapter, we describe the method of normal forms using single-degree-of-freedom (SDOF) autonomous systems that can be modeled by the following second-order nonlinear ordinary differential equation:

$$\ddot{u} + \omega^2 u = f(u, \dot{u}) \tag{1.1}$$

where $f(u, \dot{u})$ can be developed in a power series in terms of u and $\dot{u}$. In what follows, we will refer to $\dot{u} + \omega^2 u = 0$ as the *unperturbed system* and (1.1) as the *perturbed system*. We assume that (1.1) has an equilibrium at $u = 0$ and $\dot{u} = 0$. Equation 1.1 can be cast as a system of two first-order equations by letting

$$x_1 = u \quad \text{and} \quad x_2 = \dot{u} \tag{1.2}$$

The result is

$$\dot{x}_1 = x_2 \tag{1.3}$$

$$\dot{x}_2 = -\omega^2 x_1 + f(x_1, x_2) \tag{1.4}$$

It is clear that the unperturbed system

$$\dot{x}_1 = x_2 \quad \text{and} \quad \dot{x}_2 = -\omega^2 x_1$$

has a simple pair of purely imaginary eigenvalues $\pm i\omega$.

The main idea underlying the method of normal forms is to introduce a near-identify transformation

$$x_1 = y_1 + h_1(y_1, y_2) \tag{1.5a}$$

$$x_2 = y_2 + h_2(y_1, y_2) \tag{1.5b}$$

from (x_1, x_2) to (y_1, y_2) into (1.3) and (1.4) to produce the simplest possible equations (the so-called normal form). We call the transformation (1.5) near-identity

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because $x_1(t) - y_1(t)$ and $x_2(t) - y_2(t)$ are small; that is, $o(x_1(t), x_2(t))$. This procedure is also called *normalization*. To this end, we substitute (1.5) into (1.3) and (1.4) and obtain

$$\dot{y}_1 = y_2 + h_2 - \frac{\partial h_1}{\partial y_1}\dot{y}_1 - \frac{\partial h_1}{\partial y_2}\dot{y}_2 \tag{1.6a}$$

$$\dot{y}_2 = -\omega^2 y_1 - \omega^2 h_1 + f\,(y_1 + h_1, y_2 + h_2) - \frac{\partial h_2}{\partial y_1}\dot{y}_1 - \frac{\partial h_2}{\partial y_2}\dot{y}_2 \tag{1.6b}$$

Then, we choose h_1 and h_2 such that (1.6) assume their simplest form. This task is accomplished in steps. If one decomposes $f(x_1, x_2)$ as

$$f(x_1, x_2) = \sum_{n=1}^{N} f_n(x_1, x_2) \tag{1.7}$$

where f_n is a polynomial of degree n in x_1 and x_2, then one chooses h_1 and h_2 to simplify the terms resulting from the lowest-order polynomial $f_m(x_1, x_2)$, where $m \geq 2$, in $f(x_1, x_2)$. In the next step, one chooses a second near-identity transformation to simplify the polynomial terms of degree $m + 1$, and so on.

It turns out that, because the unperturbed system (1.3) and (1.4) represents an oscillator, the governing equations can conveniently be expressed as a single complex-valued equation. To this end, we follow steps similar to those used in the method of variation of parameters (Nayfeh, 1981). When $f \equiv 0$, the solution of (1.1) can be expressed as

$$u = Be^{i\omega t} + \bar{B}e^{-i\omega t} \tag{1.8}$$

where B is a constant and $\bar{B}$ is the complex conjugate of B. Hence,

$$\dot{u} = i\omega\left(Be^{i\omega t} - \bar{B}e^{-i\omega t}\right) \tag{1.9}$$

When $f \neq 0$, we continue to represent the solution of (1.1) as in (1.8) subject to the constraint (1.9) but with time-varying rather than constant B. Next, we replace $Be^{i\omega t}$ with $\zeta(t)$ and rewrite (1.8) and (1.9) as

$$u = \zeta(t) + \bar{\zeta}(t) \quad \text{and} \quad \dot{u} = i\omega\left[\zeta(t) - \bar{\zeta}(t)\right] \tag{1.10}$$

Hence, solving for ζ and $\bar{\zeta}$, we obtain

$$\zeta = \frac{1}{2}\left(u - \frac{i}{\omega}\dot{u}\right) \quad \text{and} \quad \bar{\zeta} = \frac{1}{2}\left(u + \frac{i}{\omega}\dot{u}\right) \tag{1.11}$$

Differentiating (1.11) with respect to t yields

$$\dot{\zeta} = \frac{1}{2}\left(\dot{u} - \frac{i}{\omega}\ddot{u}\right) = \frac{1}{2}\left(\dot{u} + i\omega u - \frac{i}{\omega}f\right) \tag{1.12}$$

on account of (1.1). Hence,

$$\dot{\zeta} = \frac{1}{2}i\omega\left(u - \frac{i}{\omega}\dot{u}\right) - \frac{i}{2\omega}f(u, \dot{u}) \tag{1.13}$$

which, upon using (1.10), becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i}{2\omega} f\left[\zeta + \bar{\zeta}, i\omega\left(\zeta - \bar{\zeta}\right)\right] \tag{1.14}$$

Next, we consider different polynomial forms for *f*.

1.2 Duffing Equation

The Duffing equation is

$$\ddot{u} + \omega^2 u = \alpha u^3$$

so that, in this case, $f = \alpha u^3$ and (1.14) becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{1.15}$$

We introduce a near-identity transformation from ζ to η in the form

$$\zeta = \eta + h\left(\eta, \bar{\eta}\right) \tag{1.16}$$

and obtain

$$\dot{\eta} = i\omega\eta + i\omega h - \frac{\partial h}{\partial \eta}\dot{\eta} - \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} - \frac{i\alpha}{2\omega}\left(\eta + h + \bar{\eta} + \bar{h}\right)^3 \tag{1.17}$$

Because the nonlinearity is cubic, we assume that h is third order in η and $\bar{\eta}$; that is,

$$h = \Lambda_1\eta^3 + \Lambda_2\eta^2\bar{\eta} + \Lambda_3\eta\bar{\eta}^2 + \Lambda_4\bar{\eta}^3 \tag{1.18}$$

and choose the Λ_i so that (1.17) takes the simplest possible (normal) form.

In the first step, we eliminate $\dot{\eta}$ and $\dot{\bar{\eta}}$ from the right-hand side of (1.17). This task is accomplished by iteration. To the first approximation, it follows from (1.17) that

$$\dot{\eta} = i\omega\eta \quad \text{and} \quad \dot{\bar{\eta}} = -i\omega\bar{\eta} \tag{1.19}$$

Next, we replace $\dot{\eta}$ and $\dot{\bar{\eta}}$ on the right-hand side of (1.17) using (1.19), use (1.18), keep up to third-order terms, and obtain

$$\dot{\eta} = i\omega\eta - i\omega\left(2\Lambda_1 + \frac{\alpha}{2\omega^2}\right)\eta^3 - \frac{3i\alpha}{2\omega}\eta^2\bar{\eta} + i\omega\left(2\Lambda_3 - \frac{3\alpha}{2\omega^2}\right)\eta\bar{\eta}^2 + i\omega\left(4\Lambda_4 - \frac{\alpha}{2\omega^2}\right)\bar{\eta}^3 \tag{1.20}$$

Next, we choose Λ_1, Λ_3, and Λ_4 to eliminate the terms involving η^3, $\eta\bar{\eta}^2$, and $\bar{\eta}^3$; that is,

$$\Lambda_1 = -\frac{\alpha}{4\omega^2}\,, \quad \Lambda_3 = \frac{3\alpha}{4\omega^2}\,, \quad \Lambda_4 = \frac{\alpha}{8\omega^2} \tag{1.21}$$

However, because Λ_2 does not appear in (1.20), the term involving $\eta^2\bar{\eta}$ cannot be eliminated; it is called a *resonance term*. Consequently, to the second approximation, the simplest possible form for $\dot{\eta}$ is

$$\dot{\eta} = i\omega\eta - \frac{3i\alpha}{2\omega}\eta^2\bar{\eta} \tag{1.22}$$

To show that $\eta^2\bar{\eta}$ is a resonance term, we find a solution for (1.22) by iteration. To the first approximation, $\eta = Ae^{i\omega t}$, where A is a constant. Then, (1.22) becomes

$$\dot{\eta} = i\omega\eta - \frac{3i\alpha}{2\omega}A^2\bar{A}e^{i\omega t}$$

whose solution can be written as

$$\eta = Ae^{i\omega t} - \frac{3\alpha}{2\omega}A^2\bar{A}te^{i\omega t} \tag{1.23a}$$

It is clear that this expansion, which is also a straightforward expansion, is nonuniform for large t because of the presence of a secular term created by $\eta^2\bar{\eta}$. Alternatively, we can demonstrate that the term $\zeta^2\bar{\zeta}$ is a *resonance term* in the original equation (1.15). To the first approximation, we neglect the nonlinear term in (1.15) and find that $\zeta = Ae^{i\omega t}$. Then, to the second approximation, (1.15) becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i\alpha}{2\omega}\left(A^3e^{3i\omega t} + 3A^2\bar{A}e^{i\omega t} + 3A\bar{A}^2e^{-i\omega t} + \bar{A}^3e^{-3i\omega t}\right)$$

whose solution can be written as

$$\zeta = Ae^{i\omega t} - \frac{\alpha}{4\omega^2}A^3e^{3i\omega t} - \frac{3i\alpha}{2\omega}A^2\bar{A}te^{i\omega t} + \frac{3\alpha}{4\omega^2}A\bar{A}^2e^{-i\omega t} + \frac{\alpha}{8\omega^2}\bar{A}^3e^{-3i\omega t} \tag{1.23b}$$

It is clear that this expansion is nonuniform because of the presence of a secular term created by $\zeta^2\bar{\zeta}$. The other three terms proportional to $A^3e^{3i\omega t}$, $A\bar{A}^2e^{-i\omega t}$, and $\bar{A}^3e^{-3i\omega t}$ created by ζ^3, $\zeta\bar{\zeta}^2$, and $\bar{\zeta}^3$ do not produce secular terms and hence they are *nonresonance*. Consequently, one can choose a near-identity transformation to eliminate them.

As a second alternative, starting with the original equation (1.15), we break the nonlinear part $f(\zeta, \bar{\zeta})$ into two parts as

$$f(\zeta, \bar{\zeta}) = f_1(\zeta, \bar{\zeta}) + f_2(\zeta, \bar{\zeta})$$

where

$$e^{-i\omega t}f_1\left(e^{i\omega t}, e^{-i\omega t}\right)$$

is *time invariant*, whereas

$$e^{-i\omega t} f_2\left(e^{i\omega t}, e^{-i\omega t}\right)$$

is not time invariant. In the present case,

$$f = \left(\zeta + \bar{\zeta}\right)^3, \quad f_1 = 3\zeta^2\bar{\zeta}, \quad f_2 = \zeta^3 + 3\zeta\bar{\zeta}^2 + \bar{\zeta}^3$$

Thus,

$$e^{-i\omega t} f_1\left(e^{i\omega t}, e^{-i\omega t}\right) = e^{-i\omega t}\left(3e^{2i\omega t}e^{-i\omega t}\right) = 3$$

which is time invariant, whereas

$$e^{-i\omega t} f_2\left(e^{i\omega t}, e^{-i\omega t}\right) = e^{2i\omega t} + 3e^{-2i\omega t} + e^{-4i\omega t}$$

which does not contain any time-invariant terms.

Substituting (1.16) and (1.18) into (1.10), using (1.21), and setting $\Lambda_2 = 0$ because it is arbitrary yields

$$u = \eta + \bar{\eta} - \frac{\alpha}{8\omega^2}\left(\eta^3 + \bar{\eta}^3\right) + \frac{3\alpha}{4\omega^2}\left(\eta\bar{\eta}^2 + \eta^2\bar{\eta}\right) \tag{1.24}$$

where η is given by (1.22). Next, we separate the fast from the slow variations in η by introducing the transformation

$$\eta = A(t)e^{i\omega t}$$

where ω is the natural frequency of the system and A is a function of time, into (1.22) and (1.24) and obtain

$$\dot{A} = -\frac{3i\alpha}{2\omega}A^2\bar{A} \tag{1.25}$$

$$u = Ae^{i\omega t} + \bar{A}e^{-i\omega t} - \frac{\alpha}{8\omega^2}\left(A^3e^{3i\omega t} + \bar{A}^3e^{-3i\omega t}\right) + \frac{3\alpha}{4\omega^2}\left(A^2\bar{A}e^{i\omega t} + \bar{A}^2Ae^{-i\omega t}\right) + \cdots \tag{1.26}$$

Expressing A in the polar form

$$A = \tfrac{1}{2}ae^{i\beta} \tag{1.27}$$

where a and β are functions of t, we rewrite (1.26) as

$$u = \left(a + \frac{3\alpha}{16\omega^2}a^3\right)\cos(\omega t + \beta) - \frac{\alpha a^3}{32\omega^2}\cos(3\omega t + 3\beta) + \cdots \tag{1.28}$$

Substituting (1.27) into (1.25) and separating real and imaginary parts, we have

$$\dot{a} = 0 \tag{1.29}$$

$$a\dot{\beta} = -\frac{3\alpha}{8\omega}a^3 \tag{1.30}$$

In determining the normal form (1.22), we had to use an ordering scheme to indicate the relative magnitudes of the different terms in (1.15). We based the ordering scheme on the fact that ζ and $\bar{\zeta}$ are small and hence ζ^3, $\zeta^2\bar{\zeta}$, $\zeta\bar{\zeta}^2$, and $\bar{\zeta}^3$ are much smaller than ζ and $\bar{\zeta}$. In other words, we based the ordering scheme on the degree of the terms. This worked well in this example, but there are many physical systems where the ordering does not follow from the degree of the polynomial but from the presence of certain parameters in their models. We consider such an example in the next section.

Next, we treat (1.15) by using the method of multiple scales. To this end, we introduce a small nondimensional parameter ϵ as a bookkeeping device and rewrite (1.15) as

$$\dot{\zeta} = i\omega\zeta - \frac{i\epsilon\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{1.31}$$

Then, we seek an approximate solution of (1.31) in the form

$$\zeta(t;\epsilon) = \zeta_0\left(T_0, T_1\right) + \epsilon\zeta_1\left(T_0, T_1\right) + \cdots \tag{1.32}$$

where $T_n = \epsilon^n t$ and

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \cdots = D_0 + \epsilon D_1 + \cdots \tag{1.33}$$

Substituting (1.32) and (1.33) into (1.31) and equating coefficients of like powers of ϵ yields

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.34}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 - \frac{i\alpha}{2\omega}\left(\zeta_0 + \bar{\zeta}_0\right)^3 \tag{1.35}$$

The solution of (1.34) can be expressed as

$$\zeta_0 = A(T_1)e^{i\omega T_0} \tag{1.36}$$

Then, (1.35) becomes

$$\begin{aligned} D_0\zeta_1 - i\omega\zeta_1 = -A'e^{i\omega T_0} - \frac{i\alpha}{2\omega}\big(&A^3e^{3i\omega T_0} + 3A^2\bar{A}e^{i\omega T_0} \\ &+ 3A\bar{A}^2e^{-i\omega T_0} + \bar{A}^3e^{-3i\omega T_0}\big) \end{aligned} \tag{1.37}$$

Eliminating the terms that lead to secular terms from (1.37), we have

$$A' = -\frac{3i\alpha}{2\omega}A^2\bar{A} \tag{1.38}$$

Then, a particular solution of (1.37) can be expressed as

$$\zeta_1 = -\frac{\alpha}{4\omega^2} A^3 e^{3i\omega T_0} + \frac{3\alpha}{4\omega^2} A\bar{A}^2 e^{-i\omega T_0} + \frac{\alpha}{8\omega^2} \bar{A}^3 e^{-3i\omega T_0} \tag{1.39}$$

Substituting (1.36) and (1.39) into (1.10), we obtain

$$\begin{aligned} u &= Ae^{i\omega t} + \bar{A}e^{-i\omega t} - \frac{\epsilon\alpha}{8\omega^2}\left(A^3 e^{3i\omega t} + \bar{A}^3 e^{-3i\omega t}\right) \\ &\quad + \frac{3\epsilon\alpha}{4\omega^2}\left(A^2\bar{A}e^{i\omega t} + A\bar{A}^2 e^{-i\omega t}\right) + \cdots \end{aligned} \tag{1.40}$$

Equations 1.38–1.40 are in full agreement with (1.25) and (1.26) obtained with the method of normal forms because $T_1 = \epsilon t$ and ϵ can be set equal to unity.

1.3 Rayleigh Equation

The Rayleigh equation is

$$\ddot{u} + \omega^2 u = \epsilon\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right) \tag{1.41}$$

where ϵ is a small, positive nondimensional parameter. Here

$$f = \epsilon\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right)$$

and (1.14) becomes

$$\dot{\zeta} = i\omega\zeta + \tfrac{1}{2}\epsilon\left[\zeta - \bar{\zeta} + \tfrac{1}{3}\omega^2\left(\zeta - \bar{\zeta}\right)^3\right] \tag{1.42}$$

In this example, the ordering is not based on the degree of the polynomial, but on the small nondimensional parameter ϵ. Normalization is carried out in terms of the small parameter ϵ. In fact, the perturbation contains linear as well as cubic terms.

Using the transformation (1.16), we rewrite (1.42) as

$$\begin{aligned} \dot{\eta} &= i\omega\eta + i\omega h - \frac{\partial h}{\partial \eta}\dot{\eta} - \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} + \frac{1}{2}\epsilon\left[\eta - \bar{\eta} + h - \bar{h}\right. \\ &\quad \left. + \frac{1}{3}\omega^2\left(\eta - \bar{\eta} + h - \bar{h}\right)^3\right] \end{aligned} \tag{1.43}$$

Because the perturbation in (1.43) involves linear and cubic terms, we express h in the form

$$h = \epsilon\left(\Delta_1\eta + \Delta_2\bar{\eta} + \Lambda_1\eta^3 + \Lambda_2\eta^2\bar{\eta} + \Lambda_3\eta\bar{\eta}^2 + \Lambda_4\bar{\eta}^3\right) \tag{1.44}$$

Moreover, to the first approximation, $\dot{\eta}$ and $\dot{\bar{\eta}}$ are given by (1.19). Then, substituting (1.19) and (1.44) into the right-hand side of (1.43) and keeping terms up to $O(\epsilon)$, we obtain

$$\dot{\eta} = i\omega\eta + 2i\epsilon\omega\left(\Delta_2 + \frac{i}{4\omega}\right)\bar{\eta} + \frac{1}{2}\epsilon\eta - i\epsilon\omega\left(2\Lambda_1 + \frac{1}{6}i\omega\right)\eta^3 - \frac{1}{2}\epsilon\omega^2\eta^2\bar{\eta} + i\epsilon\omega\left(2\Lambda_3 - \frac{1}{2}i\omega\right)\eta\bar{\eta}^2 + i\epsilon\omega\left(4\Lambda_4 + \frac{1}{6}i\omega\right)\bar{\eta}^3 \tag{1.45}$$

We note that (1.45) is independent of Δ_1 and Λ_2 and hence they are arbitrary. Moreover, the terms proportional to $\epsilon\eta$ and $\epsilon\eta^2\bar{\eta}$ are resonance terms and hence cannot be eliminated from (1.45). Next, we choose $\Delta_2, \Lambda_1, \Lambda_3$, and Λ_4 to eliminate the terms involving $\bar{\eta}, \eta^3, \eta\bar{\eta}^2$, and $\bar{\eta}^3$, thereby producing the simplest possible equation for η. Thus, we have

$$\Delta_2 = -\frac{i}{4\omega}\,, \quad \Lambda_1 = -\frac{1}{12}i\omega\,, \quad \Lambda_3 = \frac{1}{4}i\omega\,, \quad \Lambda_4 = -\frac{1}{24}i\omega \tag{1.46}$$

With this choice, (1.45) takes the normal form

$$\dot{\eta} = i\omega\eta + \tfrac{1}{2}\epsilon\eta - \tfrac{1}{2}\epsilon\omega^2\eta^2\bar{\eta} \tag{1.47}$$

Again, in this case, we could have identified the resonance terms in (1.42) by one of the procedures described in Section 1.2. Because the solution of the unperturbed problem is proportional to $e^{i\omega t}$, the resonance terms in

$$f(\zeta, \bar{\zeta}) = \zeta - \bar{\zeta} + \tfrac{1}{3}\omega^2(\zeta - \bar{\zeta})^3$$

are the terms proportional to $e^{i\omega t}$ or the time-invariant terms in

$$e^{-i\omega t} f\left[e^{i\omega t} - e^{-i\omega t}, i\omega\left(e^{i\omega t} - e^{-i\omega t}\right)\right]$$

A simple calculation shows that the term $1/2\epsilon(\zeta - \omega^2\zeta^2\bar{\zeta})$ is the only resonance term. Hence, keeping only the resonance terms in (1.42), we have

$$\dot{\zeta} = i\omega\zeta + \tfrac{1}{2}\epsilon\left(\zeta - \omega^2\zeta^2\bar{\zeta}\right) + \cdots$$

which is formally equivalent to (1.47).

Next, we treat (1.42) with the method of multiple scales. To this end, we substitute (1.32) and (1.33) into (1.42), equate coefficients of equal powers of ϵ, and obtain

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.48}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 + \tfrac{1}{2}\left[\zeta_0 - \bar{\zeta}_0 + \tfrac{1}{3}\omega^2\left(\zeta_0 - \bar{\zeta}_0\right)^3\right] \tag{1.49}$$

The solution of (1.48) can be expressed as

$$\zeta_0 = A(T_1)e^{i\omega T_0} \tag{1.50}$$

Then, (1.49) becomes

$$\begin{aligned} D_0\zeta_1 - i\omega\zeta_1 = {} & -A' e^{i\omega T_0} + \tfrac{1}{2}Ae^{i\omega T_0} - \tfrac{1}{2}\bar{A}e^{-i\omega T_0} + \tfrac{1}{6}\omega^2 A^3 e^{3i\omega T_0} \\ & - \tfrac{1}{2}\omega^2 A^2 \bar{A} e^{i\omega T_0} + \tfrac{1}{2}\omega^2 A\bar{A}^2 e^{-i\omega T_0} - \tfrac{1}{6}\omega^2 \bar{A}^3 e^{-3i\omega T_0} \end{aligned} \tag{1.51}$$

Eliminating the terms that lead to secular terms from (1.51), we have

$$A' = \tfrac{1}{2}A - \tfrac{1}{2}\omega^2 A^2 \bar{A} \tag{1.52}$$

Letting $\eta = Ae^{i\omega t}$ in (1.47), we obtain (1.52) because $T_1 = \epsilon t$.

1.4
Duffing–Rayleigh–van der Pol Equation

The Duffing, Rayleigh, and van der Pol equations are special cases of

$$\ddot{u} + \omega^2 u = \epsilon\left(\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right) \tag{1.53}$$

so that

$$f = \epsilon\left(\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right)$$

and (1.14) becomes

$$\begin{aligned} \dot{\zeta} = i\omega\zeta - \frac{i\epsilon}{2\omega}\Big[& i\mu\omega\left(\zeta - \bar{\zeta}\right) + \alpha_1\left(\zeta + \bar{\zeta}\right)^3 + i\omega\alpha_2\left(\zeta + \bar{\zeta}\right)^2\left(\zeta - \bar{\zeta}\right) \\ & -\omega^2\alpha_3\left(\zeta + \bar{\zeta}\right)\left(\zeta - \bar{\zeta}\right)^2 - i\omega^3\alpha_4\left(\zeta - \bar{\zeta}\right)^3\Big] \end{aligned} \tag{1.54}$$

Using the transformation (1.16), where $h = O(\epsilon)$, we rewrite (1.54) as

$$\begin{aligned} \dot{\eta} = {} & i\omega\eta + i\omega h - \frac{\partial h}{\partial \eta}\dot{\eta} - \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} \\ & - \frac{i\epsilon}{2\omega}\left[i\mu\omega\left(\eta - \bar{\eta}\right) + i\omega\alpha_2\left(\eta + \bar{\eta}\right)^2\left(\eta - \bar{\eta}\right)\right. \\ & \left. +\alpha_1\left(\eta + \bar{\eta}\right)^3 - \omega^2\alpha_3\left(\eta + \bar{\eta}\right)\left(\eta - \bar{\eta}\right)^2 - i\omega^3\alpha_4\left(\eta - \bar{\eta}\right)^3\right] \end{aligned} \tag{1.55}$$

where terms of $O(\epsilon^2)$ and higher have been neglected.

Again, because the perturbation contains linear as well as third-order terms, h has the form (1.44). Moreover, to the first approximation, $\dot{\eta}$ and $\dot{\bar{\eta}}$ are given by

(1.19). Hence, substituting (1.19) and (1.44) into (1.55) yields

$$\begin{aligned}\dot{\eta} &= i\omega\eta + 2i\epsilon\omega\left(\Delta_2 + \frac{i\mu}{4\omega}\right)\bar{\eta} \\ &\quad - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\eta^2\bar{\eta} \\ &\quad + \frac{1}{2}\epsilon\mu\eta + i\epsilon\omega\left[-2\Lambda_1 - \frac{1}{2\omega^2}\left(\alpha_1 + i\omega\alpha_2 - \omega^2\alpha_3 - i\omega^3\alpha_4\right)\right]\eta^3 \\ &\quad + i\epsilon\omega\left[2\Lambda_3 - \frac{1}{2\omega^2}\left(3\alpha_1 - i\omega\alpha_2 + \omega^2\alpha_3 - 3i\omega^3\alpha_4\right)\right]\eta\bar{\eta}^2 \\ &\quad + i\epsilon\omega\left[4\Lambda_4 - \frac{1}{2\omega^2}\left(\alpha_1 - i\omega\alpha_2 - \omega^2\alpha_3 + i\omega^3\alpha_4\right)\right]\bar{\eta}^3 \end{aligned} \tag{1.56}$$

We note that Δ_1 and Λ_2 do not appear in (1.56) and hence they are arbitrary and the terms η and $\eta^2\bar{\eta}$ are resonance terms. To produce the simplest form for (1.56), we choose $\Delta_2, \Lambda_1, \Lambda_3$, and Λ_4 to eliminate the terms involving $\bar{\eta}, \eta^3, \eta\bar{\eta}^2$, and $\bar{\eta}^3$; that is,

$$\Delta_2 = -\frac{i\mu}{4\omega} \tag{1.57}$$

$$\Lambda_1 = -\frac{1}{4\omega^2}\left(\alpha_1 + i\omega\alpha_2 - \omega^2\alpha_3 - i\omega^3\alpha_4\right) \tag{1.58}$$

$$\Lambda_3 = \frac{1}{4\omega^2}\left(3\alpha_1 - i\omega\alpha_2 + \omega^2\alpha_3 - 3i\omega^3\alpha_4\right) \tag{1.59}$$

$$\Lambda_4 = \frac{1}{8\omega^2}\left(\alpha_1 - i\omega\alpha_2 - \omega^2\alpha_3 + i\omega^3\alpha_4\right) \tag{1.60}$$

With these choices, (1.56) assumes the simple form

$$\dot{\eta} = i\omega\eta + \frac{1}{2}\epsilon\mu\eta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\eta^2\bar{\eta} \tag{1.61}$$

Again, we did not have to go through the lengthy algebra to arrive at the normal form (1.61). Because the solution of the unperturbed problem (1.54) is proportional to $e^{i\omega t}$, we could have replaced ζ with $e^{i\omega t}$ in the perturbation and identified the terms proportional to $e^{i\omega t}$. In this case, they are

$$\frac{1}{2}\epsilon\mu\zeta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\zeta^2\bar{\zeta}$$

Hence, keeping only the resonance terms in (1.54), we obtain the normal form

$$\dot{\zeta} = i\omega\zeta + \frac{1}{2}\epsilon\mu\zeta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\zeta^2\bar{\zeta}$$

which is formally equivalent to (1.61).

1.5 An Oscillator with Quadratic and Cubic Nonlinearities

We consider free oscillations of a single-degree-of-freedom system governed by

$$\ddot{u} + \omega^2 u + \delta u^2 + \alpha u^3 = 0 \tag{1.62}$$

To keep track of the different orders of magnitude, we use a nondimensional parameter ϵ that is the order of the amplitude of oscillations and hence rewrite (1.62) as

$$\ddot{u} + \omega^2 u + \epsilon \delta u^2 + \epsilon^2 \alpha u^3 = 0 \tag{1.63}$$

Thus, $f = -\epsilon \delta u^2 - \epsilon^2 \alpha u^3$ and (1.14) becomes

$$\dot{\zeta} = i\omega\zeta + \frac{i\epsilon\delta}{2\omega}\left(\zeta + \bar{\zeta}\right)^2 + \frac{i\epsilon^2\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{1.64}$$

In the next section, we use two successive transformations to produce the normal form of (1.64). In Section 1.5.3, we use a single transformation to produce the same normal form, and in Section 1.5.2, we use the method of multiple scales to determine a second-order expansion of (1.64).

1.5.1 Successive Transformations

To simplify the $O(\epsilon)$ terms in (1.64), we introduce the near-identity transformation

$$\zeta = \eta + \epsilon h_1(\eta, \bar{\eta}) \tag{1.65}$$

and rewrite (1.64) as

$$\dot{\eta} = i\omega\eta + i\epsilon\omega h_1 - \epsilon\frac{\partial h_1}{\partial \eta}\dot{\eta} - \epsilon\frac{\partial h_1}{\partial \bar{\eta}}\dot{\bar{\eta}} + \frac{i\epsilon\delta}{2\omega}\left(\eta + \bar{\eta} + \epsilon h_1 + \epsilon \bar{h}_1\right)^2 + \frac{i\epsilon^2\alpha}{2\omega}(\eta + \bar{\eta})^3 + \cdots \tag{1.66}$$

The form of the $O(\epsilon)$ terms suggests choosing h_1 in the form

$$h_1 = \Gamma_1\eta^2 + \Gamma_2\eta\bar{\eta} + \Gamma_3\bar{\eta}^2 \tag{1.67}$$

It follows from (1.66) that

$$\dot{\eta} = i\omega\eta + O(\epsilon) \quad \text{and} \quad \dot{\bar{\eta}} = -i\omega\bar{\eta} + O(\epsilon)$$

so that to $O(\epsilon)$ (1.66) becomes

$$\dot{\eta} = i\omega\eta + i\epsilon\omega\left(-\Gamma_1 + \frac{\delta}{2\omega^2}\right)\eta^2 + i\epsilon\omega\left(\Gamma_2 + \frac{\delta}{\omega^2}\right)\eta\bar{\eta} + i\epsilon\omega\left(3\Gamma_3 + \frac{\delta}{2\omega^2}\right)\bar{\eta}^2 + O(\epsilon^2) \tag{1.68}$$

The simplest possible form for (1.68) corresponds to the vanishing of the terms involving $\eta^2, \eta\bar{\eta}$, and $\bar{\eta}^2$; that is, choosing Γ_1, Γ_2, and Γ_3 to be

$$\Gamma_1 = \frac{\delta}{2\omega^2}, \quad \Gamma_2 = -\frac{\delta}{\omega^2}, \quad \Gamma_3 = -\frac{\delta}{6\omega^2} \tag{1.69}$$

Then, (1.68) reduces to

$$\dot{\eta} = i\omega\eta + O(\epsilon^2) \tag{1.70}$$

Substituting (1.67) into (1.66) and using (1.70) to eliminate $\dot{\eta}$ and $\dot{\bar{\eta}}$, we obtain

$$\dot{\eta} = i\omega\eta + \frac{i\epsilon^2}{2\omega}\left[\alpha(\eta+\bar{\eta})^3 + \frac{2\delta^2}{3\omega^2}\left(\eta^3+\bar{\eta}^3-5\eta^2\bar{\eta}-5\eta\bar{\eta}^2\right)\right]+\cdots \tag{1.71}$$

Next, we introduce a near-identity transformation from η to ξ in the form

$$\eta = \xi + \epsilon^2 h_2(\xi, \bar{\xi}) \tag{1.72}$$

and obtain

$$\begin{aligned}\dot{\xi} &= i\omega\xi + i\epsilon^2\omega h_2 - \epsilon^2\frac{\partial h_2}{\partial\xi}\dot{\xi} - \epsilon^2\frac{\partial h_2}{\partial\bar{\xi}}\dot{\bar{\xi}} \\ &\quad + \frac{i\epsilon^2}{2\omega}\left[\alpha\left(\xi+\bar{\xi}\right)^3 + \frac{2\delta^2}{3\omega^2}\left(\xi^3+\bar{\xi}^3-5\xi^2\bar{\xi}-5\xi\bar{\xi}^2\right)\right]+\cdots\end{aligned} \tag{1.73}$$

The form of the $O(\epsilon^2)$ terms suggests choosing h_2 in the form

$$h_2 = \Lambda_1\xi^3 + \Lambda_2\xi^2\bar{\xi} + \Lambda_3\xi\bar{\xi}^2 + \Lambda_4\bar{\xi}^3 \tag{1.74}$$

It follows from (1.73) that

$$\dot{\xi} = i\omega\xi + O(\epsilon^2) \quad \text{and} \quad \dot{\bar{\xi}} = -i\omega\bar{\xi} + O(\epsilon^2) \tag{1.75}$$

Therefore, substituting (1.74) and (1.75) into the right-hand side of (1.73) and keeping terms up to $O(\epsilon^2)$, we have

$$\begin{aligned}\dot{\xi} &= i\omega\xi + i\epsilon^2\omega\left(-2\Lambda_1 + \frac{\alpha}{2\omega^2} + \frac{\delta^2}{3\omega^4}\right)\xi^3 \\ &\quad + i\epsilon^2\omega\left(4\Lambda_4 + \frac{\alpha}{2\omega^2} + \frac{\delta^2}{3\omega^4}\right)\bar{\xi}^3 + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\xi^2\bar{\xi} \\ &\quad + i\epsilon^2\omega\left(2\Lambda_3 + \frac{3\alpha}{2\omega^2} - \frac{5\delta^2}{3\omega^4}\right)\xi\bar{\xi}^2 + \cdots\end{aligned} \tag{1.76}$$

We note that (1.76) is independent of Λ_2 and hence it is arbitrary and the term $\xi^2\bar{\xi}$ is a resonance term. Equation 1.76 takes the simplest possible form if

$$\Lambda_1 = \frac{\alpha}{4\omega^2} + \frac{\delta^2}{6\omega^4}, \quad \Lambda_3 = -\frac{3\alpha}{4\omega^2} + \frac{5\delta^2}{6\omega^4}, \quad \Lambda_4 = -\frac{\alpha}{8\omega^2} - \frac{\delta^2}{12\omega^4} \tag{1.77}$$

Then, (1.76) becomes

$$\dot{\xi} = i\omega\xi + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\xi^2\bar{\xi} + \cdots \tag{1.78}$$

Substituting (1.65) into (1.10), we have

$$u = \zeta + \bar{\zeta} = \eta + \bar{\eta} + \epsilon h_1(\eta, \bar{\eta}) + \epsilon \bar{h}_1(\eta, \bar{\eta}) \tag{1.79}$$

Then, substituting (1.72) into (1.79) yields

$$u = \xi + \bar{\xi} + \epsilon h_1(\xi, \bar{\xi}) + \epsilon \bar{h}_1(\xi, \bar{\xi}) + \cdots \tag{1.80}$$

Substituting for h_1 from (1.67) into (1.80) and using (1.69), we obtain

$$u = \xi + \bar{\xi} + \frac{\epsilon\delta}{3\omega^2}\left(\xi^2 - 6\xi\bar{\xi} + \bar{\xi}^2\right) + \cdots \tag{1.81}$$

Substituting the polar form

$$\xi = \tfrac{1}{2} a e^{i(\omega t + \beta)}$$

into (1.81), we find that

$$u = a\cos(\omega t + \beta) + \frac{\epsilon\delta a^2}{6\omega^2}\left[\cos(2\omega t + 2\beta) - 3\right] + \cdots \tag{1.82}$$

Substituting the polar form into (1.78) and separating real and imaginary parts, we have

$$\dot{a} = 0 \tag{1.83}$$

$$a\dot{\beta} = \epsilon^2 a^3\left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right) \tag{1.84}$$

Equations 1.82–1.84 are in full agreement with those obtained by using the method of multiple scales, as shown in the next section.

1.5.2
The Method of Multiple Scales

Using the method of multiple scales, we seek a second-order uniform expansion of the solution of (1.64) in the form

$$\zeta(t;\epsilon) = \sum_{n=0}^{2} \epsilon^n \zeta_n(T_0, T_1, T_2) + \cdots \tag{1.85}$$

where $T_n = \epsilon^n t$. In terms of these scales, the time derivative becomes

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \quad D_n = \frac{\partial}{\partial T_n} \tag{1.86}$$

Substituting (1.85) and (1.86) into (1.64) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.87}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 + \frac{i\delta}{2\omega}\left(\zeta_0 + \bar{\zeta}_0\right)^2 \tag{1.88}$$

Order (ϵ^2)

$$D_0\zeta_2 - i\omega\zeta_2 = -D_2\zeta_0 - D_1\zeta_1 + \frac{i\delta}{\omega}\left(\zeta_0 + \bar{\zeta}_0\right)\left(\zeta_1 + \bar{\zeta}_1\right) + \frac{i\alpha}{2\omega}\left(\zeta_0 + \bar{\zeta}_0\right)^3 \tag{1.89}$$

The general solution of (1.87) can be expressed as

$$\zeta_0 = A\left(T_1, T_2\right) e^{i\omega T_0} \tag{1.90}$$

where A is an undetermined function of T_1 and T_2 at this order; it is determined by eliminating the secular terms at the next orders of approximation.

Substituting (1.90) into (1.88) yields

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1Ae^{i\omega T_0} + \frac{i\delta}{2\omega}\left(A^2e^{2i\omega T_0} + 2A\bar{A} + \bar{A}^2e^{-2i\omega T_0}\right) \tag{1.91}$$

Eliminating the terms that produce secular terms in (1.91) demands that $D_1A = 0$ or $A = A(T_2)$. Then, the solution of (1.91) can be expressed as

$$\zeta_1 = \frac{\delta A^2}{2\omega^2}e^{2i\omega T_0} - \frac{\delta A\bar{A}}{\omega^2} - \frac{\delta\bar{A}^2}{6\omega^2}e^{-2i\omega T_0} \tag{1.92}$$

where the solution of the homogeneous equation has not been included so that the amplitude of the term at the frequency of oscillation is uniquely defined by the zeroth-order solution (1.90). We note that the coefficients in (1.92) are the same as the Γ_i defined in (1.69).

Substituting (1.90) and (1.92) into (1.89) and using the fact that $D_1A = 0$, we have

$$D_0\zeta_2 - i\omega\zeta_2 = -D_2Ae^{i\omega T_0} + \frac{i}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)A^2\bar{A}e^{i\omega T_0} + \text{NST} \tag{1.93}$$

where NST stands for the terms that do not produce secular terms. Eliminating the terms that produce secular terms from (1.93), we obtain

$$D_2A = \frac{i}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)A^2\bar{A} \tag{1.94}$$

Putting $\xi = Ae^{i\omega t}$ in (1.78) and using the fact that $D_2 A = \epsilon^2 dA/dt$, we obtain exactly (1.94).

We note that, for a uniform second approximation, we do not need to solve for ζ_2, but we only need to inspect (1.93) and eliminate the terms that produce secular terms. Similarly, to determine a uniform second approximation by using the method of normal forms, we do not need to determine h_2 in (1.72), but we need only keep the resonance terms in (1.73).

1.5.3
A Single Transformation

Instead of using successive transformations to produce the normal form of (1.64), one can formulate the process as a perturbation method. Thus, we expand ζ in a power series of ϵ in terms of a new variable η in the form

$$\zeta = \eta + \epsilon h_1(\eta, \bar{\eta}) + \epsilon^2 h_2(\eta, \bar{\eta}) + \cdots \tag{1.95}$$

$$\dot{\eta} = i\omega\eta + \epsilon g_1(\eta, \bar{\eta}) + \epsilon^2 g_2(\eta, \bar{\eta}) + \cdots \tag{1.96}$$

where h_1 and h_2 are smooth functions of η and $\bar{\eta}$ and g_1 and g_2 contain all of the resonance and near-resonance terms. Substituting (1.95) and (1.96) into (1.64) and equating coefficients of like powers of ϵ, we obtain

$$g_1 + i\omega\left(\eta\frac{\partial h_1}{\partial\eta} - \bar{\eta}\frac{\partial h_1}{\partial\bar{\eta}} - h_1\right) = \frac{i\delta}{2\omega}(\eta + \bar{\eta})^2 \tag{1.97}$$

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) = -g_1\frac{\partial h_1}{\partial\eta} - \bar{g}_1\frac{\partial h_1}{\partial\bar{\eta}} + \frac{i\alpha}{2\omega}(\eta + \bar{\eta})^3 + \frac{i\delta}{\omega}(\eta + \bar{\eta})\left(h_1 + \bar{h}_1\right) \tag{1.98}$$

Equations 1.97 and 1.98 are the so-called homology equations for h_1 and h_2.

Next, we need to determine g_1 and h_1 from (1.97). In order that h_1 be nonsingular (smooth), we choose g_1 to eliminate all of the resonance and near-resonance terms; otherwise, h_1 will be singular (i.e., have secular terms) if there are resonance terms and near singular (i.e., have small divisors) if there are near-resonance terms. In the present case, there are no resonance terms in (1.97). To see this, we note from (1.96) that $\eta = Be^{i\omega t}$ and hence the perturbation terms on the right-hand side of (1.97) contain terms proportional to $e^{\pm 2i\omega t}$ and a constant. Because none of these terms is proportional to $e^{i\omega t}$, which is the solution of the first-order problem in (1.96), there are no resonance terms. If we are in doubt, we seek a function h_1 that can be used to eliminate all of the perturbation terms. If we are successful in finding a smooth function h_1 that eliminates all of the perturbation terms, then $g_1 = 0$. Otherwise, we choose g_1 to eliminate all terms that produced the troublesome terms (i.e., singular and near-singular terms) in h_1. Because the perturbation terms are of second degree, we let

$$h_1 = \Gamma_1\eta^2 + \Gamma_2\eta\bar{\eta} + \Gamma_3\bar{\eta}^2 \tag{1.99}$$

choose Γ_1, Γ_2, and Γ_3 to eliminate $i\delta(\eta+\bar{\eta})^2/2\omega$ and obtain (1.69). Because the obtained Γ_m are regular, there are no resonance terms and $g_1 = 0$.

Substituting (1.99) and $g_1 = 0$ into (1.98) and using (1.69), we obtain

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial \eta} - \bar{\eta}\frac{\partial h_2}{\partial \bar{\eta}} - h_2\right) = \frac{i\delta^2}{3\omega^3}(\eta+\bar{\eta})\left(\eta^2+\bar{\eta}^2-6\eta\bar{\eta}\right) + \frac{i\alpha}{2\omega}(\eta+\bar{\eta})^3 \tag{1.100}$$

As stated earlier, we do not need to determine h_2 and all that we need is to inspect (1.100) and choose g_2 to eliminate all of the resonance and near-resonance terms. Again, because $\eta \propto e^{i\omega t}$, only the term proportional to $\eta^2\bar{\eta}$ is a resonance term. Hence, choosing g_2 to eliminate this term, we obtain

$$g_2 = \frac{i}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} \tag{1.101}$$

Substituting (1.95) and (1.99) into (1.10) and using (1.69), we obtain

$$u = \eta + \bar{\eta} + \frac{\epsilon\delta}{3\omega^2}\left(\eta^2+\bar{\eta}^2-6\eta\bar{\eta}\right) + \cdots \tag{1.102}$$

Substituting for g_2 from (1.101) into (1.96) and using the fact that $g_1 = 0$, we have

$$\dot{\eta} = i\omega\eta + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} + \cdots \tag{1.103}$$

in agreement through second order with the expansions obtained in Sections 1.5.1 and 1.5.2.

1.6
A General System with Quadratic and Cubic Nonlinearities

We consider free oscillations of a single-degree-of-freedom system governed by

$$\ddot{u} + \omega^2 u + \epsilon\left(\delta_1 u^2 + \delta_2 \dot{u}^2\right) + \epsilon^2\left(2\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right) = 0 \tag{1.104}$$

so that

$$f = -\epsilon\left(\delta_1 u^2 + \delta_2 \dot{u}^2\right) - \epsilon^2\left(2\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right) \tag{1.105}$$

We note that a small nondimensional parameter ϵ has been introduced as a bookkeeping device. The quadratic terms have been ordered as $O(\epsilon)$, whereas the cubic terms and the linear damping term have been ordered as $O(\epsilon^2)$. Then, (1.14) becomes

$$\begin{aligned}\dot{\zeta} &= i\omega\zeta + \frac{i\epsilon}{2\omega}\left[\delta_1\left(\zeta+\bar{\zeta}\right)^2 - \delta_2\omega^2\left(\zeta-\bar{\zeta}\right)^2\right] - \epsilon^2\mu\left(\zeta-\bar{\zeta}\right)\\ &+ \frac{i\epsilon^2}{2\omega}\left[\alpha_1\left(\zeta+\bar{\zeta}\right)^3 + i\omega\alpha_2\left(\zeta+\bar{\zeta}\right)^2\left(\zeta-\bar{\zeta}\right)\right.\\ &\left.-\omega^2\alpha_3\left(\zeta+\bar{\zeta}\right)\left(\zeta-\bar{\zeta}\right)^2 - i\omega^3\alpha_4\left(\zeta-\bar{\zeta}\right)^3\right] \end{aligned} \tag{1.106}$$

As in Section 1.5.3, we seek an expansion for (1.106) in the form (1.95) and (1.96), equate coefficients of like powers of ϵ, and obtain

$$g_1 + i\omega\left(\eta\frac{\partial h_1}{\partial\eta} - \bar{\eta}\frac{\partial h_1}{\partial\bar{\eta}} - h_1\right) = \frac{i}{2\omega}\left[\delta_1(\eta+\bar{\eta})^2 - \delta_2\omega^2(\eta-\bar{\eta})^2\right] \tag{1.107}$$

$$\begin{aligned} g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) &= -g_1\frac{\partial h_1}{\partial\eta} - \bar{g}_1\frac{\partial h_1}{\partial\bar{\eta}} - \mu(\eta-\bar{\eta})\\ + \frac{i}{\omega}\left[\delta_1(\eta+\bar{\eta})\left(h_1+\bar{h}_1\right) - \delta_2\omega^2(\eta-\bar{\eta})\left(h_1-\bar{h}_1\right)\right] &+ \frac{i}{2\omega}\alpha_1(\eta+\bar{\eta})^3\\ + \frac{i}{2\omega}\left[i\omega\alpha_2(\eta+\bar{\eta})^2(\eta-\bar{\eta}) - i\omega^3\alpha_4(\eta-\bar{\eta})^3 - \omega^2\alpha_3(\eta+\bar{\eta})(\eta-\bar{\eta})^2\right] \end{aligned} \tag{1.108}$$

The right-hand side of (1.107) does not contain resonance or near-resonance terms, and hence we put $g_1 = 0$, seek h_1 in the form (1.99), and obtain

$$\Gamma_1 = \frac{\delta_1}{2\omega^2} - \frac{1}{2}\delta_2\,, \quad \Gamma_2 = -\frac{\delta_1}{\omega^2} - \delta_2\,, \quad \Gamma_3 = -\frac{\delta_1}{6\omega^2} + \frac{1}{6}\delta_2 \tag{1.109}$$

Because all of the Γ_m are regular, our conclusion that there are no resonance or near-resonance terms in (1.107) and hence $g_1 = 0$ is justified a posteriori.

Substituting (1.99) and (1.109) into (1.108) and using the fact that $g_1 = 0$, we obtain

$$\begin{aligned} g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) &= -\frac{2}{3}i\delta_2\omega\left(\frac{\delta_1}{\omega^2} - \delta_2\right)(\eta-\bar{\eta})\left(\eta^2-\bar{\eta}^2\right)\\ -\mu(\eta-\bar{\eta}) + \frac{i\delta_1}{3\omega}(\eta+\bar{\eta})&\left[\left(\frac{\delta_1}{\omega^2} - \delta_2\right)\left(\eta^2+\bar{\eta}^2\right) - 6\left(\frac{\delta_1}{\omega^2} + \delta_2\right)\eta\bar{\eta}\right]\\ + \frac{i}{2\omega}\left[\alpha_1(\eta+\bar{\eta})^3 + i\omega\alpha_2(\eta+\bar{\eta})^2(\eta-\bar{\eta}) - \omega^2\alpha_3(\eta+\bar{\eta})(\eta-\bar{\eta})^2\right.&\\ \left.-i\omega^3\alpha_4(\eta-\bar{\eta})^3\right] + \cdots \end{aligned} \tag{1.110}$$

Inspecting the right-hand side of (1.110), we conclude that the terms proportional to η and $\eta^2\bar{\eta}$ are the only resonance terms and there are no near-resonance terms. Consequently, choosing g_2 to eliminate the resonance terms, we obtain

$$\begin{aligned} g_2 = -\mu\eta + \frac{i}{2\omega}&\left[3\alpha_1 + \omega^2\alpha_3 - \frac{2}{3}\left(\frac{5\delta_1^2}{\omega^2} + 5\delta_1\delta_2 + 2\delta_2^2\omega^2\right)\right.\\ &\left.+i\omega\left(\alpha_2 + 3\omega^2\alpha_4\right)\right]\eta^2\bar{\eta} \end{aligned} \tag{1.111}$$

Substituting (1.95) and (1.99) into (1.10) and using (1.109), we obtain

$$u = \eta + \bar{\eta} + \epsilon \left[\left(\frac{\delta_1}{3\omega^2} - \frac{1}{3}\delta_2 \right) (\eta^2 + \bar{\eta}^2) - \left(\frac{2\delta_1}{\omega^2} + 2\delta_2 \right) \eta\bar{\eta} \right] + \cdots \quad (1.112)$$

Substituting for g_2 from (1.111) into (1.96) and using the fact that $g_1 = 0$, we obtain

$$\dot{\eta} = i\omega\eta - \epsilon^2\mu\eta + \frac{i\epsilon^2}{2\omega} \left[3\alpha_1 + \omega^2\alpha_3 - \frac{2}{3} \left(\frac{5\delta_1^2}{\omega^2} + 5\delta_1\delta_2 + 2\delta_2^2\omega^2 \right) + i\omega \left(\alpha_2 + 3\omega^2\alpha_4 \right) \right] \eta^2\bar{\eta} \quad (1.113)$$

To compare the expansion (1.112) and (1.113) with that obtained by using the method of multiple scales (Nayfeh, 1984), we substitute the polar form

$$\eta = \tfrac{1}{2} a e^{i(\omega t + \beta)} \quad (1.114)$$

in (1.112) and (1.113) and obtain

$$u = a\cos(\omega t + \beta) + \frac{1}{6}\epsilon a^2 \left[\left(\frac{\delta_1}{\omega^2} - \delta_2 \right) \cos(2\omega t + 2\beta) - 3 \left(\frac{\delta_1}{\omega^2} + \delta_2 \right) \right] + \cdots \quad (1.115)$$

where

$$\dot{a} = -\epsilon^2\mu a - \tfrac{1}{8}\epsilon^2 \left(\alpha_2 + 3\omega^2\alpha_4 \right) a^3 + \cdots \quad (1.116)$$

$$a\dot{\beta} = \frac{\epsilon^2}{8\omega} \left[3\alpha_1 + \omega^2\alpha_3 - \frac{2}{3} \left(\frac{5\delta_1}{\omega^2} + 5\delta_1\delta_2 + 2\delta_2^2\omega^2 \right) \right] a^3 + \cdots \quad (1.117)$$

which is formally equivalent to that obtained by using the method of multiple scales.

1.7 The van der Pol Oscillator

In this section, we construct a second-order approximation of the normal form of the van der Pol oscillator

$$\ddot{u} + \omega^2 u = \epsilon(1 - u^2)\dot{u} \quad (1.118)$$

Using the transformation (1.10), we rewrite (1.118) as

$$\dot{\zeta} = i\omega\zeta + \tfrac{1}{2}\epsilon(\zeta - \bar{\zeta}) \left[1 - (\zeta + \bar{\zeta})^2 \right] \quad (1.119)$$

1.7.1 The Method of Normal Forms

As in the preceding two sections, we seek a second-order expansion of (1.119) in the form (1.95) and (1.96), equate coefficients of like powers of ϵ, and obtain

$$g_1 + i\omega\left(\eta\frac{\partial h_1}{\partial\eta} - \bar{\eta}\frac{\partial h_1}{\partial\bar{\eta}} - h_1\right) = \frac{1}{2}(\eta - \bar{\eta}) - \frac{1}{2}\left(\eta^3 + \eta^2\bar{\eta} - \bar{\eta}^2\eta - \bar{\eta}^3\right) \tag{1.120}$$

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) = -g_1\frac{\partial h_1}{\partial\eta} - \bar{g}_1\frac{\partial h_1}{\partial\bar{\eta}} + \frac{1}{2}\left(h_1 - \bar{h}_1\right)$$
$$-\frac{3}{2}\eta^2 h_1 - \eta\bar{\eta}(h_1 - \bar{h}_1) - \frac{1}{2}\eta^2\bar{h}_1 + \frac{1}{2}\bar{\eta}^2 h_1 + \frac{3}{2}\bar{\eta}^2\bar{h}_1 \tag{1.121}$$

Choosing g_1 to eliminate the resonance terms in (1.120), we have

$$g_1 = \tfrac{1}{2}\eta - \tfrac{1}{2}\eta^2\bar{\eta} \tag{1.122}$$

Then, we seek h_1 in the form

$$h_1 = \Delta_1\bar{\eta} + \Lambda_1\eta^3 + \Lambda_2\eta\bar{\eta}^2 + \Lambda_3\bar{\eta}^3 \tag{1.123}$$

Substituting (1.123) and (1.122) into (1.120) yields

$$\left(2i\omega\Delta_1 - \tfrac{1}{2}\right)\bar{\eta} - \left(2i\omega\Lambda_1 + \tfrac{1}{2}\right)\eta^3 + \left(2i\omega\Lambda_2 + \tfrac{1}{2}\right)\eta\bar{\eta}^2$$
$$+ \left(4i\omega\Lambda_3 + \tfrac{1}{2}\right)\bar{\eta}^3 = 0 \tag{1.124}$$

Hence,

$$\Delta_1 = -\frac{i}{4\omega}, \quad \Lambda_1 = \frac{i}{4\omega}, \quad \Lambda_2 = \frac{i}{4\omega}, \quad \Lambda_3 = \frac{i}{8\omega} \tag{1.125}$$

Therefore,

$$h_1 = -\frac{1}{4\omega}i\left(\bar{\eta} - \eta^3 - \eta\bar{\eta}^2 - \frac{1}{2}\bar{\eta}^3\right) \tag{1.126}$$

Substituting (1.122) and (1.126) into (1.121) yields

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) = -\frac{i}{16\omega}\left(2\eta + 5\eta^3 + 5\eta^5 - 12\eta^2\bar{\eta}\right.$$
$$\left.-2\eta^4\bar{\eta} - 4\eta\bar{\eta}^2 + 11\eta^3\bar{\eta}^2 + 2\bar{\eta}^3 + 5\eta^2\bar{\eta}^3 + \eta\bar{\eta}^4 + 5\bar{\eta}^5\right) \tag{1.127}$$

Choosing g_2 to eliminate the resonance terms (terms proportional to η, $\eta^2\bar{\eta}$, and $\eta^3\bar{\eta}^2$) from (1.127), we have

$$g_2 = -\frac{1}{16\omega}i\left(2\eta - 12\eta^2\bar{\eta} + 11\eta^3\bar{\eta}^2\right) \tag{1.128}$$

Substituting (1.122) and (1.128) into (1.96), we obtain, to the second approximation, the normal form

$$\dot{\eta} = i\omega\eta + \frac{1}{2}\epsilon\left(\eta - \eta^2\bar{\eta}\right) - \frac{1}{16\omega}i\epsilon^2\left(2\eta - 12\eta^2\bar{\eta} + 11\eta^3\bar{\eta}^2\right) \tag{1.129}$$

Substituting the polar form (1.27) into (1.129) and separating real and imaginary parts, we obtain

$$\dot{a} = \tfrac{1}{2}\epsilon\left(a - \tfrac{1}{4}a^3\right) \tag{1.130}$$

$$\dot{\beta} = -\frac{1}{8\omega}\epsilon^2\left(1 - \frac{3}{2}a^2 + \frac{11}{32}a^4\right) \tag{1.131}$$

in agreement with those obtained with the generalized method of averaging (Nayfeh, 1973).

1.7.2 The Method of Multiple Scales

We seek a second-order expansion of the solution of (1.119) in the form (1.85). Substituting (1.85) and (1.86) into (1.119) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.132}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 + \tfrac{1}{2}\left(\zeta_0 - \bar{\zeta}_0\right) - \tfrac{1}{2}\left(\zeta_0^3 + \zeta_0^2\bar{\zeta}_0 - \zeta_0\bar{\zeta}_0^2 - \bar{\zeta}_0^3\right) \tag{1.133}$$

Order (ϵ^2)

$$\begin{aligned} D_0\zeta_2 - i\omega\zeta_2 = {} & -D_2\zeta_0 - D_1\zeta_1 + \tfrac{1}{2}\left(\zeta_1 - \bar{\zeta}_1\right) - \tfrac{1}{2}\left(3\zeta_0^2 + 2\zeta_0\bar{\zeta}_0 - \bar{\zeta}_0^2\right)\zeta_1 \\ & - \tfrac{1}{2}\left(\zeta_0^2 - 2\zeta_0\bar{\zeta}_0 - 3\bar{\zeta}_0^2\right)\bar{\zeta}_1 \end{aligned} \tag{1.134}$$

The general solution of (1.132) can be expressed as in (1.90). Then, (1.133) becomes

$$\begin{aligned} D_0\zeta_1 - i\omega\zeta_1 = {} & -D_1Ae^{i\omega T_0} + \tfrac{1}{2}Ae^{i\omega T_0} - \tfrac{1}{2}\bar{A}e^{-i\omega T_0} - \tfrac{1}{2}A^3e^{3i\omega T_0} \\ & - \tfrac{1}{2}A^2\bar{A}e^{i\omega T_0} + \tfrac{1}{2}A\bar{A}^2e^{-i\omega T_0} + \tfrac{1}{2}\bar{A}^3e^{-3i\omega T_0} \end{aligned} \tag{1.135}$$

Eliminating the terms that lead to secular terms from (1.135) yields

$$D_1A = \tfrac{1}{2}\left(A - A^2\bar{A}\right) \tag{1.136}$$

Then, the solution of (1.135) can be expressed as

$$\zeta_1 = \frac{i}{8\omega}\left(-2\bar{A}e^{-i\omega T_0} + 2A^3e^{3i\omega T_0} + 2A\bar{A}^2e^{-i\omega T_0} + \bar{A}^3e^{-3i\omega T_0}\right) \tag{1.137}$$

Substituting (1.90), (1.136), and (1.137) into (1.134), we have

$$D_0\zeta_2 - i\omega\zeta_2 = -D_2Ae^{i\omega T_0} - \frac{i}{16\omega}\left[2A - 12A^2\bar{A} + 11A^3\bar{A}^2\right]e^{i\omega T_0} + \text{NST} \tag{1.138}$$

Eliminating the terms that lead to secular terms from (1.138) yields

$$D_2A = -\frac{i}{16\omega}\left[2A - 12A^2\bar{A} + 11A^3\bar{A}^2\right] \tag{1.139}$$

Using the method of reconstitution, we obtain from (1.136) and (1.139) that

$$\dot{A} = \frac{1}{2}\epsilon\left(A - A^2\bar{A}\right) - \frac{i}{16\omega}\epsilon^2\left[2A - 12A^2\bar{A} + 11A^3\bar{A}^2\right] \tag{1.140}$$

Letting $\zeta = Ae^{i\omega t}$ in (1.129), we obtain (1.140), which means that the results obtained with the methods of normal forms and multiple scales are the same.

1.8 Exercises

1.8.1 Use the methods of normal forms and multiple scales to determine the normal forms of

a) $\ddot{u} + \omega^2 u + \alpha\dot{u}^3 = 0$,
b) $\ddot{u} + \omega^2 u + \alpha u^2\dot{u} = 0$,
c) $\ddot{u} + \omega^2 u + \alpha u^5 = 0$,
d) $\ddot{u} + \omega^2 u + \alpha u^3\dot{u}^2 = 0$,
e) $\ddot{u} + \omega^2 u + \epsilon\dot{u}^5 = 0$.

1.8.2 Use the methods of multiple scales and normal forms to construct a second-order approximation to the normal form of

$$\ddot{u} + \omega^2 u + \alpha u^3 + 2\mu u^2\dot{u} = 0$$

1.8.3 Use the methods of multiple scales and normal forms to construct a first-order approximation to the normal form of

$$\ddot{u} + \omega^2 u + \alpha u^2\ddot{u} = 0$$

1.8.4 Use the methods of normal forms and multiple scales to construct a second-order approximation to the normal form of

$$\ddot{u} + \omega^2 u + \alpha\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right) = 0$$

1.8.5 Consider the equation

$$\ddot{x} + x + 3x^2 + 2x^3 = 0$$

Determine the equilibrium points. Determine, to second order, the normal form of the system near each of these equilibrium points.

1.8.6 Consider

$$\ddot{x} + x + ax^2 + 2x^3 = 0$$

Show that there is only one equilibrium point when $a < 2\sqrt{2}$ and that there are three equilibrium points when $a > 2\sqrt{2}$. Determine, to second order, the normal forms near the equilibrium points.

1.8.7 Consider

$$\ddot{x} + x - \frac{a}{1-x} = 0$$

Show that there is only one equilibrium point when $a \le 1/4$. Determine, to second order, the normal form near this equilibrium point.

1.8.8 Consider

$$\ddot{x} - 3x + x^3 = -2$$

Determine the equilibrium points and the normal forms near them.

1.8.9 Consider

$$\ddot{u} - u + u^4 = 0$$

Determine the equilibrium points and the normal forms near them.

1.8.10 Consider

$$\ddot{x} - 2x - x^2 + x^3 = 0$$

Determine the equilibrium points and the normal forms near them.

1.8.11 Consider

$$\ddot{u} + u - \frac{3}{16(1-u)} = 0$$

Determine the equilibrium points and the normal forms near them.

1.8.12 Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for the general solution of

$$\ddot{\theta} + \omega^2 \sin\theta + \frac{4\sin^2\theta}{1 + 4(1-\cos\theta)}\dot{\theta} = 0$$

for small but finite θ.

1.8.13 Consider the equation

$$\ddot{u} + \omega_0^2 u + \frac{\mu \dot{u}}{1 - u^2} = 0$$

Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for small u.

1.8.14 Consider the equation

$$\left(l^2 + r^2 - 2rl\cos\theta\right)\ddot{\theta} + rl\sin\theta\,\dot{\theta}^2 + gl\sin\theta = 0$$

where g, r, and l are constants. Determine a first-order expansion for small but finite θ by using the methods of multiple scales and normal forms.

1.8.15 Consider the equation

$$\left(\tfrac{1}{12}l^2 + r^2\theta^2\right)\ddot{\theta} + r^2\theta\,\dot{\theta}^2 + gr\theta\cos\theta = 0$$

where r, l, and g are constants. Determine a first-order uniform expansion for small but finite θ by using the methods of multiple scales and normal forms.

1.8.16 The motion of a simple pendulum is governed by

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for small but finite θ.

1.8.17 Consider the equation

$$\ddot{\theta} = \Omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta$$

Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for small but finite θ.

1.8.18 The motion of a particle on a rotating parabola is governed by

$$\left(1 + 4p^2x^2\right)\ddot{x} + \Lambda x + 4p^2\dot{x}^2 = 0$$

where p and Λ are constants. Use the methods of multiple scales and normal forms to determine a first-order expansion for small but finite x.

1.8.19 Consider the equation

$$\left(1 + \frac{u^2}{1 - u^2}\right)\ddot{u} + \frac{u\dot{u}^2}{(1 - u^2)^2} + \omega_0^2 u + \frac{g}{l}\frac{u}{\sqrt{1 - u^2}} = 0$$

Use the methods of multiple scales and normal forms to determine a first-order expansion for small u.