17
Calculus of Variations

Tomáš Roubíček

17.1
Introduction

The history of the calculus of variations dates back several thousand years, fulfilling the ambition of mankind to seek lucid principles that govern the Universe. Typically, one tries to identify scalar-valued functionals having a clear physical interpretation, for example, time, length, area, energy, and entropy, whose extremal (critical) points (sometimes under some constraints) represent solutions of the problem in question. Rapid development was initiated between the sixteenth and nineteenth centuries when practically every leading scholar, for example, J. Bernoulli, B. Bolzano, L. Euler, P. Fermat, J.L. Lagrange, A.-M. Legendre, G.W. Leibniz, I. Newton, K. Weierstrass and many others, contributed to variational calculus; at that time, the focus was rather on one-dimensional problems cf. also [1–3]. There has been progress through the twentieth century, which is still continuing, informed by the historically important project of Hilbert [4], Problems 19, 20, and 23] and accelerated by the development of functional analysis, theory of partial differential equations, and efficient computational algorithms supported by rigorous numerical analysis and computers of ever-increasing power. Modern methods allow simple formulations in abstract spaces where technicalities are suppressed, cf. Section 17.2, although concrete problems ultimately require additional tools, cf. Section 17.3. An important "side effect" has been the development of a sound theory of optimization and optimal control and of its foundations, convex and nonsmooth analysis.

17.2
Abstract Variational Problems

Variational problems typically deal with a real-valued functional $\Phi : V \to \mathbb{R}$ on an abstract space V that is equipped with a linear structure to handle variations and a topological structure to handle various continuity/stability/localization concepts. In the simplest and usually sufficiently general scenario, V is a *Banach space*[1] [5] or,

1) A linear space equipped with a norm $\|\cdot\|$, that is, $0 \le \|u+v\| \le \|u\|+\|v\|$, $\|u\|=0 \Rightarrow u=0$, $\|\lambda u\| = \lambda\|u\|$ for any $\lambda \ge 0$ and $u, v \in V$, is

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in physics, often even a *Hilbert space*.[2] The Banach space structure allows us to define basic notions, such as linearity, continuity, and convexity: Φ is called *continuous* if $\Phi(u_k) \to \Phi(u)$ for any $u_k \to u$, *convex* if $\Phi(\lambda u + (1-\lambda)v) \le \lambda\Phi(u) + (1-\lambda)\Phi(v)$ for any $u, v \in V$ and $0 \le \lambda \le 1$, *concave* if $-\Phi$ is convex, or *linear* if it is convex, concave, and $\Phi(0) = 0$.

Yet it should be pointed out that the linear structure imposed on a problem is the result of our choice; it serves rather as a mathematical tool used to define variations or laws of evolution, or to devise numerical algorithms, and so on. Often, this choice is rather artificial, especially if it leads to nonquadratic or even nonconvex functionals possibly with nonlinear constraints.

17.2.1 Smooth (Differentiable) Case

The Banach space structure allows further to say that Φ is *directionally differentiable* if the directional derivative at u in the direction of (variation) v, defined as

$$\mathrm{D}\Phi(u, v) = \lim_{\varepsilon \searrow 0} \frac{\Phi(u + \varepsilon v) - \Phi(u)}{\varepsilon}, \quad (17.1)$$

exists for any $u, v \in V$, and is *smooth* if it is directionally differentiable and $\mathrm{D}\Phi(u, \cdot) : V \to \mathbb{R}$ is a linear continuous functional; then the *Gâteaux differential* $\Phi'(u) \in V^*$, with V^* being the *dual space*,[3] is defined by

$$\left\langle \Phi'(u), v \right\rangle = \mathrm{D}\Phi(u, v). \quad (17.2)$$

If $\Phi' : V \to V^*$ is continuous, then Φ is called *continuously differentiable*. Furthermore, $u \in V$ is called a *critical point* if

$$\Phi'(u) = 0, \quad (17.3)$$

which is an abstract version of the *Euler–Lagrange equation*. In fact, (17.3) is a special case of the abstract operator equation

$$A(u) = f \quad \text{with } A : V \to V^*, f \in V^*, \quad (17.4)$$

provided $A = \Phi' + f$ for some *potential* Φ whose existence requires some symmetry of A: if A itself is Gâteaux differentiable and *hemicontinuous*,[4] it has a potential if, and only if, it is *symmetric*, that is,

$$\left\langle [A'(u)](v), w \right\rangle = \left\langle [A'(u)](w), v \right\rangle \quad (17.5)$$

for any $u, v, w \in V$; up to a constant; this potential is given by the formula

$$\Phi(u) = \int_0^1 \left\langle A(\lambda u), u \right\rangle \mathrm{d}\lambda\,. \quad (17.6)$$

Equation (17.3) is satisfied, for example, if Φ attains its minimum[5] or maximum at u. The former case is often connected with a *minimum-energy principle* that is assumed

called a *Banach space* if it is complete, that is, any Cauchy sequence $\{u_k\}_{k\in\mathbb{N}}$ converges: $\lim_{\max(k,l)\to\infty} \|u_k - u_l\| = 0$ implies that there is $u \in V$ such that $\lim_{k\to\infty} \|u_k - u\| = 0$; then we write $u_k \to u$.

2) This is a Banach space V whose norm makes the functional $V \to \mathbb{R} : u \mapsto \|u+v\|^2 - \|u-v\|^2$ linear for any $v \in V$; in this case, we define the scalar product by $(u, v) = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$.

3) The dual space V^* is the Banach space of all linear continuous functionals f on V with the norm $\|f\|_* = \sup_{\|u\|\le 1}\langle f, u\rangle$, with the duality pairing $\langle\cdot,\cdot\rangle : V \times V^* \to \mathbb{R}$ being the bilinear form defined by $\langle f, u\rangle = f(u)$.

4) This is a very weak mode of continuity, requiring that $t \mapsto \langle A(u+tv), w\rangle$ is continuous.

5) The proof is simple: suppose $\Phi(u) = \min \Phi(\cdot)$ but $\Phi'(u) \neq 0$, then for some $v \in V$ we would have $\langle \Phi'(u), v\rangle = \mathrm{D}\Phi(u, v) < 0$ so that, for a sufficiently small $\varepsilon > 0$, $\Phi(u+\varepsilon v) = \Phi(u) + \varepsilon\left\langle \Phi'(u), v\right\rangle + o(\varepsilon) < \Phi(u)$, a contradiction.

to govern many steady-state physical problems. The existence of solutions to (17.3) can thus often be based on the existence of a minimizer of Φ, which can rely on the *Bolzano – Weierstrass theorem*, which states that a *lower* (resp. upper) *semicontinuous* functional[6] on a *compact*[7] set attains its minimum (resp. maximum).

In infinite-dimensional Banach spaces, it is convenient to use this theorem with respect to *weak* convergence*: assuming $V = (V')^*$ for some Banach space V' (called the *pre-dual*), we say that a sequence $\{u_k\}_{k\in\mathbb{N}}$ converges weakly* to u if $\lim_{k\to\infty}\langle u_k, z\rangle = \langle u, z\rangle$ for any $z \in V'$. If V^* is taken instead of V', this mode of convergence is called *weak convergence*. Often $V' = V^*$ (such spaces are called *reflexive*), and then the weak* and the weak convergences coincide. The Bolzano – Weierstrass theorem underlies the *direct method*,[8] invented essentially in [6], for proving existence of a solution to (17.3). We say that Φ is *coercive* if $\lim_{\|u\|\to\infty} \Phi(u)/\|u\| = +\infty$.

Theorem 17.1 (Direct method) [9] *Let V have a pre-dual and $\Phi : V \to \mathbb{R}$ be weakly* lower semicontinuous, smooth, and coercive. Then* (17.3) *has a solution.*

6) Lower semicontinuity of Φ means that $\liminf_{k\to\infty} \Phi(u_k) \ge \Phi(u)$ for any sequence $\{u_k\}_{k\in\mathbb{N}}$ converging (in a sense to be specified) to u; more precisely, this is sequential lower semicontinuity, but we will confine ourselves to the sequential concept throughout the chapter, which is sufficiently general provided the related topologies are metrizable.

7) A set is compact if any sequence has a converging (in the same sense as used for the semicontinuity of the functional) subsequence.

8) This means that no approximation and subsequent convergence is needed.

9) The proof relies on coercivity of Φ, which allows for a localization on bounded sets and then, due to weak* compactness of convex closed bounded sets in V, on the Bolzano – Weierstrass theorem.

AS continuous convex functionals are also weakly* lower semicontinuous, one gets a useful modification:

Theorem 17.2 (Direct method II) *Let V have a pre-dual and let $\Phi : V \to \mathbb{R}$ be continuous, smooth, coercive, and convex. Then* (17.3) *has a solution.*

If Φ is furthermore *strictly convex* in the sense that $\Phi(\lambda u+(1-\lambda)v) < \lambda\Phi(u) + (1-\lambda)\Phi(v)$ for any $u \neq v$ and $0 < \lambda < 1$, then (17.3) has at most one solution.

We say that a nonlinear operator $A : V \to V^*$ is *monotone* if $\langle A(u)-A(v), u-v\rangle \ge 0$ for any $u, v \in V$. Monotonicity of a potential nonlinear operator implies convexity of its potential, and then Theorem 17.2 implies the following.

Theorem 17.3 *Let V be reflexive and $A : V \to V^*$ be monotone, hemicontinuous, coercive in the sense that $\lim_{\|u\|\to\infty}\langle A(u), u\rangle = \infty$, and possess a potential. Then, for any $f \in V^*$,* (17.4) *has a solution.*

In fact, Theorem 17.3 holds even for mappings not having a potential but its proof, due to Brézis [7], then relies on an approximation and on implicit, nonconstructive fixed-point arguments.

The solutions to (17.3) do not need to represent the global minimizers that we have considered so far. Local minimizers, being consistent with physical principles of minimization of energy, would also serve well. The same holds for maximizers. Critical points may, however, have a more complicated saddle-like character. One intuitive example is the following: let the origin, being at the level 0, be surrounded by a range of mountains all of height $h > 0$ at distance ρ from the origin, but assume that there is at least one location v beyond that circle, which has lower altitude. Going from the origin to v, one is tempted to minimize climbing and takes a mountain pass. The Ambrosetti – Rabinowitz *mountain pass theorem* [8] says that there is such a mountain pass and Φ' vanishes there. More rigorously, we have Theorem 17.4.

Theorem 17.4 (Mountain pass) *Let Φ be continuously differentiable, satisfy the Palais – Smale property*[10] *and satisfy the following three conditions:*

$$\Phi(0) = 0, \tag{17.7a}$$

$$\exists \rho, h > 0\colon \; \|u\| = \rho \;\Rightarrow\; \Phi(u) \geq h, \tag{17.7b}$$

$$\exists v \in V\colon \quad \|v\| > \rho, \qquad \Phi(v) < h. \tag{17.7c}$$

Then Φ has a critical point $u \neq 0$.

A similar assertion relies on a Cartesian structure, leading to a *von Neumann's saddle-point theorem*.

Theorem 17.5 (Saddle point)[11] *Let $V = Y \times Z$ be reflexive, $\Phi(y, \cdot) : Z \to \mathbb{R}$ be concave continuous and $\Phi(\cdot, z) : Y \to \mathbb{R}$ be convex continuous for any $(y, z) \in Y \times Z$, $\Phi(\cdot, \bar{z}) : Y \to \mathbb{R}$ and let $-\Phi(\bar{y}, \cdot) : Z \to \mathbb{R}$ be coercive for some $(\bar{y}, \bar{z}) \in Y \times Z$. Then there is $(y, z) \in Y \times Z$ so that*

$$\forall \tilde{y} \in Y \forall \tilde{z} \in Z : \;\; \Phi(\tilde{y}, z) \geq \Phi(y, z) \geq \Phi(y, \tilde{z})$$

and, if Φ is smooth, then $\Phi'(y, z) = 0$.

10) More specifically, $\{\Phi(u_k)\}_{k\in\mathbb{N}}$ bounded and $\lim_{k\to\infty} ||\Phi'(u_k)||_{V^*} = 0$ imply that $\{u_k\}_{k\in\mathbb{N}}$ has a convergent subsequence.

11) The proof is nonconstructive, based on a fixed-point argument, see, for example, [9, Theorems 9D and 49A with Prop. 9.9]. The original von Neumann's version [10] dealt with the finite-dimensional case only.

17.2.2 Nonsmooth Case

For $\Phi : V \mapsto \mathbb{R} \cup \{+\infty\}$ convex, we define the *subdifferential* of Φ at u as

$$\partial\Phi(u) = \Big\{ f \in V^*;\; \forall v \in V :\; \Phi(v) + \langle f, u - v\rangle \geq \Phi(u) \Big\}. \tag{17.8}$$

If Φ is Gâteaux differentiable, then $\partial\Phi(u) = \{\Phi'(u)\}$, hence this notion is indeed a generalization of the conventional differential. Instead of the abstract Euler – Lagrange equation (17.3), it is natural to consider the abstract inclusion $0 \in \partial\Phi(u)$. More generally, assuming $\Phi = \Phi_0 + \Phi_1$ with Φ_0 smooth and Φ_1 convex, instead of (17.3), we consider the *inclusion*

$$\partial\Phi_1(u) + \Phi_0'(u) \ni 0. \tag{17.9}$$

In view of (17.8), this inclusion can equally be written as a *variational inequality*

$$\forall v \in V : \;\; \Phi_1(v) + \langle \Phi_0'(u), v - u\rangle \geq \Phi_1(u). \tag{17.10}$$

Theorems 17.1 and 17.2 can be reformulated, for example, as follows.

Theorem 17.6 *Let V have a pre-dual and let $\Phi_0 : V \to \mathbb{R}$ be weakly* lower semicontinuous and smooth, $\Phi_1 : V \mapsto \mathbb{R} \cup \{+\infty\}$ convex and lower semicontinuous, and let $\Phi_0 + \Phi_1$ be coercive. Then* (17.9) *has a solution.*[12]

Introducing the *Fréchet subdifferential*

$$\partial_{\mathrm{F}}\Phi(u) = \Big\{ f \in V^*;\; \liminf_{\|v\|\to 0} \frac{\Phi(u+v) - \Phi(u) - \langle f, v\rangle}{\|v\|} \geq 0 \Big\}, \tag{17.11}$$

12) The proof relies on existence of a minimizer of $\Phi_0 + \Phi_1$ as in Theorem 17.1; then one shows that any such a minimizer satisfies (17.9).

the inclusion (17.9) can be written simply as $\partial_{_F}\Phi(u)\ni 0$; in fact, a calculus for Fréchet subdifferentials can be developed for a wider class of (sometimes called *amenable*) functionals than that considered in (17.9), cf. [11, 12].

Example 17.1 Let us consider the *indicator function* δ_K of a set $K \subset V$ defined as

$$\delta_K(u) = \begin{cases} 0 & \text{if } u{\in}K, \\ +\infty & \text{if } u{\notin}K. \end{cases} \qquad (17.12)$$

Clearly, δ_K is convex or lower semicontinuous if (and only if) K is convex or closed, respectively. Assuming K convex closed, it is not difficult to check that $\partial\delta_K(u) = \{f{\in}V^*;\ \forall v{\in}K:\ \langle f, v-u\rangle \le 0\}$ if $u \in K$, otherwise $\partial\delta_K(u) = \emptyset$. The set $\partial\delta_K(u)$ is called the *normal cone* to K at u; denoted also by $N_K(u)$. For the very special case $\Phi_1 = \delta_K$, the variational inequality (17.10) (i.e. here also $\Phi_0'(u){\in}{-N_K(u)}$) represents the problem of finding $u{\in}K$ satisfying

$$\forall v{\in}K:\ \ \big\langle \Phi_0'(u), v{-}u \big\rangle \ge 0. \qquad (17.13)$$

17.2.3 Constrained Problems

In fact, we saw in Example 17.1 a variational problem for Φ_0 with the constraint formed by a convex set K. Sometimes, there still is a need to involve constraints of the type $R(u) = 0$ (or possibly more general $R(u) \le 0$) for a nonlinear mapping $R: V \to \Lambda$ with Λ a Banach space that is possibly ordered; we say that Λ is *ordered* by "$\ge$" if $\{\lambda \ge 0\}$ forms a closed convex *cone*[13] in Λ. Then the constrained minimization problems reads as follows:

$$\text{Minimize } \Phi(u) \text{ subject to } R(u) \le 0,\ u{\in}K. \qquad (17.14)$$

Let us define the *tangent cone* $T_K(u)$ to K at u as the closure of $\cup_{a\ge 0} a(K{-}u)$. For $A: V \to \Lambda$ linear continuous, the *adjoint operator* $A^*: \Lambda^* \to V^*$ is defined by $\langle A^*\lambda^*, u\rangle = \langle \lambda^*, Au\rangle$ for all $\lambda^*{\in}\Lambda^*$ and $u{\in}V$. Assuming R to be smooth, the first-order necessary optimality *Karush—Kuhn–Tucker*[14] *condition* takes the following form:

Theorem 17.7 (First-order condition) *Let $u{\in}V$ solve* (17.14) *and let*[15]

$$\exists \tilde u{\in}T_K(u):\quad [R'(u)](\tilde u) < 0 \qquad (17.15)$$

hold. Then there exists $\lambda^* \overset{*}{\ge} 0$[16] *such that*[17]

$$\big\langle \lambda^*, R(u)\big\rangle = 0 \qquad \textit{and} \qquad (17.16a)$$

$$\Phi'(u) + R'(u)^*\lambda^* + N_K(u) \ni 0. \qquad (17.16b)$$

The condition (17.15) is called the *Mangasarian–Fromovitz constraint qualification*, while (17.16a) is called the *complementarity* (or sometimes *orthogonality* or *transversality*) *condition* and the triple

$$R(u) \le 0, \quad \lambda^* \overset{*}{\ge} 0, \quad \langle \lambda^*, R(u)\rangle = 0 \qquad (17.17)$$

is called a *complementarity problem*. Defining the *Lagrangean* by

$$\mathscr{L}(u, \lambda^*) = \Phi(u) + \lambda^* \circ R(u), \qquad (17.18)$$

13) A cone C is a set such that $a\lambda \in C$ whenever $\lambda \in C$ and $a \ge 0$.

14) Conditions of this kind were first formulated in Karush's thesis [13] and later independently in [14].

15) The inequality "<" in (17.15) means that a neighborhood of $[R'(u)](\tilde u)$ still lies in the cone $\{\lambda \le 0\}$.

16) The so-called dual ordering $\overset{*}{\ge}$ on Λ^* means that $\lambda^* \overset{*}{\ge} 0$ if, and only if, $\langle \lambda^*, v\rangle \ge 0$ for all $v \ge 0$.

17) The linear operator $R'(u)^*: \Lambda^* \to V^*$ is adjoint to $R'(u): V \to \Lambda$ and (17.16b) is meant in V^*.

we can write the inclusion (17.16b) simply as $\mathscr{L}'_u(u,\lambda^*) + N_K(u) \ni 0$. The optimality condition *à la* Example 17.1 for maximization of $\mathscr{L}(u,\cdot) : \Lambda^* \to \mathbb{R}$ over the cone $\{\lambda^* \overset{*}{\geq} 0\}$ is simply $R(u) \leq 0$.

If R is a *convex mapping*[18] and K is a convex set, then (17.15) is equivalent to the simpler *Slater constraint qualification*: $\exists u_0 \in K : R(u_0) < 0$. If Φ is also convex, then (17.16) represents the first-order sufficient optimality condition in the sense that if (17.16) is satisfied, u solves (17.14). Moreover, the couple (u,λ^*) represents a saddle point for $\mathscr{L}$ on the set $K \times \{\lambda^* \overset{*}{\geq} 0\}$, and its existence can be proved by using Theorem 17.5.

Minimization problems without the constraint $R(u) \leq 0$ may be much easier to solve in specific cases. In particular, one can explicitly calculate the value $D(\lambda^*) = \min_{u\in K} \mathscr{L}(u,\lambda^*)$. The functional $D : \Lambda^* \to \mathbb{R} \cup \{-\infty\}$ is concave and

$$\text{maximize } D(\lambda^*) \text{ subject to } \lambda^* \overset{*}{\geq} 0 \quad (17.19)$$

is called the *dual problem*. The supremum of (17.19) is always below the infimum of (17.14). Under additional conditions, they can be equal to each other, and (17.19) has a solution λ^* that can serve as the multiplier for (17.16). For duality theory, see, for example, [12, Chapter 12].

In the general nonconvex case, (17.16) is no longer a sufficient condition and construction of such conditions is more involved. A prototype is a *sufficient second-order condition* that uses the *approximate critical cone* C_ε:

$$C_\varepsilon(u) = \{h \in T_K(u);\ \Phi'(u)h \leq \varepsilon\|h\|,\ \operatorname{dist}\big(R'(u)h, T_{-D}(R(u))\big) \leq \varepsilon\|h\|\}$$

for some $\varepsilon > 0$:

18) In this Banach-valued case, convexity means $R(\lambda u+(1-\lambda)v) \leq \lambda R(u) + (1-\lambda)R(v)$ for any $u, v \in V$ and $0 \leq \lambda \leq 1$ with $\leq$ referring to the ordering in Λ.

Theorem 17.8 (Second-order condition) *Let Φ and R be twice differentiable and let the first-order necessary condition* (17.16) *with a multiplier $\lambda^* \overset{*}{\geq} 0$ hold at some u and let*

$$\exists\, \varepsilon,\delta > 0\ \forall h \in C_\varepsilon(u) : \quad \mathscr{L}''_u(u,\lambda^*)(h,h) \geq \delta\|h\|^2. \quad (17.20)$$

Then u is a local minimizer for (17.14).

A very special case is when $R \equiv 0$ and $K = V$: in this unconstrained case, $N_K = \{0\}$, $C_\varepsilon = V$, and (17.16) and (17.20) become, respectively, the well-known classical condition $\Phi'(u) = 0$ and $\Phi''(u)$ is positive definite.

17.2.4 Evolutionary Problems

Imposing a linear structure allows us not only to define differentials by using (17.1) and (17.2) but also to defining the derivatives $\mathrm{d}u/\mathrm{d}t$ of trajectories $t \mapsto u(t) : \mathbb{R} \to V$.

17.2.4.1 Variational Principles

Minimization of the energy Φ is related to a gradient flow, that is, a process u evolving in time, governed by the gradient Φ' in the sense that the velocity $\mathrm{d}u/\mathrm{d}t$ is always in the direction of steepest descent $-\Phi'$ of Φ. Starting from a given initial condition u_0 and generalizing it for a time-dependent potential $\Phi - f(t)$ with $f(t) \in V^*$, one considers the initial-value problem (a *Cauchy problem*) for the *abstract parabolic equation*:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \Phi'(u) = f(t), \qquad u(0) = u_0. \quad (17.21)$$

It is standard to assume $V \subset H$, with H a Hilbert space, this embedding being *dense*[19] and continuous. Identifying H with its own dual, we obtain a *Gelfand-triple* $V \subset H \subset V^*$. Then, with the coercivity/growth assumption

$$\exists \epsilon > 0 : \quad \epsilon \|u\|_V^p \leq \Phi(t,u) \leq \frac{1+\|u\|_V^p}{\epsilon} \tag{17.22}$$

for some $1 < p < +\infty$ and fixing a time horizon $T > 0$, the solution to (17.21) is sought in the affine manifold

$$\left\{ v \in L^p(I;V);\ v(0) = u_0,\ \frac{\mathrm{d}v}{\mathrm{d}t} \in L^{p'}(I;V^*) \right\} \tag{17.23}$$

with $I = [0,T]$, where $L^p(I;V)$ stands for a Lebesgue space of abstract functions with values in a Banach space (here V), which is called a *Bochner space.*

By continuation, we obtain a solution u to (17.21) on $[0,+\infty)$. If Φ is convex and f is constant in time, there is a relation to the variational principle for $\Phi - f$ in Section 17.2.1: the function $t \mapsto [\Phi - f](u(t))$ is nonincreasing and convex, and $u(t)$ converges weakly as $t \to \infty$ to a minimizer of $\Phi - f$ on V.

The variational principle for (17.21) on the bounded time interval I uses the functional $\mathfrak{F}$ defined by

$$\mathfrak{F}(u) = \int_0^T \Phi(t,u(t)) + \Phi^*\left(t, f(t) - \frac{\mathrm{d}u}{\mathrm{d}t}\right) - \langle f(t), u(t)\rangle \,\mathrm{d}t + \frac{1}{2}\|u(T)\|_H^2, \tag{17.24}$$

where $\Phi^*(t,\cdot) : V^* \to \mathbb{R} \cup \{+\infty\}$ is the *Legendre conjugate* to $\Phi(t,\cdot) : V \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Phi^*(t,f) = \sup_{v \in V} \langle f, v\rangle - \Phi(v); \tag{17.25}$$

the construction $\Phi(t,\cdot) \mapsto \Phi^*(t,\cdot)$ is called the *Legendre transformation*. Omitting t for the moment, Φ^* is convex and

$$\Phi^*(f) + \Phi(v) \geq \langle f, v\rangle, \tag{17.26}$$

which is *Fenchel's inequality*. If Φ, resp. Φ^*, is smooth, the equality in (17.26) holds if, and only if, $f \in \Phi'(v)$, resp. $v \in [\Phi^*]'(f)$. Moreover, if $\Phi(\cdot)$ is lower semicontinuous, it holds $\Phi^{**} = \Phi$.

The infimum of $\mathfrak{F}$ on (17.24) is equal to $\frac{1}{2}\|u_0\|_H^2$. If u from (17.23) minimizes $\mathfrak{F}$ from (17.24), that is, $\mathfrak{F}(u) = \frac{1}{2}\|u_0\|_H^2$, then u solves the Cauchy problem (17.21); this is the *Brezis–Ekeland–Nayroles principle* [15, 16]. It can also be used in the direct method, see [17] or [18, Section 8.10]:

Theorem 17.9 (Direct method for (17.21)**)** *Let $\Phi : [0,T] \times V \to \mathbb{R}$ be a Carathéodory function such that $\Phi(t,\cdot)$ is convex, both $\Phi(t,\cdot)$ and $\Phi^*(t,\cdot)$ are smooth,* (17.22) *holds, $u_0 \in H$, and $f \in L^{p'}(I;V^*)$. Then $\mathfrak{F}$ from* (17.24) *attains a minimum on* (17.23) *and the (unique) minimizer solves the Cauchy problem* (17.21).

One can consider another side-condition instead of the initial condition, for example, the periodic condition $u(0) = u(T)$, having the meaning that we are seeking periodic solutions with an a priori prescribed period T assuming f is periodic with the period T. Instead of (17.21), one thus considers

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \Phi'(u) = f(t), \quad u(0) = u(T). \tag{17.27}$$

Then, instead of (17.23), solutions are sought in the linear (in fact, Banach) space

$$\left\{ v \in L^p(I;V);\ v(0) = v(T),\ \frac{\mathrm{d}v}{\mathrm{d}t} \in L^{p'}(I;V^*) \right\}. \tag{17.28}$$

19) A subset is dense if its closure is the whole space, here H.

The direct method now uses, instead of (17.24), the functional

$$\mathfrak{F}(u) = \int_0^T \Phi(t, u(t)) - \langle f(t), u(t)\rangle + \Phi^*\Big(t, f(t) - \frac{\mathrm{d}u}{\mathrm{d}t}\Big)\,\mathrm{d}t, \quad (17.29)$$

and an analog of Theorem 17.9 but using (17.28) and (17.29); the minimum is 0 and the minimizer need not be unique, in general.

Often, physical and mechanical applications use a convex (in general, nonquadratic) *potential of dissipative forces* $\Psi : H \to \mathbb{R} \cup \{+\infty\}$ leading to a *doubly nonlinear Cauchy problem*:

$$\Psi'\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big) + \Phi'(u) = f(t), \quad u(0) = u_0. \quad (17.30)$$

In fact, the hypothesis that (here abstract) dissipative forces, say $A(\mathrm{d}u/\mathrm{d}t)$, have a potential needs a symmetry of A, cf. (17.5), which has been under certain conditions justified in continuum-mechanical (even anisothermal) linearly responding systems (so that the resulting Ψ is quadratic); this is *Onsager's* (or *reciprocal*) *symmetry* condition [19],[20] cf. [20, Section 12.3]. Sometimes, (17.30) is also equivalently written as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = [\Psi^*]'\big(f(t) - \Phi'(u)\big), \quad u(0) = u_0, \quad (17.31)$$

where Ψ^* again denotes the conjugate functional, that is, here $\Psi^*(v^*) = \sup_{v\in H}\langle v^*, v\rangle - \Psi(v)$. If Ψ is also proper in the sense that $\Psi > -\infty$ and $\Psi \not\equiv +\infty$, then $[\Psi^*]' = [\Psi']^{-1}$, which was used in (17.31). For $\Psi = \frac{1}{2}\|\cdot\|_H^2$, we get $\mathrm{d}u/\mathrm{d}t = f - \Phi'(u)$, cf. (17.21). Thus, for $f = 0$, (17.31) represents a *generalized gradient flow*. For a general f, a *Stefanelli's variational principle* [21] for (17.30) employs the functional

$$\mathfrak{F}(u, w) = \Big(\int_0^T \Psi\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big) - \Big\langle f, \frac{\mathrm{d}u}{\mathrm{d}t}\Big\rangle + \Psi^*(w)\,\mathrm{d}t + \Phi(u(T)) - \Phi(u_0)\Big)^+ + \int_0^T \Phi(u) - \langle f - w, u\rangle + \Phi^*(f - w)\,\mathrm{d}t \quad (17.32)$$

to be minimized on the affine manifold

$$\Big\{(u, w) \in L^\infty(I; V);\;\; u(0) = u_0,\;\; \frac{\mathrm{d}u}{\mathrm{d}t} \in L^q(I; H),\;\; w \in L^{q'}(I; H)\Big\}, \quad (17.33)$$

where $1 < q < +\infty$ refers to a coercivity/growth condition for Ψ. On the set (17.33), $\mathfrak{F}(u, w) \geq 0$ always holds, and $\mathfrak{F}(u, w) = 0$ means that $w = \Psi'(\mathrm{d}u/\mathrm{d}t)$ and $f - w = \Phi'(u)$ a.e. (almost everywhere) on I, that is, u solves (17.30).

Another option is to use the conjugation and Fenchel inequality only for Ψ, which leads to[21]

$$\mathfrak{G}(u) = \int_0^T \Psi\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big) + \Psi^*\big(f - \Phi'(u)\big) + \Big\langle \frac{\mathrm{d}f}{\mathrm{d}t}, u\Big\rangle\,\mathrm{d}t + \Phi(u(T)) \quad (17.34)$$

to be minimized on a submanifold $\{u = w\}$ of (17.33). The infimum is $\Phi(u_0) - f(0) + f(T)$ and any minimizer u is a solution to (17.30). Sometimes, this is known under the name *principle of least dissipation*, cf. [22] for Ψ quadratic. The relation

$$\mathfrak{G}(u) = \Phi(u_0) - f(0) + f(T) \quad (17.35)$$

20) A Nobel prize was awarded to Lars Onsager in 1968 "for the discovery of the reciprocal relations bearing his name, which are fundamental for the thermodynamics of irreversible processes."

21) Here (17.26) reads as $\Psi(\mathrm{d}u/\mathrm{d}t) + \Psi^*(f - \Phi'(u)) \geq \langle f - \Phi'(u), \mathrm{d}u/\mathrm{d}t\rangle = \langle f, \mathrm{d}u/\mathrm{d}t\rangle - [\mathrm{d}\Phi/\mathrm{d}t](u)$, from which (17.34) results by integration over $[0, T]$.

is sometimes called *De Giorgi's formulation* of (17.30); rather than for existence proofs by the direct method, this formulation is used for various passages to a limit. Note that for f constant, the only time derivative involved in (17.34) is $\Psi\big(\mathrm{d}u/\mathrm{d}t\big)$, which allows for an interpretation even if V is only a metric space and thus $\mathrm{d}u/\mathrm{d}t$ itself is not defined, which leads to a theory of *gradient flows in metric spaces*, cf. [23, Theorem 2.3.3].

A combination of (17.27) and (17.30) leads to

$$\Psi'\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \Phi'(u) = f(t), \qquad u(T) = u(0),$$

and the related variational principle uses $\mathfrak{F}$ from (17.32) but with $\Phi(u(T)) - \Phi(u_0)$ omitted, to be minimized on the linear manifold (17.33) with u_0 replaced by $u(T)$.

Many physical systems exhibit oscillatory behavior combined possibly with attenuation by nonconservative forces having a (pseudo)potential Ψ, which can be covered by the (Cauchy problem for the) abstract second-order evolution equation

$$\mathscr{T}'\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \Psi'\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \Phi'(u) = f(t),$$
$$u(0) = u_0, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = v_0, \quad (17.36)$$

where $\mathscr{T} : H \to \mathbb{R}$ is the positive (semi)definite quadratic form representing the kinetic energy. The *Hamilton variational principle* extended to dissipative systems says that the solution u to (17.36) is a critical point of the integral functional

$$\int_0^T \mathscr{T}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) - \Phi(u) + \big\langle f - \mathfrak{f}, u\big\rangle \,\mathrm{d}t \quad (17.37)$$

with a nonconservative force $\mathfrak{f} = \Psi'(\mathrm{d}u/\mathrm{d}t)$ considered fixed on the affine manifold $\{u \in L^\infty(I; V);\ \mathrm{d}u/\mathrm{d}t \in L^\infty(I; H),\ \mathrm{d}^2u/\mathrm{d}t^2 \in L^2(I; V^*),\ u(0) = u_0,\ \mathrm{d}u/\mathrm{d}t = v_0\}$, cf. [24].

17.2.4.2 Evolution Variational Inequalities

For nonsmooth potentials, the above evolution equations turn into inclusions. Instead of the Legendre transformation, we speak about the *Legendre–Fenchel transformation*. For example, instead of $[\Psi^*]' = [\Psi']^{-1}$, we have $\partial\Psi^* = [\partial\Psi]^{-1}$. Note that variational principles based on $\mathfrak{F}$ from (17.24), (17.29), or (17.32) do not involve any derivatives of Φ and Ψ and are especially designed for nonsmooth problems, and also $\mathfrak{G}$ from (17.34) allows for Ψ to be nonsmooth. For example, in the case of (17.30), with convex but nonsmooth Ψ and Φ, we have the *doubly nonlinear inclusion*

$$\partial\Psi\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \partial\Phi(u) \ni f(t), \ u(0) = u_0, \quad (17.38)$$

and $\mathfrak{F}(u, w) = 0$ in (17.32) and (17.33) means exactly that $w \in \partial\Psi(\mathrm{d}u/\mathrm{d}t)$ and $f - w \in \partial\Phi(u)$ hold a.e. on I,[22] which (in the spirit of Section 17.2.2) can be written as a system of two variational inequalities for u and w:

$$\forall v: \quad \Psi(v) + \left\langle w, v - \frac{\mathrm{d}u}{\mathrm{d}t}\right\rangle \geq \Psi\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right), \quad (17.39a)$$

$$\forall v: \quad \Phi(v) + \langle f - w, v - u\rangle \geq \Phi(u). \quad (17.39b)$$

For a systematic treatment of such multiply nonlinear inequalities, see [25].

In applications, the nonsmoothness of Ψ occurs typically at 0 describing activation phenomena: the abstract driving force $f - \partial\Phi(u)$ must pass a threshold, that is, the boundary of the convex set $\partial\Psi(0)$, in order to trigger the evolution of u. Often, any

22) The idea behind the principle in (17.32) and (17.33) is to apply two Fenchel inequalities to (17.38) written as $w \in \partial\Psi(\mathrm{d}u/\mathrm{d}t)$ and $f - w \in \partial\Phi(u)$.

rate dependence is neglected, and then Ψ is degree-1 positively homogeneous.[23] In this kind of *rate-independent* case, $\Psi^* = \delta_{\partial\Psi(0)}$, while $\Psi = \delta^*_{\partial\Psi(0)}$, and the De Giorgi formulation (17.35) leads to the *energy equality*

$$E(T, u(T)) + \int_0^T \Psi\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) \mathrm{d}t = E(0, u_0) - \int_0^T \left\langle \frac{\mathrm{d}f}{\mathrm{d}t}, u \right\rangle \mathrm{d}t \quad \text{for } E(t, u) = \Phi(u) - \langle f(t), u\rangle \tag{17.40a}$$

together with $f(t)-\Phi'(u(t))\in\partial\Psi(0)$ for a.a. (almost all) $t\in[0, T]$; here, in accordance with (17.35), we assume Φ to be smooth for the moment. This inclusion means $\Psi(v) - \langle f - \Phi'(u), v\rangle \geq \Psi(0) = 0$ and, as Φ is convex, we obtain the *stability condition*,[24]

$$\forall t\in[0, T]\ \forall v\in V: \quad E(t, u(t)) \leq E(t, v) + \Psi(v-u(t)). \tag{17.40b}$$

Moreover, in this rate-independent case, $\partial\Psi^* = N_{\partial\Psi(0)}$ and (17.31) reads $\mathrm{d}u/\mathrm{d}t \in N_{\partial\Psi(0)}(f - \Phi'(u))$. By (17.13), it means that $\langle \mathrm{d}u/\mathrm{d}t, v - f + \Phi'(u)\rangle \leq 0$ for any $v\in \partial\Psi(0)$, that is,

$$\max_{v\in\partial\Psi(0)} \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, v \right\rangle = \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, f - \Phi'(u) \right\rangle, \tag{17.41}$$

which says that the dissipation due to the driving force $f - \Phi'(u)$ is maximal compared to all admissible driving forces provided the rate $\mathrm{d}u/\mathrm{d}t$ is kept fixed; this is the *maximum dissipation principle*.

In fact, (17.40) does not contain Φ' and thus works for Φ convex nonsmooth, too. Actually, (17.40) was invented in [26], where it is called the *energetic formulation* of (17.38), cf. also [27].

23) This means $\Psi(\lambda v) = \lambda\Psi(v)$ for any $\lambda \geq 0$.

24) By convexity of Φ, we have $\Phi(v) \geq \Phi(u) + \langle \Phi'(u), v-u\rangle$, and adding it with $\Psi(v-u) - \langle f-\Phi'(u), v-u\rangle \geq 0$, we get (17.40b).

17.2.4.3 Recursive Variational Problems Arising by Discretization in Time

The variational structure related to the potentials of Section 17.2.4.1 can be exploited not only for formulation of "global" in time-variational principles, but, perhaps even more efficiently, to obtain recursive (incremental) variational problems when discretizing the abstract evolution problems in time by using some (semi) implicit formulae. This can serve as an efficient theoretical method for analyzing evolution problems (the *Rothe method*, [28]) and for designing efficient conceptual algorithms for numerical solution of such problems.

Considering a uniform partition of the time interval with the time step $\tau > 0$ with T/τ integer, we discretize (17.21) as

$$\frac{u_\tau^k - u_\tau^{k-1}}{\tau} + \Phi'(u_\tau^k) = f(k\tau), \quad k = 1, \dots, \frac{T}{\tau}, \quad u_\tau^0 = u_0. \tag{17.42}$$

This is also known as the *implicit Euler formula* and u_τ^k for $k = 1, \dots, T/\tau$ approximate respectively the values $u(k\tau)$. One can apply the direct method by employing the recursive variational problem for the functional

$$\Phi(u) + \frac{1}{2\tau}\left\|u-u_\tau^{k-1}\right\|_H^2 - \big\langle f(k\tau), u\big\rangle \tag{17.43}$$

to be minimized for u. Obviously, any critical point u (and, in particular, a minimizer) of this functional solves (17.42) and we put $u = u_\tau^k$. Typically, after ensuring existence of the approximate solutions $\{u_\tau^k\}_{k=1}^{T/\tau}$, a priori estimates have to be derived[25] and then convergence as $\tau \to 0$ is to be proved by

25) For this, typically, testing (17.42) (or its difference from $k-1$ level) by u_τ^k or by $u_\tau^k - u_\tau^{k-1}$ (or $u_\tau^k - 2u_\tau^{k-1} + u_\tau^{k-2}$) is used with Young's and (discrete) Gronwall's inequalities, and so on.

various methods.[26] Actually, Φ does not need to be smooth and, referring to (17.11), we can investigate the set-valued variational inclusion $\mathrm{d}u/\mathrm{d}t+\partial_{\mathrm{F}}\Phi(u) \ni f$.

In specific situations, the fully implicit scheme (17.42) can be advantageously modified in various ways. For example, in case $\Phi = \Phi_1 + \Phi_2$ and $f = f_1 + f_2$, one can apply the *fractional-step method*, alternatively to be understood as a Lie–Trotter (or *sequential*) *splitting* combined *with the implicit Euler formula*:

$$\frac{u_\tau^{k-1/2} - u_\tau^{k-1}}{\tau} + \Phi_1'(u_\tau^{k-1/2}) = f_1(k\tau), \quad (17.44a)$$

$$\frac{u_\tau^{k} - u_\tau^{k-1/2}}{\tau} + \Phi_2'(u_\tau^{k}) \ni f_2(k\tau), \quad (17.44b)$$

with $k = 1, \dots, T/\tau$. Clearly, (17.44) leads to two variational problems that are to be solved in alternation.

Actually, we have needed rather the splitting of the underlying operator $A = \Phi_1' + \Phi_2' : V \to V^*$ and not of its potential $\Phi = \Phi_1 + \Phi_2 : V \to \mathbb{R}$. In case $\Phi : V = Y \times Z \to \mathbb{R}$, $u = (y, z)$ and $f = (g, h)$ where (17.21) represents a system of two equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \Phi_y'(y, z) = g, \quad y(0) = y_0, \quad (17.45a)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \Phi_z'(y, z) = h, \quad z(0) = z_0, \quad (17.45b)$$

with Φ_y' and Φ_z' denoting partial differentials, one can thus think also about the splitting $\Phi' - f = (\Phi_y' - g, \Phi_z' - h) = (\Phi_y' - g, 0) + (0, \Phi_z' - h)$. Then the fractional method such as (17.44) yields a *semi-implicit scheme*[27]

$$\frac{y_\tau^k - y_\tau^{k-1}}{\tau} + \Phi_y'(y_\tau^k, z_\tau^{k-1}) = g(k\tau), \quad (17.46a)$$

$$\frac{z_\tau^k - z_\tau^{k-1}}{\tau} + \Phi_z'(y_\tau^k, z_\tau^k) = h(k\tau), \quad (17.46b)$$

again for $k = 1, \dots, T/\tau$. Note that the use of z_τ^{k-1} in (17.46a) decouples the system (17.46), in contrast to the fully implicit formula which would use z_τ^k in (17.46a) and would not decouple the original system (17.45). The underlying variational problems for the functionals $y \mapsto \Phi(y, z_\tau^{k-1}) + \frac{1}{2\tau}\|y - y_\tau^{k-1}\|^2 - \langle g(k\tau), y\rangle$ and $z \mapsto \Phi(y_\tau^k, z) + \frac{1}{2\tau}\|z - z_\tau^{k-1}\|^2 - \langle h(k\tau), z\rangle$ represent recursive alternating variational problems; these particular problems can be convex even if Φ itself is not; only separate convexity[28] of Φ suffices. Besides, under certain relatively weak conditions, this semi-implicit discretization is "numerically" stable; cf. [18, Remark 8.25]. For a convex/concave situation as in Theorem 17.5, (17.46) can be understood as an iterative *algorithm of Uzawa*'s type (with a damping by the implicit formula) for finding a saddle point.[29]

Of course, this decoupling method can be advantageously applied to nonsmooth situations and for u with more than two components, that is, for systems of more than two equations or inclusions. Even more, the splitting as in (17.45) may yield a variational structure of the decoupled incremental problems even if the original problem of the type $\mathrm{d}u/\mathrm{d}t + A(u) \ni 0$ itself does not have it. An obvious example for this is $A(y, z) = (\Phi_1'(\cdot, z)](y), \Phi_2'(y, \cdot)](z))$, which does not need to satisfy the symmetry (17.5) if $\Phi_1 \neq \Phi_2$ although the corresponding semi-implicit scheme

26) Typically, a combination of the arguments based on weak lower semicontinuity or compactness is used.

27) Indeed, in (17.44), one has $u_\tau^{k-1} = (y_\tau^{k-1}, z_\tau^{k-1})$, $u_\tau^{k-1/2} = (y_\tau^k, z_\tau^{k-1})$, and eventually $u_\tau^k = (y_\tau^k, z_\tau^k)$.

28) This means that only $\Phi(y, \cdot)$ and $\Phi(\cdot, z)$ are convex but not necessarily $\Phi(\cdot, \cdot)$.

29) This saddle point is then a steady state of the underlying evolution system (17.45).

(17.46) still possesses a "bi-variational" structure.

Similarly to (17.42), the doubly nonlinear problem (17.38) uses the formula

$$\partial\Psi\Big(\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\Big) + \partial\Phi(u_\tau^k) \ni f(k\tau) \quad (17.47)$$

and, instead of (17.43), the functional

$$\Phi(u) + \tau\Psi\Big(\frac{u - u_\tau^{k-1}}{\tau}\Big) - \big\langle f(k\tau), u\big\rangle. \quad (17.48)$$

Analogously, for the second-order doubly nonlinear problem (17.36) in the nonsmooth case, that is, $\mathscr{T}'\mathrm{d}^2u/\mathrm{d}t^2 + \partial\Psi\big(\mathrm{d}u/\mathrm{d}t\big) + \partial\Phi(u) \ni f(t)$, we would use

$$\mathscr{T}'\frac{u_\tau^k - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2} + \partial\Psi\Big(\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\Big) + \partial\Phi(u_\tau^k) \ni f(k\tau) \quad (17.49)$$

and the recursive variational problem for the functional

$$\Phi(u) + \tau\Psi\Big(\frac{u - u_\tau^{k-1}}{\tau}\Big) - \big\langle f(k\tau), u\big\rangle + \tau^2\mathscr{T}\Big(\frac{u - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2}\Big). \quad (17.50)$$

The fractional-step method and, in particular, various semi-implicit variants of (17.47) and (17.49) are widely applicable, too.

17.3 Variational Problems on Specific Function Spaces

We now use the abstract framework from Section 17.2 for concrete variational problems formulated on specific function spaces.

17.3.1 Sobolev Spaces

For this, we consider a bounded domain $\Omega \subset \mathbb{R}^d$ equipped with the Lebesgue measure, having a smooth boundary $\Gamma := \partial\Omega$. For $1 \le p < \infty$, we will use the standard notation

$$L^p(\Omega;\mathbb{R}^n) = \Big\{u : \Omega \to \mathbb{R}^n \text{ measurable}; \int_\Omega |u(x)|^p\,\mathrm{d}x < \infty\Big\}$$

for the *Lebesgue space*; the addition and the multiplication understood pointwise makes it a linear space, and introducing the norm

$$\|u\|_p = \Big(\int_\Omega |u(x)|^p\,\mathrm{d}x\Big)^{1/p}$$

makes it a Banach space. For $p = \infty$, we define $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\Omega}|u(x)| = \inf_{N\subset\Omega,\,\mathrm{meas}_d(N)=0} \sup_{x\in\Omega\setminus N} |u(x)|$. For $1 < p < \infty$, $L^p(\Omega;\mathbb{R}^n)$ is reflexive. For $1 \le p < \infty$, $L^p(\Omega;\mathbb{R}^n)^* = L^{p'}(\Omega;\mathbb{R}^n)$ with $p' = p/(p-1)$ if the duality is defined naturally as $\langle f, u\rangle = \int_\Omega f(x)\cdot u(x)\,\mathrm{d}x$. For $p = 2$, $L^p(\Omega;\mathbb{R}^n)$ becomes a Hilbert space. For $n = 1$, we write for short $L^p(\Omega)$ instead of $L^p(\Omega;\mathbb{R})$.

Denoting the kth order gradient of u by $\nabla^k u = (\partial^k/\partial x_{i_1}\cdots\partial x_{i_k}u)_{1\le i_1,\ldots,i_k\le d}$, we define the *Sobolev space* by

$$W^{k,p}(\Omega;\mathbb{R}^n) = \Big\{u \in L^p(\Omega;\mathbb{R}^n);\ \nabla^k u \in L^p(\Omega;\mathbb{R}^{n\times d^k})\Big\},$$

$$\text{normed by } \|u\|_{k,p} = \sqrt[p]{\|u\|_p^p + \|\nabla^k u\|_p^p}.$$

If $n = 1$, we will again use the shorthand notation $W^{k,p}(\Omega)$. If $p = 2$, $W^{k,p}(\Omega;\mathbb{R}^n)$ is a Hilbert space and we will write $H^k(\Omega;\mathbb{R}^n) = W^{k,2}(\Omega;\mathbb{R}^n)$. Moreover, we occasionally use a subspace of $W^{k,p}(\Omega;\mathbb{R}^n)$

with vanishing traces on the boundary Γ, denoted by

$$W_0^{k,p}(\Omega;\mathbb{R}^n) = \big\{u \in W^{k,p}(\Omega;\mathbb{R}^n);\ \nabla^l u = 0 \text{ on } \Gamma,\ l = 0,\dots,k-1\big\}. \tag{17.51}$$

To give a meaningful interpretation to traces $\nabla^l u$ on Γ, this boundary has to be sufficiently regular; roughly speaking, piecewise C^{l+1} is enough.

We denote by $C^k(\bar{\Omega})$ the space of *smooth functions* whose gradients up to the order k are continuous on the closure $\bar{\Omega}$ of Ω. For example, we have obviously embeddings $C^k(\bar{\Omega}) \subset W^{k,p}(\Omega) \subset L^p(\Omega)$; in fact, these embeddings are dense.

An important phenomenon is the compactifying effect of derivatives. A prototype for it is the Rellich–Kondrachov theorem, saying that $H^1(\Omega)$ is *compactly*[30] embedded into $L^2(\Omega)$. More generally, we have

Theorem 17.10 (Compact embedding)
For the Sobolev critical exponent

$$p^* \begin{cases} = dp/(d-p) & \textit{for } p < d, \\ \in [1,+\infty) \textit{ arbitrary} & \textit{for } p = d, \\ = +\infty & \textit{for } p > d, \end{cases}$$

the embedding $W^{1,p}(\Omega) \subset L^{p^*-\epsilon}(\Omega)$ *is compact for any* $0 < \epsilon \le p^*-1$.

Iterating this theorem, we can see, for example, that, for $p < d/2$, the embedding $W^{2,p}(\Omega) \subset L^{[p^*]^*-\epsilon}(\Omega)$ is compact; note that $[p^*]^* = dp/(d-2p)$.

Another important fact is the compactness of the trace operator $u \mapsto u|_\Gamma$:

Theorem 17.11 (Compact trace operator)
For the boundary critical exponent

$$p^\sharp \begin{cases} = (dp-p)/(d-p) & \textit{for } p < d, \\ \in [1,+\infty) \textit{ arbitrary} & \textit{for } p = d, \\ = +\infty & \textit{for } p > d, \end{cases}$$

the trace operator $u \mapsto u|_\Gamma : W^{1,p}(\Omega) \subset L^{p^\sharp-\epsilon}(\Gamma)$ *is compact for any* $0 < \epsilon \le p^\sharp-1$.

For example, the trace operator from $W^{2,p}(\Omega)$ is compact into $L^{[p^*]^\sharp-\epsilon}(\Gamma)$.[31]

17.3.2
Steady-State Problems

The above abstract functional-analysis scenario gives a lucid insight into concrete variational problems leading to boundary-value problems for quasilinear equations in divergence form which is what we will now focus on. We consider a bounded domain $\Omega \subset \mathbb{R}^d$ with a sufficiently regular boundary Γ divided into two disjoint relatively open parts Γ_D and Γ_N whose union is dense in Γ. An important tool is a generalization of the superposition operator, the *Nemytskiĭ mapping* $\mathscr{N}_a$, induced by a *Carathéodory*[32] *mapping* $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ by prescribing $[\mathscr{N}_a(u)](x) = a(x, u(x))$.

Theorem 17.12 (Nemytskiĭ mapping)
Let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ *be a Carathéodory mapping and* $p, q \in [1,\infty)$. *Then* $\mathscr{N}_a$ *maps* $L^p(\Omega;\mathbb{R}^n)$ *into* $L^q(\Omega;\mathbb{R}^m)$ *and is continuous if, and only if, for some* $\gamma \in L^q(\Omega)$ *and* $C < \infty$, *we have that*

$$\big|a(x,u)\big| \le \gamma(x) + C\big|u\big|^{p/q}.$$

30) This means that the embedding is a compact mapping in the sense that weakly converging sequences in $H^1(\Omega)$ converge strongly in $L^2(\Omega)$.

31) To see this, we use Theorem 17.10 to obtain $W^{2,p}(\Omega) \subset W^{1,p^*-\epsilon}(\Omega)$, and then Theorem 17.11 with $p^* - \epsilon$ in place of p.

32) The Carathéodory property means measurability in the x-variable and continuity in all other variables.

17.3.2.1 Second Order Systems of Equations

First, we consider the integral functional

$$\Phi(u) = \int_\Omega \varphi(x, u, \nabla u)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \phi(x, u)\,\mathrm{d}S \tag{17.52a}$$

involving Carathéodory integrands $\varphi : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}$ and $\phi : \Gamma_\mathrm{N}\times\mathbb{R}^n \to \mathbb{R}$. The functional Φ is considered on an affine closed manifold

$$\left\{u\in W^{1,p}(\Omega;\mathbb{R}^n);\ \ u|_{\Gamma_\mathrm{D}} = u_\mathrm{D}\right\} \tag{17.52b}$$

for a suitable given u_D; in fact, existence of $\overline{u}_\mathrm{D} \in W^{1,p}(\Omega;\mathbb{R}^n)$ such $u_\mathrm{D} = \overline{u}_\mathrm{D}|_{\Gamma_\mathrm{D}}$ is to be required. Equipped with the theory of $W^{1,p}$-Sobolev spaces,[33] one considers a p-polynomial-type coercivity of the highest-order term and the corresponding growth restrictions on the partial derivatives φ'_F, φ'_u, and ϕ'_u with some $1 < p < \infty$, that is,

$$\varphi(x, u, F){:}F \geq \epsilon|F|^p + |u|^\epsilon - \frac{1}{\epsilon}, \tag{17.53a}$$

$$\exists\gamma\in L^{p'}(\Omega):\ \ |\varphi'_F(x, u, F)| \leq \gamma(x) + C|u|^{(p^*-\epsilon)/p'} + C|F|^{p-1}, \tag{17.53b}$$

$$\exists\gamma\in L^{p^{*\prime}}(\Omega):\ \ |\varphi'_u(x, u, F)| \leq \gamma(x) + C|u|^{p^*-1-\epsilon} + C|F|^{p/p^{*\prime}}, \tag{17.53c}$$

$$\exists\gamma\in L^{p^{\sharp\prime}}(\Gamma):\ \ |\phi'_u(x, u)| \leq \gamma(x) + C|u|^{p^\sharp-1-\epsilon} \tag{17.53d}$$

for some $\epsilon > 0$ and $C < \infty$; we used F as a placeholder for ∇u. A generalization of Theorem 17.12 for Nemytskiǐ mappings of several arguments says that (17.53b) ensures just continuity of $\mathscr{N}_{\varphi'_F} : L^{p^*-\epsilon}(\Omega;\mathbb{R}^n)\times L^p(\Omega;\mathbb{R}^{n\times d}) \to L^{p'}(\Omega;\mathbb{R}^{n\times d})$, and analogously also (17.53c) works for $\mathscr{N}_{\varphi'_u}$, while (17.53d) gives continuity of $\mathscr{N}_{\phi'_u} : L^{p^\sharp-\epsilon}(\Gamma;\mathbb{R}^n) \to L^{p^{\sharp\prime}}(\Gamma;\mathbb{R}^n)$. This, together with Theorems 17.10 and 17.11, reveals the motivation for the growth conditions (17.53b–d).

For $\epsilon \geq 0$, (17.53b–d) ensures that the functional Φ from (17.52a) is *Gâteaux differentiable* on $W^{1,p}(\Omega;\mathbb{R}^n)$. The abstract *Euler–Lagrange equation* (17.3) then leads to the integral identity

$$\int_\Omega \varphi'_{\nabla u}(x, u, \nabla u){:}\nabla v + \varphi'_u(x, u, \nabla u){\cdot}v\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \phi'_u(x, u){\cdot}v\,\mathrm{d}S = 0 \tag{17.54}$$

for any $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $v|_{\Gamma_\mathrm{D}} = 0$; the notation " : " or " · " means summation over two indices or one index, respectively. Completed by the Dirichlet condition on Γ_D, this represents a *weak formulation* of the *boundary-value problem* for a system of second-order elliptic quasilinear equations:[34]

$$\operatorname{div}\varphi'_{\nabla u}(u, \nabla u) = \varphi'_u(u, \nabla u)\ \text{ in } \Omega, \tag{17.55a}$$

$$\varphi'_{\nabla u}(u, \nabla u){\cdot}\vec{n} + \phi'_u(u) = 0\ \text{ on } \Gamma_\mathrm{N}, \tag{17.55b}$$

$$u|_\Gamma = u_\mathrm{D}\ \text{ on } \Gamma_\mathrm{D}, \tag{17.55c}$$

where x-dependence has been omitted for notational simplicity. The conditions (17.55b) and (17.55c) are called the *Robin* and the *Dirichlet boundary conditions*, respectively, and (17.55) is called the *classical formulation* of this boundary-value problem. Any $u \in C^2(\overline{\Omega};\mathbb{R}^n)$ satisfying (17.55) is called a *classical solution*, while $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ satisfying (17.54) for any $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $v|_{\Gamma_\mathrm{D}} = 0$ is

33) More general nonpolynomial growth and coercivity conditions would require the theory of Orlicz spaces instead of the Lebesgue ones, cf. [9, Chapter 53].

34) Assuming sufficiently smooth data as well as u, this can be seen by multiplying (17.55a) by v, using the Green formula $\int_\Omega(\operatorname{div} a)v + a{\cdot}v\,\mathrm{d}x = \int_\Gamma(a\cdot\vec{n})v\,\mathrm{d}S$, and using $v = 0$ on Γ_D and the boundary conditions (17.55b) on Γ_N.

called a *weak solution*; note that much less smoothness is required for weak solutions.

Conversely, taking general Carathéodory integrands $a : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}^{n\times d}$, $b : \Gamma_N\times\mathbb{R}^n \to \mathbb{R}^n$, and $c : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}^n$, one can consider a boundary-value problem for a system of second-order elliptic quasilinear equations

$$\operatorname{div} a(u, \nabla u) = c(u, \nabla u) \quad \text{in } \Omega, \tag{17.56a}$$

$$a(u, \nabla u)\cdot\vec{n} + b(u) = 0 \quad \text{on } \Gamma_N, \tag{17.56b}$$

$$u|_{\Gamma_D} = u_D \quad \text{on } \Gamma_D. \tag{17.56c}$$

Such a problem does not need to be induced by any potential Φ; nevertheless, it possesses a weak formulation as in (17.54), namely, $\int_\Omega a(u, \nabla u) : \nabla v + c(u, \nabla u) \cdot v \,\mathrm{d}x + \int_{\Gamma_N} b(u)\cdot v \,\mathrm{d}S = 0$ for any "variation" v as in (17.54), and related methods are sometimes called variational in spite of absence of a potential Φ. The existence of such a potential requires a certain *symmetry* corresponding to that in (17.5) for the underlying nonlinear operator $A : W^{1,p}(\Omega; \mathbb{R}^n) \to W^{1,p}(\Omega; \mathbb{R}^n)^*$ given by $\langle A(u), v\rangle = \int_\Omega a(u, \nabla u) : \nabla v + c(u, \nabla u)\cdot v \,\mathrm{d}x + \int_{\Gamma_N} b(u)\cdot v \,\mathrm{d}S$, namely,

$$\frac{\partial a_{il}(x, u, F)}{\partial F_{jk}} = \frac{\partial a_{kj}(x, u, F)}{\partial F_{li}}, \tag{17.57a}$$

$$\frac{\partial a_{il}(x, u, F)}{\partial u_j} = \frac{\partial c_j(x, u, F)}{\partial F_{li}}, \tag{17.57b}$$

$$\frac{\partial c_j(x, u, F)}{\partial u_l} = \frac{\partial c_l(x, u, F)}{\partial u_j} \tag{17.57c}$$

for all $i, k = 1, \ldots, d$ and $j, l = 1, \ldots, n$ and for a.a. $(x, u, F)\in\Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d}$, and also

$$\frac{\partial b_j(x, u)}{\partial u_l} = \frac{\partial b_l(x, u)}{\partial u_j}. \tag{17.57d}$$

for all $j, l = 1, \ldots, n$ and for a.a. $(x, u) \in \Gamma \times \mathbb{R}^n$. Note that (17.57a–c) just means a symmetry for the Jacobian of the mapping $(F, u) \mapsto (a(x, u, F), c(x, u, F)) : \mathbb{R}^{n\times d} \times \mathbb{R}^d \to \mathbb{R}^{n\times d} \times \mathbb{R}^d$ while (17.57d) is the symmetry for the Jacobian of $b(x, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$.

Then (17.6) leads to the formula (17.52a) with

$$\varphi(x, u, F) = \int_0^1 a(x, \lambda u, \lambda F) : F + c(x, \lambda u, \lambda F)\cdot u \,\mathrm{d}\lambda, \tag{17.58a}$$

$$\phi(x, u) = \int_0^1 b(x, \lambda u)\cdot u \,\mathrm{d}\lambda. \tag{17.58b}$$

Relying on the minimization-of-energy principle described above, which is often a governing principle in steady-state mechanical and physical problems, and on Theorem 17.1 or 17.2, one can prove existence of weak solutions to the boundary-value problem by the *direct method*; cf. e.g. [29–32]. Theorem 17.2 imposes a strong (although often applicable) structural restriction that $\varphi(x, \cdot, \cdot) : \mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}$ and $\phi(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$ are convex for a.a. x.

Yet, in general, Theorem 17.1 places fewer restrictions on φ and ϕ by requiring only weak lower semicontinuity of Φ. The precise condition (i.e., sufficient and necessary) that guarantees such semicontinuity of $u \mapsto \int_\Omega \varphi(x, u, \nabla u) \,\mathrm{d}x$ on $W^{1,p}(\Omega; \mathbb{R}^n)$ is called $W^{1,p}$*-quasiconvexity*, defined in a rather nonexplicit way by requiring

$$\forall x\in\Omega\ \forall u\in\mathbb{R}^n\ \forall F\in\mathbb{R}^{n\times d} : \quad \varphi(x, u, F) = \\ = \inf_{v\in W_0^{1,p}(O;\mathbb{R}^d)} \int_O \frac{\varphi(x, u, F+\nabla v(\xi))}{\mathrm{meas}_d(O)} \,\mathrm{d}\xi,$$

where $O \subset \mathbb{R}^d$ is an arbitrary smooth domain. This condition cannot be verified efficiently except for very special cases, unlike, for example, polyconvexity which is a (strictly) stronger condition. Subsequently, another type of convexity, called *rank-one convexity*, was introduced by Morrey [33] by requiring

$\lambda \mapsto \varphi(x, u, F+\lambda a \otimes b) : \mathbb{R} \to \mathbb{R}$ to be convex for any $a \in \mathbb{R}^d$, $b \in \mathbb{R}^n$, $[a \otimes b]_{ij} = a_i b_j$. For smooth $\varphi(x, u, \cdot)$, rank-one convexity is equivalent to the *Legendre–Hadamard condition* $\varphi''_{FF}(x, u, F)(a \otimes b, a \otimes b) \geq 0$ for all $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^d$. Since Morrey's [33] introduction of quasiconvexity, the question of its coincidence with rank-one convexity had been open for many decades and eventually answered negatively by Šverák [34] at least if $n \geq 3$ and $d \geq 2$. Weak lower semicontinuity of the boundary integral $u \mapsto \int_\Omega \phi(x, u)\,\mathrm{d}S$ in (17.52a) does not entail any special structural condition because one can use compactness of the trace operator, cf. Theorem 17.11. Here, Theorem 17.1 leads to the following theorem:

Theorem 17.13 (Direct method) *Let* (17.53) *hold with $\epsilon > 0$, let $\varphi(x, u, \cdot)$ be quasiconvex, and let $u_{\mathrm{D}} \in W^{1-1/p,p}(\Gamma_{\mathrm{D}}; \mathbb{R}^n)$.*[35] *Then* (17.54) *has a solution, that is, the boundary-value problem* (17.55) *has a weak solution.*

For $n = d$, an example for a quasiconvex function is $\varphi(x, u, F) = \mathfrak{f}(x, u, F, \det F)$ with a convex function $\mathfrak{f}(x, u, \cdot, \cdot) : \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}$. The weak lower semicontinuity of Φ from (17.52a) is then based on the weak continuity of the nonlinear mapping induced by $\det : \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}$ if restricted to gradients, that is,

$$u_k \to u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^d) \quad \Rightarrow \quad \det\nabla u_k \to \det\nabla u \text{ weakly in } L^{p/d}(\Omega), \tag{17.59}$$

which holds for $p > d$; note that nonaffine mappings on Lebesgue spaces such as $G \mapsto \det G$ with $G \in L^p(\Omega; \mathbb{R}^{d\times d}) \to L^{p/d}(\Omega)$ can be continuous[36] but not weakly continuous, so (17.59) is not entirely trivial. Even less trivial, it holds for $p = d$ locally (i.e., in $L^1(K)$ for any compact $K \subset \Omega$) if $\det\nabla u_k \geq 0$.[37] Invented by Ball [36], such functions $\varphi(x, u, \cdot)$ are called *polyconvex*, and in general this property requires

$$\varphi(x, u, F) = \mathfrak{f}\Big(x, u, (\mathrm{adj}_i F)_{i=1}^{\min(n,d)}\Big) \tag{17.60}$$

for some $\mathfrak{f} : \Omega \times \mathbb{R}^n \times \prod_{i=1}^{\min(n,d)} \mathbb{R}^{\kappa(i,n,d)} \to \mathbb{R} \cup \{\infty\}$ such that $\mathfrak{f}(x, u, \cdot)$ is convex, where $\kappa(i, n, d)$ is the number of all minors of the ith order and where $\mathrm{adj}_i F$ denotes the determinants of all $(i\times i)$-submatrices. Similarly, as in (17.59), we have that $\mathrm{adj}_i \nabla u_k \to \mathrm{adj}_i \nabla u$ weakly in $L^{p/i}(\Omega; \mathbb{R}^{\kappa(i,n,d)})$ provided $p > i \leq \min(n, d)$, and Theorem 17.13 directly applies if $\mathfrak{f}$ from (17.60) gives φ satisfying (17.53a–c).

Yet, this special structure allows for much weaker restrictions on φ if one is concerned with the minimization of Φ itself rather than the satisfaction of the Euler–Lagrange equation (17.54):

Theorem 17.14 (Direct method, polyconvex) *Let φ be a normal integrand*[38] *satisfying* (17.53a) *with $\varphi(x, u, \cdot) : \mathbb{R}^{n\times d} \to \mathbb{R} \cup \{\infty\}$ polyconvex, and let $u_{\mathrm{D}} \in W^{1-1/p,p}(\Gamma_{\mathrm{D}}; \mathbb{R}^n)$. Then the minimization problem* (17.52) *has a solution.*

Obviously, polyconvexity (and thus also quasi- and rank-one convexity) is weaker than usual convexity. Only for $\min(n, d) = 1$, all mentioned modes coincide with usual convexity of $\varphi(x, u, \cdot)$.

35) Without going into detail concerning the so-called Sobolev–Slobodetskiǐ spaces with fractional derivatives, this condition means exactly that u_{D} allows an extension onto Ω belonging to $W^{1,p}(\Omega; \mathbb{R}^n)$.

36) For $p \geq d$, Theorem 17.12 gives this continuity.

37) Surprisingly, not only $\{\det\nabla u_k\}_{k\in\mathbb{N}}$ but even $\{\det\nabla u_k \ln(2+\det\nabla u_k)\}_{k\in\mathbb{N}}$ stays bounded in $L^1(K)$, as proved by S. Müller in [35].

38) This means φ is measurable but $\varphi(x, \cdot, \cdot)$ is only lower semicontinuous.

Example 17.2 [Oscillation effects.] A simple one-dimensional counterexample for *nonexistence* of a solution due to oscillation effects is based on

$$\Phi(u) = \int_0^L \Big(\big(\frac{\mathrm{d}u}{\mathrm{d}x}\big)^2 - 1\Big)^2 + u^2\,\mathrm{d}x \qquad (17.61)$$

to be minimized for $u \in W^{1,4}([0,L])$. A minimizing sequence $\{u_k\}_{k\in\mathbb{N}}$ is, for example,[39]

$$u_k(0) = \frac{1}{k},\quad \frac{\mathrm{d}u_k}{\mathrm{d}x} = \begin{cases} 1 \text{ if } \sin(kx){>}0, \\ -1 \text{ otherwise.}\end{cases} \qquad (17.62)$$

Then $\Phi(u_k) = \mathcal{O}(1/k^2) \to 0$ for $k{\to}\infty$, so that $\inf\Phi = 0$. Yet, there is no u such that $\Phi(u) = 0$.[40] We can observe that Theorem 17.1 (resp. Theorem 17.2) cannot be used due to lack of weak lower semicontinuity (resp. convexity) of Φ which is due to nonconvexity of the *double-well potential* density $F \mapsto \varphi(x,u,F) = (|F|^2{-}1)^2 + u^2$; cf. also (17.105) below for a "fine limit" of the fast oscillations from Figure 17.1.

Example 17.3 [Concentration effects.] The condition that V in Theorems 17.1 and 17.2 has a pre-dual, is essential. A simple one-dimensional counterexample for *nonexistence* of a solution in the situation where V is not reflexive and even does not have any pre-dual, is based on

$$\Phi(u) = \int_{-1}^1 (1{+}x^2)\Big|\frac{\mathrm{d}u}{\mathrm{d}x}\Big|\,\mathrm{d}x + (u(-1){+}1)^2 + (u(1){-}1)^2 \qquad (17.63)$$

for $u \in W^{1,1}([-1,1])$. If u were a minimizer, then u must be nondecreasing (otherwise, it obviously would not be optimal), and we can always take some "part" of the nonnegative derivative of this function and add the corresponding area in a neighborhood of 0. This does not affect $u(\pm1)$ but makes $\int_{-1}^1(1{+}x^2)|\mathrm{d}u/\mathrm{d}x|\,\mathrm{d}x$ lower, contradicting the original assumption that u is a minimizer. In fact, as $1{+}x^2$ in (17.63) attains its minimum only at a single point $x = 0$, any minimizing sequence $\{u_k\}_{k\in\mathbb{N}}$ is forced to concentrate its derivative around $x = 0$. For example considering, for $k \in \mathbb{N}$ and $\ell \in \mathbb{R}$, the sequence given by

$$u_k(x) = \frac{\ell k x}{1 + k|x|} \qquad (17.64)$$

yields $\Phi(u_k) = 2\ell + 2(\ell{-}1)^2 + \mathcal{O}(1/k^2)$. Obviously, the sequence $\{u_k\}_{k\in\mathbb{N}}$ will minimize Φ provided $\ell = 1/2$; then $\lim_{k\to\infty}\Phi(u_k) = 3/2 = \inf\Phi$; see Figure 17.2. On the other hand, this value $\inf\Phi$ cannot be achieved, otherwise such u must have simultaneously $|\mathrm{d}u/\mathrm{d}x| = 0$ a.e. and $u(\pm1) = \pm1/2$, which is not possible.[41] A similar effect occurs for $\varphi(F) = \sqrt{1 + |F|^2}$ for which $\int_\Omega \varphi(\nabla u)\,\mathrm{d}x$ is the area of the parameterized hypersurface $\{(x, u(x));\ x \in \Omega\}$ in $\mathbb{R}^{d+1}$. Minimization of such a functional is known as the *Plateau variational problem.* Hyper-*surfaces of minimal area* typically do not exists in $W^{1,1}(\Omega)$, especially if Ω is not convex and the concentration of the gradient typically occurs on Γ rather than inside Ω, cf. e.g. [37, Chapter V].

Example 17.4 [Lavrentiev phenomenon.] Coercivity in Theorems 17.1 and 17.2 is also essential even if Φ is bounded from below. An innocent-looking one-dimensional

39) Actually, $u_k(0) \neq 0$ was used in (17.62) only for a better visualization on Figure 17.1.

40) Indeed, then both $\int_0^L((\mathrm{d}u/\mathrm{d}x)^2{-}1)^2\mathrm{d}x$ and $\int_0^L u^2\mathrm{d}x$ would have to be zero, so that $u = 0$, but then also $\mathrm{d}u/\mathrm{d}x = 0$, which however contradicts $\int_0^L((\mathrm{d}u/\mathrm{d}x)^2{-}1)^2\mathrm{d}x = 0$.

41) This is because of the concentration effect. More precisely, the sequence $\{\mathrm{d}u_k/\mathrm{d}x\}_{k\in\mathbb{N}} \subset L^1(-1,1)$ is not uniformly integrable.

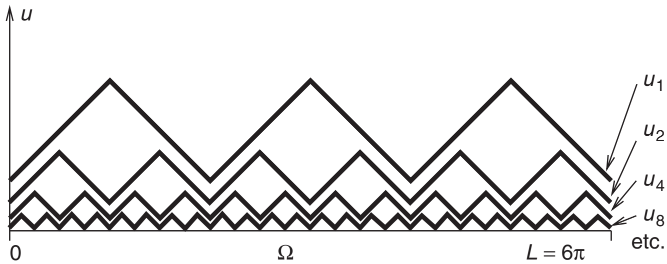

Figure 17.1 A minimizing sequence (17.62) for Φ from (17.61) whose gradient exhibits faster and faster spatial oscillations.

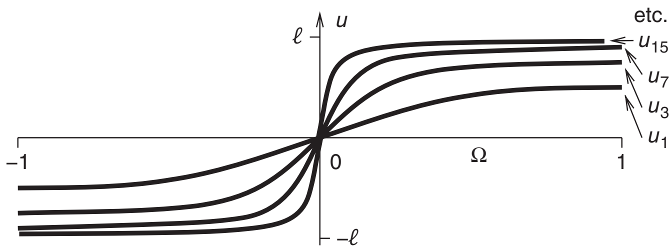

Figure 17.2 A minimizing sequence (17.64) for Φ from (17.63) whose gradient concentrates around the point $x = 0$ inside Ω.

counterexample for nonexistence of a solution in the situation where V is reflexive and $\Phi \geq 0$ is continuous and weakly lower semicontinuous is based on

$$\Phi(u) = \int_0^1 (u^3 - x)^2 \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^6 \mathrm{d}x \quad \text{subject to} \quad u(0) = 0, \; u(1) = 1, \tag{17.65}$$

for $u \in W^{1,6}([0,1]) = V$. The minimum of (17.65) is obviously 0, being realized on $u(x) = x^{1/3}$. Such $u \in W^{1,1}([0,1])$, however, does not belong to $W^{1,6}([0,1])$ because $|\mathrm{d}u/\mathrm{d}x|^6 = 3^{-6}x^{-4}$ is not integrable owing to its singularity at $x = 0$. Thus (17.65) attains the minimum on $W^{1,p}([0,1])$ with $1 \leq p < 3/2$ although Φ is not (weakly lower semi-) continuous and even not finite on this space, and thus abstract Theorem 17.1 cannot be used. A surprising and not entirely obvious phenomenon is that the infimum (17.65) on $W^{1,6}([0,1])$ is positive, that is, greater than the infimum on $W^{1,p}([0,1])$ with $p < 3/2$; this effect was first observed in [38], cf. also, e.g. [1, Section 4.3.]. Note that $W^{1,6}([0,1])$ is dense in $W^{1,p}([0,1])$ but one cannot rely on $\Phi(u_k) \to \Phi(u)$ if $u_k \to u$ in $W^{1,p}([0,1])$ for $p < 6$; it can even happen that $\Phi(u) = 0$ while $\Phi(u_k) \to \infty$ for $u_k \to u$, a *repulsive effect*, cf. [39, Section 7.3]. Here $\varphi(x, u, \cdot)$ is not uniformly convex, yet the Lavrentiev phenomenon can occur even for uniformly convex φ's, cf. [40].

17.3.2.2 **Fourth Order Systems**

Higher-order problems can be considered analogously but the complexity of the problem grows with the order. Let us therefore use for illustration fourth-order problems only, governed by an integral functional

$$\Phi(u) = \int_\Omega \varphi(x, u, \nabla u, \nabla^2 u)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \phi(x, u, \nabla u)\,\mathrm{d}S \qquad (17.66\mathrm{a})$$

involving Carathéodory integrands $\varphi : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^{n\times d\times d} \to \mathbb{R}$ and $\phi : \Gamma_\mathrm{N}\times\mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}$. The functional Φ is considered on an affine closed manifold

$$\Big\{u\in W^{2,p}(\Omega;\mathbb{R}^n);\ u|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},1},\ \frac{\partial u}{\partial\vec{n}}|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},2}\Big\} \qquad (17.66\mathrm{b})$$

for a given suitable $u_{\mathrm{D},1}$ and $u_{\mathrm{D},2}$. Instead of (17.54), the abstract Euler–Lagrange equation (17.3) now leads to the integral identity:

$$\int_\Omega \Big(\varphi'_{\nabla^2 u}(x, u, \nabla u, \nabla^2 u) \vdots \nabla^2 v + \varphi'_{\nabla u}(x, u, \nabla u, \nabla^2 u) : \nabla v + \varphi'_u(x, u, \nabla u, \nabla^2 u)\cdot v\Big)\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \Big(\phi'_{\nabla u}(x, u, \nabla u)\cdot\frac{\partial v}{\partial\vec{n}} + \phi'_u(x, u, \nabla u)\cdot v\Big)\,\mathrm{d}S = 0 \qquad (17.67)$$

for any $v\in W^{2,p}(\Omega;\mathbb{R}^n)$ such that $v|_{\Gamma_\mathrm{D}} = 0$ and $\partial u/\partial\vec{n}|_{\Gamma_\mathrm{D}} = 0$; the notation " $\vdots$ " stands for summation over three indices. Completed by the Dirichlet conditions on Γ_D, this represents a *weak formulation* of the *boundary-value problem* for a system of fourth-order elliptic quasilinear equations

$$\mathrm{div}^2\, \varphi'_{\nabla^2 u}(u, \nabla u, \nabla^2 u) - \mathrm{div}\, \varphi'_{\nabla u}(u, \nabla u, \nabla^2 u) + \varphi'_u(u, \nabla u, \nabla^2 u) = 0 \text{ in } \Omega, \qquad (17.68\mathrm{a})$$

with two natural (although quite complicated) boundary conditions prescribed on each part of the boundary, namely,

$$\big(\mathrm{div}\, \varphi'_{\nabla^2 u}(u,\nabla u,\nabla^2 u) - \varphi'_{\nabla u}(u,\nabla u,\nabla^2 u)\big)\cdot\vec{n} + \mathrm{div}_\mathrm{S}\big(\varphi'_{\nabla^2 u}(u, \nabla u, \nabla^2 u)\vec{n}\big) - \big(\mathrm{div}_\mathrm{S}\vec{n}\big)\big(\vec{n}^\top \varphi'_{\nabla^2 u}(u, \nabla u, \nabla^2 u)\vec{n}\big) + \phi'_{\nabla u}(u, \nabla u) = 0 \quad \text{on } \Gamma_\mathrm{N}, \quad (17.68\mathrm{b})$$

$$\varphi'_{\nabla^2 u}(u, \nabla u, \nabla^2 u) : (\vec{n}\otimes\vec{n}) + \phi'_u(u, \nabla u) = 0 \quad \text{on } \Gamma_\mathrm{N}, \quad (17.68\mathrm{c})$$

$$u|_\Gamma = u_{\mathrm{D},1}, \quad \frac{\partial u}{\partial\vec{n}}\Big|_\Gamma = u_{\mathrm{D},2} \quad \text{on } \Gamma_\mathrm{D}. \quad (17.68\mathrm{d})$$

Again, (17.68) is called the *classical formulation* of the boundary-value problem in question, and its derivation from (17.67) is more involved than in Section 17.3.2.1. One must use a general decomposition $\nabla v = \partial v/\partial\vec{n} + \nabla_\mathrm{S} v$ on Γ with $\nabla_\mathrm{S} v = \nabla v - \partial v/\partial\vec{n}$ being the *tangential gradient* of v. On a smooth boundary Γ, one can use another (now $(d{-}1)$-dimensional) Green-type formula on tangent spaces:[42)]

$$\int_\Gamma a : (\vec{n}\otimes\nabla v)\,\mathrm{d}S = \int_\Gamma \big(\vec{n}^\top a\vec{n}\big)\frac{\partial v}{\partial\vec{n}} + a : (\vec{n}\otimes\nabla_\mathrm{S} v)\,\mathrm{d}S = \int_\Gamma \big(\vec{n}^\top a\vec{n}\big)\frac{\partial v}{\partial\vec{n}} - \mathrm{div}_\mathrm{S}\big(a\vec{n}\big)v + \big(\mathrm{div}_\mathrm{S}\vec{n}\big)\big(\vec{n}^\top a\vec{n}\big)v\,\mathrm{d}S, \quad (17.69)$$

where $a = \varphi'_{\nabla^2 u}(x, u, \nabla u, \nabla^2 u)$ and $\mathrm{div}_\mathrm{S} = \mathrm{tr}(\nabla_\mathrm{S})$ with $\mathrm{tr}(\cdot)$ being the trace of a $(d{-}1)\times(d{-}1)$-matrix, denotes the $(d{-}1)$-dimensional *surface divergence* so that $\mathrm{div}_\mathrm{S}\vec{n}$ is (up to a factor $-1/2$) the mean curvature of the surface Γ. Comparing the variational formulation as critical points of (17.66) with the weak formulation (17.67) and with the classical formulation (17.68), one can see that although formally all formulations are equivalent to each other, the advantage of the variational formulations such as (17.66) in its simplicity is obvious.

42) This "surface" Green-type formula reads $\int_\Gamma w : ((\nabla_\mathrm{S} v)\otimes\vec{n})\,\mathrm{d}S = \int_\Gamma (\mathrm{div}_\mathrm{S}\vec{n})(w : (\vec{n}\otimes\vec{n}))v - \mathrm{div}_\mathrm{S}(w\cdot\vec{n})v\,\mathrm{d}S$.

As in (17.56), taking general Carathéodory integrands $a : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^{n\times d\times d} \to \mathbb{R}^{n\times d\times d}$, $b : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^{n\times d\times d} \to \mathbb{R}^{n\times d}$, $c : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^{n\times d\times d} \to \mathbb{R}^n$, $d : \Gamma_{\rm N}\times\mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}^{n\times d}$, and finally $e : \Gamma_{\rm N}\times\mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}^n$, one can consider a boundary-value problem for a system of fourth-order elliptic quasilinear equations:

$$\begin{aligned}\mathrm{div}^2 a(u,\nabla u,\nabla^2 u) - \mathrm{div}\, b(u,\nabla u,\nabla^2 u)&\\ + c(u,\nabla u,\nabla^2 u) = 0 \quad \text{in } \Omega,&\quad (17.70\text{a})\end{aligned}$$

with the boundary conditions (17.68d) and

$$\begin{aligned}&a(u,\nabla u,\nabla^2 u):(\vec n\otimes\vec n)\\ &\qquad + d(u,\nabla u) = 0 \qquad \text{on } \Gamma_{\rm N}, \quad (17.70\text{b})\\ &\big(\mathrm{div}\, a(u,\nabla u,\nabla^2 u) - b(u,\nabla u,\nabla^2 u)\big)\cdot\vec n\\ &\qquad + \mathrm{div}_{\rm S}\big(a(u,\nabla u,\nabla^2 u)\vec n\big)\\ &\qquad - \big(\mathrm{div}_{\rm S}\vec n\big)\big(\vec n^\top a(u,\nabla u,\nabla^2 u)\vec n\big)\\ &\qquad + e(u,\nabla u) = 0 \qquad \text{on } \Gamma_{\rm N}. \quad (17.70\text{c})\end{aligned}$$

Existence of a potential for which the boundary-value problem (17.70) would represent the Euler–Lagrange equation (17.3) requires the *symmetry* for the Jacobians of the mapping $(H,F,u) \mapsto (a(x,u,F,H), b(x,u,F,H), c(x,u,F,H)) : \mathbb{R}^{n\times d\times d}\times\mathbb{R}^{n\times d}\times\mathbb{R}^d \to \mathbb{R}^{n\times d\times d}\times\mathbb{R}^{n\times d}\times\mathbb{R}^d$ and of the mapping $(F,u) \mapsto (d(x,u,F), e(x,u,F)) : \mathbb{R}^{n\times d}\times\mathbb{R}^d \to \mathbb{R}^{n\times d}\times\mathbb{R}^d$; we used H as a placeholder for $\nabla^2 u$. The formula (17.58) then takes the form:

$$\begin{aligned}\varphi(x,u,F,H) = \int_0^1 & a(x,\lambda u,\lambda F,\lambda H)\vdots H\\ &+ b(x,\lambda u,\lambda F,\lambda H):F\\ &+ c(x,\lambda u,\lambda F,\lambda H)\cdot u\,\mathrm{d}\lambda,\end{aligned} \quad (17.71\text{a})$$

$$\begin{aligned}\phi(x,u,F) = \int_0^1 & d(x,\lambda u,\lambda F):F\\ &+ e(x,\lambda u,\lambda F)\cdot u\,\mathrm{d}\lambda. \quad (17.71\text{b})\end{aligned}$$

Analogously to Theorem 17.13, one can obtain existence of weak solutions by the direct method under a suitable coercivity/growth conditions on φ and an analogue of quasiconvexity of $\varphi(x,u,F,\cdot) : \mathbb{R}^{n\times d\times d} \to \mathbb{R}$, and $u_{\rm D,1} \in W^{2-1/p,p}(\Gamma_{\rm D};\mathbb{R}^n)$ and $u_{\rm D,2} \in W^{1-1/p,p}(\Gamma_{\rm D};\mathbb{R}^n)$.

So far, we considered two Dirichlet-type conditions (17.68d) on $\Gamma_{\rm D}$ (dealing with zeroth- and first-order derivatives) and two Robin-type conditions (17.68b,c) on $\Gamma_{\rm N}$ (dealing with second- and third-order derivatives). These arise either by fixing both $u|_\Gamma$ or $\partial u/\partial\vec n|_\Gamma$ or neither of them, cf. (17.66b). One can, however, think about fixing only $u|_\Gamma$ or only $\partial u/\partial\vec n|_\Gamma$, which gives other two options of natural boundary conditions, dealing with zeroth- and second-order derivatives or first- and third-order derivatives, respectively.

The other two combinations, namely the zeroth- and the third-order derivatives or the first- and the second-order derivatives, are not natural from the variational viewpoint because they overdetermine some of the two boundary terms arising in the weak formulation (17.67).

17.3.2.3 Variational Inequalities

Merging the previous Sections 17.3.2.1–17.3.2.2 with the abstract scheme from Sections 17.2.2–17.2.3, has important applications. Let us use as illustration $\Phi_0 = \Phi$ from (17.52) and $\Phi_1(v) = \int_\Omega \zeta(v)\,\mathrm{d}x + \int_{\Gamma_{\rm N}} \xi(v)\,\mathrm{d}S$ as in Remark 17.1, now with some convex $\zeta,\xi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$. In view of the abstract inequality (17.10), the weak formulation (17.54) gives the variational inequality

$$\begin{aligned}\int_\Omega & \zeta(v) + \varphi'_{\nabla u}(x,u,\nabla u):\nabla(v-u)\\ &+ \varphi'_u(x,u,\nabla u)\cdot(v-u)\,\mathrm{d}x\\ &+ \int_\Gamma \xi(v) + \phi'_u(x,u)\cdot(v-u)\,\mathrm{d}S\end{aligned}$$

$$\geq \int_\Omega \zeta(u)\,\mathrm{d}x + \int_\Gamma \xi(u)\,\mathrm{d}S \quad (17.72)$$

for any $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $v|_{\Gamma_\mathrm{D}} = 0$. The passage from the weak formulation to the classical boundary-value problem analogous to (17.54) $\longrightarrow$ (17.55) leads to the differential inclusion on Ω:

$$\operatorname{div} \varphi'_{\nabla u}(u,\nabla u) - \varphi'_u(u,\nabla u) - \partial\zeta(u) \ni 0 \quad \text{in } \Omega, \quad (17.73\text{a})$$

with another differential inclusion in the boundary conditions

$$\varphi'_{\nabla u}(u,\nabla u)\cdot\vec{n} + \phi'_u(u) + \partial\xi(u) \ni 0 \quad \text{on } \Gamma_\mathrm{N}, \quad (17.73\text{b})$$

$$u|_\Gamma = u_\mathrm{D} \quad \text{on } \Gamma_\mathrm{D}. \quad (17.73\text{c})$$

There is an extensive literature on mathematical methods in variational inequalities, cf. e.g. [18, 41–44].

17.3.3 Some Examples

Applications of the previous general boundary-value problems to more specific situations in continuum physics are illustrated in the following examples.

17.3.3.1 Nonlinear Heat-Transfer Problem

The steady-state temperature distribution θ in an anisotropic nonlinear heat-conductive body $\Omega \subset \mathbb{R}^3$ is described by the balance law

$$\operatorname{div} j = f \quad \text{with} \quad j = -\kappa(\theta)\mathbb{K}\nabla\theta \quad \text{on } \Omega, \quad (17.74\text{a})$$

$$\vec{n}\cdot j + b(\theta) = g \quad \text{on } \Gamma, \quad (17.74\text{b})$$

where $b(\cdot) > 0$ is a boundary heat outflow, g the external heat flux, f the bulk heat source, and with the heat flux j governed by the Fourier law involving a symmetric positive definite matrix $\mathbb{K} \in \mathbb{R}^{d\times d}$ and a nonlinearity $\kappa : \mathbb{R} \to \mathbb{R}^+$. In terms of a and c in (17.56a), we have $n = 1$, $a(x,u,F) = \kappa(u)\mathbb{K}F$ and $c(x,u,F) = f(x)$ and the symmetry (17.57b) fails, so that (17.74) does not have the variational structure unless κ is constant. Yet, a simple rescaling of θ, called the *Kirchhoff transformation*, can help: introducing the substitution $u = \widehat{\kappa}(\theta) = \int_0^\theta \kappa(\vartheta)\,\mathrm{d}\vartheta$, we have $j = -\mathbb{K}\nabla u$ and (17.74) transforms to

$$\operatorname{div}(\mathbb{K}\nabla u) + f = 0 \quad \text{on } \Omega, \quad (17.75\text{a})$$

$$\vec{n}^\top \mathbb{K}\nabla u + b(\widehat{\kappa}^{-1}(u)) = g \quad \text{on } \Gamma, \quad (17.75\text{b})$$

which already fits in the framework of (17.52) with $\varphi(x,u,F) = \frac{1}{2}F^\top\mathbb{K}F - f(x)u$ and $\phi(x,u) = \widetilde{b}(u) - g(x)u$ where $\widetilde{b}$ is a primitive of $b\circ\widehat{\kappa}^{-1}$. Eliminating the nonlinearity from the bulk to the boundary, we thus gain a variational structure at least if $f \in L^{6/5}(\Omega)$ and $g \in L^{4/3}(\Gamma)$ [43] and thus, by the direct method, we obtain existence of a solution $u \in H^1(\Omega)$ to (17.75) as well as a possibility of its efficient numerical approximation, cf. Section 17.4.1; then $\theta = \widehat{\kappa}^{-1}(u)$ yields a solution to (17.74). Our optimism should however be limited because, in the heat-transfer context, the natural integrability of the heat sources is only $f \in L^1(\Omega)$ and $g \in L^1(\Gamma)$, but this is not consistent with the variational structure if $d > 1$:

Example 17.5 [Nonexistence of minimizers] Consider the heat equation $-\operatorname{div}\nabla u = 0$ for $d = 3$ and with, for simplicity, zero Dirichlet boundary conditions, so that the underlying variational problem is to minimize $\int_\Omega \frac{1}{2}|\nabla u|^2 - fu\,\mathrm{d}x$ on $H^1_0(\Omega)$. Yet,

43) If $d = 3$, this integrability of the heat sources is necessary and sufficient to ensure the functional $(u \mapsto \int_\Omega fu\,\mathrm{d}x + \int_\Gamma gu\,\mathrm{d}S)$ to belong to $H^1(\Omega)^*$.

$$\inf_{u\in H_0^1(\Omega)} \int_\Omega \frac{1}{2}|\nabla u|^2 - fu\,\mathrm{d}x = -\infty, \quad (17.76)$$

whenever $f\in L^1(\Omega)\backslash L^{6/5}(\Omega)$.[44)]

17.3.3.2 Elasticity at Large Strains

A prominent application of multidimensional ($d > 1$) vectorial ($n > 1$) variational calculus is to elasticity under the hypothesis that the stress response on the deformation gradient is a gradient of some potential; such *materials* are called *hyperelastic.* The problem is very difficult especially at large strains where the geometry of the standard Cartesian coordinates may be totally incompatible with the largely deformed geometry of the specimen Ω.

Here, u will stand for the deformation (although the usual notation is rather y) and we will consider $n = d$ and $\varphi = \varphi(F)$ taking possibly also the value $+\infty$. The ultimate requirement is frame indifference, that is, $\varphi(RF) = \varphi(F)$ for all $R \in \mathbb{R}^{d\times d}$ in the special orthogonal group SO(d). One could try to rely on Theorem 17.13 or 17.14. In the former case, one can consider nonpolyconvex materials and one also has the Euler–Lagrange equation (17.54) at one's disposal, but the growth condition (17.53b) does not allow an infinite increase of energy when the volume of the material is locally shrunk to 0, that is, we cannot satisfy the condition

$$\varphi(F) \to +\infty \qquad \text{if } \det F \to 0+ \quad (17.77a)$$

$$\varphi(F) = +\infty \qquad \text{if } \det F \le 0. \quad (17.77b)$$

An example for a polyconvex frame-indifferent φ satisfying (17.77) is the *Ogden material*

$$\varphi(F) = a_1 \mathrm{tr}(F^\top F)^{b_1} + a_2|\mathrm{tr}(\mathrm{cof}(F^\top F))|^{b_2} + \gamma(\det F) \quad (17.78)$$

with $a_1, a_2, b_1, b_2 > 0$ and $\gamma : \mathbb{R}_+ \to \mathbb{R}$ convex such that $\gamma(\delta) = +\infty$ for $\delta \le 0$ and $\lim_{\delta\to 0_+} \gamma(\delta) = +\infty$, and where $\mathrm{cof}(A) = (\det A)A^\top$, considering $d = 3$, and where $\mathrm{tr}\,A = \sum_{i=1}^d A_{ii}$. Particular Ogden materials are Mooney–Rivlin materials with $\varphi(F) = |F|^2 + |\det F|^2 - \ln(\det F)$ or compressible neo-Hookean materials with $\varphi(F) = a|F|^2 + \gamma(\det F)$. The importance of polyconvexity is that the existence of energy-minimizing deformation can be based on Theorem 17.14, which allows us to handle local nonpenetration

$$\det(\nabla u) > 0 \qquad \text{a.e. on } \Omega \quad (17.79a)$$

involved in φ via (17.77). Handling of nonpenetration needs also the global condition

$$\int_\Omega \det(\nabla u)\,\mathrm{d}x \le \mathrm{meas}_d\big(u(\Omega)\big); \quad (17.79b)$$

(17.79) is called the *Ciarlet–Nečas nonpenetration condition* [45]. Whether it can deal with quasiconvex but not polyconvex materials remains an open question, however cf. [46].

An interesting observation is that polyconvex materials allow for an energy-controlled stress $|\varphi'(F)F^\top| \le C(1 + \varphi(F))$ even though the so-called Kirchhoff stress $\varphi'(F)F^\top$ itself does not need to be bounded. This can be used, for example, in sensitivity analysis and to obtain modified Euler–Lagrange equations to overcome the possible failure of (17.54) for such materials, cf. [46]. It is worth noting that even such spatially homogeneous, frame-indifferent, and polyconvex materials can exhibit the *Lavrentiev phenomenon*, cf. [47].

Another frequently used ansatz is just a quadratic form in terms of the

44) The proof is very simple: $f\notin H_0^1(\Omega)^*$ means $\|f\|_{H_0^1(\Omega)^*} = \sup_{u\in W_0^{1,\infty}(\Omega),\ \|u\|_{1,2}\le 1} \int_\Omega fu\,\mathrm{d}x = +\infty$, which further means $\int_\Omega fu_k\,\mathrm{d}x \to +\infty$ for some $u_k\in W_0^{1,\infty}(\Omega)$ such that $\int_\Omega \frac{1}{2}|\nabla u_k|^2\,\mathrm{d}x \le \frac{1}{2}$, so that $\int_\Omega \frac{1}{2}|\nabla u_k|^2 - fu_k\,\mathrm{d}x \to -\infty$.

Green – Lagrange strain tensor

$$E = \tfrac{1}{2}F^\top F - \tfrac{1}{2}\mathbb{I}. \qquad (17.80)$$

An example is an isotropic material described by only two elastic constants; in terms of the *Lamé constants* μ and λ, it takes the form

$$\varphi(F) = \tfrac{1}{2}\lambda|\mathrm{tr}E|^2 + \mu|E|^2, \qquad (17.81)$$

and is called a *St.Venant – Kirchhoff's material*, cf. [48, Volume I, Section 4.4]. If $\mu > 0$ and $\lambda > -\frac{2}{d}\mu$, φ from (17.81) is coercive in the sense of $\varphi(F) \geq \epsilon_0|F|^4 - 1/\epsilon_0$ for some $\epsilon_0 > 0$ but not quasiconvex (and even not rank-one convex), however. Therefore, existence of an energy-minimizing deformation in $W^{1,4}(\Omega;\mathbb{R}^d)$ is not guaranteed.

A way to improve solvability for non-quasiconvex materials imitating additional interfatial-like energy is to augment φ with some small convex higher-order term, for example, $\varphi_1(F,H) = \varphi(F) + \sum_{i,k,l,m,n=1}^d \mathbb{H}_{klmn}(x)H_{ikl}H_{imn}$ with $\mathbb{H}$ a (usually only small) fourth-order positive definite tensor, and to employ the fourth-order framework of Section 17.3.2.2. This is the idea behind the mechanics of *complex* (also called *nonsimple* or a special *micropolar*) *continua*, cf. e.g., [49].

17.3.3.3 Small-Strain Elasticity, Lamé System, Signorini Contact

Considering the deformation y and displacement $u(x) = y(x) - x$, we write $F = \nabla y = \nabla u + \mathbb{I}$ and, for $|\nabla u|$ small, the tensor E from (17.80) is

$$E = \frac{F^\top F - \mathbb{I}}{2} = \frac{(\nabla u)^\top + \nabla u + (\nabla u)^\top \nabla u}{2}$$
$$= \tfrac{1}{2}(\nabla u)^\top + \tfrac{1}{2}\nabla u + o\big(|\nabla u|\big), \qquad (17.82)$$

which leads to the definition of the linearized-strain tensor, also called *small-strain tensor*, as $e(u) = \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u$. In fact, a vast amount of engineering or also, for example, geophysical models and calculations are based on the small-strain concept. The specific energy in homogeneous materials is then $\varphi : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ where $\mathbb{R}^{d\times d}_{\mathrm{sym}} = \{A \in \mathbb{R}^{d\times d};\ A^\top = A\}$. In linearly responding materials, φ is quadratic. An example is an isotropic material; in terms of Lamé's constants as in (17.81), it takes the form

$$\varphi_{\mathrm{Lame}}(e) = \tfrac{1}{2}\lambda|\mathrm{tr}\, e|^2 + \mu|e|^2. \qquad (17.83)$$

Such φ_{Lame} is positive definite on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ if $\mu > 0$ and $\lambda > -(2d/\mu)$. The positive definiteness of the functional $\int_\Omega \varphi_{\mathrm{Lame}}(e(u))\,\mathrm{d}x$ is a bit delicate as the rigid-body motions (translations and rotations) are not taken into account by it. Yet, fixing positive definiteness by suitable boundary conditions, coercivity can then be based on the *Korn inequality*

$$\forall v \in W^{1,p}(\Omega;\mathbb{R}^d),\ \ v|_{\Gamma_{\mathrm{D}}} = 0:$$
$$\|v\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \leq C_p\|e(v)\|_{L^p(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \qquad (17.84)$$

to be used for $p = 2$; actually, (17.84) holds for $p > 1$ on connected smooth domains with Γ_{D} of nonzero measure, but notably counterexamples exist for $p = 1$. Then, by the direct method based on Theorem 17.2, one proves existence and uniqueness of the solution to the *Lamé system* arising from (17.55) by considering $\varphi(x,u,F) = \varphi_{\mathrm{Lame}}(\frac{1}{2}F^\top + \frac{1}{2}F)$ and $\phi(x,u) = g(x)u$:

$$\mathrm{div}\,\sigma + f = 0 \qquad \text{in } \Omega$$
$$\text{with } \sigma = 2\mu e(u) + \lambda(\mathrm{div}\, u)\mathbb{I}, \qquad (17.85a)$$
$$\sigma\vec{n} = g \qquad \text{on } \Gamma_{\mathrm{N}}, \qquad (17.85b)$$
$$u\big|_\Gamma = u_{\mathrm{D}} \qquad \text{on } \Gamma_{\mathrm{D}}. \qquad (17.85c)$$

The Lamé potential (17.83) can be obtained by an asymptotic expansion of an

Ogden material (17.78), see [48, Volume I, Theorem 4.10-2].

An illustration of a very specific variational inequality is a *Signorini (frictionless) contact* on a third part Γ_C of the boundary Γ; so now we consider Γ divided into three disjoint relatively open parts Γ_D, Γ_N, and Γ_C whose union is dense in Γ. This is a special case of the general problem (17.72) with $\zeta \equiv 0$ and $\xi(x,u) = 0$ if $u{\cdot}\vec{n}(x) \le 0$ on Γ_C, otherwise $\xi(x,u) = +\infty$. In the classical formulation, the boundary condition on Γ_C can be identified as

$$\left.\begin{aligned}&(\sigma\vec{n}){\cdot}\vec{n} \le 0,\\ &u{\cdot}\vec{n} \le 0,\\ &\big((\sigma\vec{n}){\cdot}\vec{n}\big)\big(u{\cdot}\vec{n}\big) = 0\end{aligned}\right\} \quad \text{on } \Gamma_C, \qquad (17.85\text{d})$$

$$(\sigma\vec{n}) - ((\sigma\vec{n}){\cdot}\vec{n})\vec{n} = 0 \qquad \text{on } \Gamma_C; \qquad (17.85\text{e})$$

note that (17.85d) has a structure of a complementarity problem (17.17) for the normal stress $(\sigma\vec{n}){\cdot}\vec{n}$ and the normal displacement $u{\cdot}\vec{n}$, while (17.85e) is the equilibrium condition for the tangential stress.

This very short excursion into a wide area of contact mechanics shows that it may have a simple variational structure: In terms of Example 17.1, the convex set K is a cone (with the vertex not at origin if $u_D \neq 0$):

$$K = \big\{u{\in}H^1(\Omega;\mathbb{R}^n);\ u|_\Gamma = u_D \text{ on } \Gamma_D,\ u{\cdot}\vec{n} \le 0 \text{ on } \Gamma_C\big\} \qquad (17.86)$$

and then (17.85) is just the classical formulation of the first-order condition (17.13) for the simple problem

$$\text{minimize } \Phi(u) = \int_\Omega \frac{\lambda}{2}|\mathrm{div}\, u|^2 + \mu|e(u)|^2 \mathrm{d}x$$
$$\text{subject to } u{\in}K \text{ from (17.86).}$$

As in Section 17.3.2.2, it again demonstrates one of the advantages of the variational formulation as having a much simpler form in comparison with the classical formulation.

17.3.3.4 Sphere-Valued Harmonic Maps

Another example is a minimization problem with $\varphi(x,u,F) = \omega(u) + h(x){\cdot}u + \frac{\epsilon}{2}|F|^2$ and the nonconvex constraint $|u| = 1$, that is,

$$\begin{aligned}&\text{minimize } \Phi = \int_\Omega \omega(u) + \frac{\epsilon}{2}|\nabla u|^2 - h{\cdot}u\,\mathrm{d}x\\ &\text{subject to } R(u) = |u|^2 - 1 = 0 \text{ on } \Omega,\end{aligned} \qquad (17.87)$$

which has again the structure (17.14) now with $V = K = H^1(\Omega;\mathbb{R}^3)$ and $\Lambda = L^2(\Omega)$ with the ordering by the trivial cone $\{0\}$ of "nonnegative" vectors. This may serve as a simplified model of static micromagnetism[45] in ferromagnetic materials at low temperatures so that the *Heisenberg constraint* $|u| = 1$ is well satisfied pointwise by the magnetization vector u. Alternatively, it may also serve as a simplified model of liquid crystals. The weak formulation of this minimization problem is

$$\int_\Omega \epsilon\nabla u{:}\nabla(v - (u{\cdot}v)u) + \big(\omega'(u) - h\big){\cdot}\big(v - (u{\cdot}v)u\big)\,\mathrm{d}x = 0 \qquad (17.88)$$

for any $v \in H^1(\Omega;\mathbb{R}^3)$ with $|v|^2 = 1$ a.e. on Ω; for $\omega \equiv 0$ cf. [50, Section 8.4.3]. The corresponding classical formulation then has the form

$$-\,\mathrm{div}(\epsilon\nabla u) + \omega'(u) - h = \big(|\nabla u|^2 + (\omega'(u) - h){\cdot}u\big)u \quad \text{in } \Omega, \qquad (17.89\text{a})$$

$$\epsilon\nabla u{\cdot}\vec{n}\big|_\Gamma = 0 \qquad \text{on } \Gamma. \qquad (17.89\text{b})$$

45) In this case, $\omega : \mathbb{R}^3 \to \mathbb{R}$ is an anisotropy energy with minima at easy-axis magnetization and $\epsilon > 0$ is an exchange-energy constant, and h is an outer magnetic field. The demagnetizing field is neglected.

Comparing it with (17.16) with $N_K \equiv \{0\}$ and $R' = 2\times$identity on $L^2(\Omega;\mathbb{R}^3)$, we can see that $\lambda^* = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\omega'(u)-h)\cdot u$ plays the role of the Lagrange multiplier for the constraint $|u|^2 = 1$ a.e. on Ω.

Let us remark that, in the above-mentioned micromagnetic model, R is more complicated than (17.87) and involves also the differential constraints

$$\operatorname{div}(h-u) = 0 \quad \text{and} \quad \operatorname{rot} h = j \tag{17.90}$$

with j assumed fixed, which is the (steady state) Maxwell system where j is the electric current, and the minimization in (17.87) is to be done over the couples (u, h); in fact, (17.90) is considered on the whole $\mathbb{R}^3$ with j fixed and u vanishing outside Ω.

17.3.3.5 Saddle-Point-Type Problems

In addition to minimization principles, other principles also have applications. For example, for the usage of the mountain pass Theorem 17.4 for potentials such as $\varphi(u, F) = \frac{1}{2}|F|^2 + c(u)$ see [50, Section 8.5] or [51, Section II.6].

Seeking saddle points of Lagrangeans such as (17.18) leads to *mixed formulations* of various constrained problems. For example, the Signorini problem (17.85) uses $\Phi(u) = \int_\Omega \varphi_{\text{Lame}}(e(u))\,\mathrm{d}x + \int_{\Gamma_N} g\cdot u\,\mathrm{d}S$, $R : u \mapsto u\cdot\vec{n} : H^1(\Omega;\mathbb{R}^d) \to H^{1/2}(\Gamma_C)$, and $D = \{v \in H^{1/2}(\Gamma_C);\ \ v \le 0\}$. The saddle point $(u, \lambda^*) \in H^1(\Omega;\mathbb{R}^d) \times H^{-1/2}(\Gamma_C)$ with $u|_{\Gamma_D} = g$ and $\lambda^* \le 0$ on Γ_C exists and represents the mixed formulation of (17.85); then $\lambda^* = (\sigma\vec{n})\cdot\vec{n}$ and cf. also the Karush–Kuhn–Tucker conditions (17.16) with $N_K \equiv \{0\}$, cf. e.g. [37, 43].

Another example is a saddle point on $H_0^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ for $\mathscr{L}(u, \lambda^*) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \lambda^* \operatorname{div} u\,\mathrm{d}x$ that leads to the system

$$-\Delta u + \nabla\lambda^* = 0 \quad \text{and} \quad \operatorname{div} u = 0, \tag{17.91}$$

which is the *Stokes system* for a steady flow of viscous incompressible fluid. The primal formulation minimizes $\int_\Omega \frac{1}{2}|\nabla u|^2 \mathrm{d}x$ on $H_0^1(\Omega;\mathbb{R}^d)$ subject to $\operatorname{div} u = 0$. The Karush—Kuhn–Tucker conditions (17.16) with $R : u \mapsto \operatorname{div} u : H_0^1(\Omega;\mathbb{R}^d) \to L^2(\Omega)$ and the ordering of $D = \{0\}$ as the cone of nonnegative vectors gives (17.91).

17.3.4 Evolutionary Problems

Let us illustrate the *Brezis–Ekeland–Nayroles principle* on the initial-boundary-value problem for a quasilinear parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = g \quad \text{in } Q, \tag{17.92a}$$

$$u = 0 \quad \text{on } \Sigma, \tag{17.92b}$$

$$u(0,\cdot) = u_0 \quad \text{in } \Omega \tag{17.92c}$$

with $Q = [0,T]\times\Omega$ and $\Sigma = [0,T]\times\Gamma$. We consider $V = W_0^{1,p}(\Omega)$ equipped with the norm $\|u\|_{1,p} = \|\nabla u\|_p$, $\Phi(u) = (1/p)\|\nabla u\|_p^p$ and $\langle f(t), u\rangle = \int_\Omega g(t,x)u(x)\,\mathrm{d}x$. Let us use the notation $\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{-1,p}(\Omega)^*$ for the p-Laplacian; this means $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We have that

$$\begin{aligned}\Phi^*(\xi) &= \frac{1}{p'}\|\xi\|^{p'}_{W^{-1,p'}(\Omega)}\\ &= \frac{1}{p'}\|\Delta_p^{-1}\xi\|^p_{1,p} = \frac{1}{p'}\|\nabla\,\Delta_p^{-1}\xi\|_p^p\,.\end{aligned}$$

Thus we obtain the following explicit form of the functional $\mathfrak{F}$ from (17.24):

$$\begin{aligned}\mathfrak{F}(u) = \int_Q &\frac{1}{p}|\nabla u|^p + \Big(\frac{\partial u}{\partial t} - g\Big)u\\ &+ \frac{1}{p'}\Big|\nabla\,\Delta_p^{-1}\Big(\frac{\partial u}{\partial t} - g\Big)\Big|^p \mathrm{d}x\mathrm{d}t\\ &+ \int_\Omega \frac{1}{2}\big|u(T)\big|^2\,\mathrm{d}x.\end{aligned} \tag{17.93}$$

We can observe that the integrand in (17.93) is nonlocal in space, which is, in fact, an inevitable feature for parabolic problems, cf. [52].

Wide generalization to *self-dual problems* of the type $(Lx, Ax) \in \partial\mathscr{L}(Ax, Lx)$ are in [53], covering also nonpotential situations such as the Navier–Stokes equations for incompressible fluids and many others.

Efficient usage of the "global" variational principles such as (17.24), (17.32), or (17.34) for parabolic equations or inequalities is, however, limited to theoretical investigations. Of much wider applicability are recursive variational problems arising by implicit or various semi-implicit time discretization as in Section 17.2.4.3, possibly combined also with spatial discretization and numerical algorithms leading to computer implementations, mentioned in Section 17.4.1 below.

Other, nonminimization principles have applications in hyperbolic problems of the type $\rho\partial^2 u/\partial t^2 - \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = g$ where the *Hamilton principle* (17.37) leads to seeking a critical point of the functional $\int_0^T \int_\Omega \rho|\partial u/\partial t|^2 - (1/p)|\nabla u|^p + g{\cdot}u \,\mathrm{d}x\mathrm{d}t$.

17.4 Miscellaneous

The area of the calculus of variations is extremely wide and the short excursion above presented a rather narrow selection. Let us at least briefly touch a few more aspects.

17.4.1 Numerical Approximation

Assuming $\{V_k\}_{k\in\mathbb{N}}$ is a nondecreasing sequence of finite-dimensional linear subspaces of V whose union is dense, that is,

$$\forall v \in V\ \exists v_k \in V_k : \quad v_k \to v, \tag{17.94}$$

we can restrict the original variational problem of V to V_k. Being finite dimensional, V_k possesses a basis and, in terms of coefficients in this basis, the restricted problem then becomes implementable on computers. This is the simplest idea behind numerical approximation, called the *Ritz method* [54] or, rather in a more general nonvariational context, also the *Galerkin method* [55]. This idea can be applied on an abstract level to problems in Section 17.2. In the simplest (conformal) version instead of (17.3), one is to seek $u_k \in V_k$ such that

$$\forall v \in V_k : \quad \langle \Phi'(u_k), v\rangle = 0; \tag{17.95}$$

such u_k is a critical point of the restriction of Φ on V_k, that is, of $\Phi : V_k \to \mathbb{R}$.

One option is to solve numerically the system of nonlinear equations (17.95) iteratively, for example, by the (quasi) Newton method. Yet, the variational structure can advantageously be exploited in a number of other options such as the conjugate-gradient or the variable-metric methods, cf. e.g. [56, Section 7]. Then approximate satisfaction of optimality conditions typically serves as a stopping criterion for an iterative strategy; in this unconstrained case, it is the residual in (17.95) that is to be small. For constrained problems, the methods of Section 17.2.3 can be adapted.

Application to concrete problems in Section 17.3.2 on function spaces opens further interesting possibilities. Typically, V_k are chosen as linear hulls of piecewise polynomial functions $\{v_{kl}\}_{l=1,\dots,L_k}$ whose supports in Ω are only very small sets so that, for most pairs (l_1, l_2), we have that $\int_\Omega \nabla^i v_{kl_1} : \nabla^j v_{kl_2} \mathrm{d}x = 0$; more precisely, this holds for each pair (l_1, l_2) for which $\mathrm{supp}(v_{kl_1}) \cap \mathrm{supp}(v_{kl_2}) = \emptyset$. For integral functionals Φ from (17.52) or (17.66a), the system of algebraic equations resulting from (17.95) is sparse, which facilitates its implementation and numerical solution on computers; this is the essence of the *finite-element method*.

Phenomena discussed in Examples 17.2 – 17.5 can make the approximation issue quite nontrivial. For Lavrentiev-type phenomena, see, for example, [39]. An example sure to lead to this phenomenon is the constraint $|u|^2 = 1$ in Section 17.3.3.4, which is not compatible with any polynomial approximation so that plain usage of standard finite-element approximation cannot converge.

All this can be applied to time-discretized evolution problems from Section 17.2.4.3, leading to implementable numerical strategies for evolution problems from Sections 17.2.4.1 and 17.2.4.2.

17.4.2
Extension of Variational Problems

Historically, variational problems have been considered together with the Euler – Lagrange equations in their classical formulations, that is, in particular, the solutions are assumed to be continuously differentiable. Here (17.55) or (17.70) are to be considered holding pointwise for $u \in C^2(\overline{\Omega};\mathbb{R}^d)$ and $C^4(\overline{\Omega};\mathbb{R}^d)$, respectively. Yet, such classical solutions do not need to exist[46] and thus the weak formulations (17.54) and (17.67) represent a natural extension (using the density $C^{2k}(\overline{\Omega}) \subset W^{k,p}(\Omega)$) of the original problems defined for smooth functions. The adjective "natural" here means the extension by continuity, referring to the continuity of $\Phi : C^{2k}(\overline{\Omega};\mathbb{R}^d) \to \mathbb{R}$ with respect to the norm of $W^{k,p}(\Omega;\mathbb{R}^d)$ provided natural growth conditions on φ and ϕ are imposed; for $k = 1$; see (17.53). Weak solutions thus represent a natural generalization of the concept of classical solutions.

46) Historically the first surprising example for a minimizer of a (17.52) with $\varphi = \varphi(F)$ smooth uniformly convex, which was only in $W^{1,\infty}(\Omega;\mathbb{R}^m)$ but not smooth is due to Nečas [57], solving negatively the 19th Hilbert's problem [4] if $d > 1$ and $m > 1$ are sufficiently high. An example for even $u \notin W^{1,\infty}(\Omega;\mathbb{R}^m)$ for $d = 3$ and $m = 5$ is in [58].

In general, the method of extension by (lower semi)continuity is called *relaxation*. It may provide a natural concept of generalized solutions with some good physical meaning. One scheme, related to the minimization principle, deals with situations when Theorem 17.1 cannot be applied owing to the lack of weak* lower semicontinuity. The relaxation then replaces Φ by its *lower semicontinuous envelope* $\bar{\Phi}$ defined by

$$\bar{\Phi}(u) = \liminf_{v\to u \text{ weakly*}} \Phi(v). \tag{17.96}$$

Theorem 17.1 then applies to $\bar{\Phi}$ instead of the original Φ, yielding thus a generalized solution to the original variational problem. The definition (17.96) is only conceptual and more explicit expressions are desirable and sometimes actually available. In particular, if $n=1$ or $d=1$, the integral functional (17.52a) admits the formula

$$\bar{\Phi}(u) = \int_\Omega \varphi^{**}(x,u,\nabla u)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \phi(x,u)\,\mathrm{d}S, \tag{17.97}$$

where $\varphi^{**}(x,u,\cdot) : \mathbb{R}^{n\times d} \to \mathbb{R}$ denotes the convex envelope of $\varphi(x,u,\cdot)$, that is, the maximal convex minorant of $\varphi(x,u,\cdot)$. In Example 17.2, $\bar{\Phi}$ is given by (17.97) with

$$\varphi^{**}(u,F) = \begin{cases} u^2 + \left(|F|^2-1\right)^2 & \text{if } |F| \geq 1, \\ u^2 & \text{if } |F| < 1, \end{cases}$$

cf. Figure 17.3, and with $\phi = 0$, and the only minimizer of $\bar{\Phi}$ on $W^{1,4}(\Omega)$ is $u = 0$, which is also a natural $W^{1,4}$-weak limit of all minimizing sequences for Φ, cf. Figure 17.1.

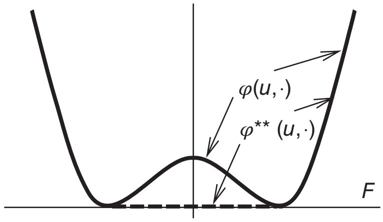

Figure 17.3 A convex envelope $\varphi^{**}(u,\cdot)$ of the double-well potential $\varphi(u,\cdot)$.

Fast oscillations of gradients of these minimizing sequences can be interpreted as *microstructure*, while the minimizers of $\bar{\Phi}$ bear only "macroscopical" information. This reflects the multiscale character of such variational problems. In general, if both $n > 1$ and $d > 1$, (17.97) involves the *quasiconvex envelope* $\varphi^\sharp(x,u,\cdot) : \mathbb{R}^{n\times d} \to \mathbb{R}$ rather than φ^{**}; this is defined by

$$\forall x\in\Omega\ \forall u\in\mathbb{R}^n\ \forall F\in\mathbb{R}^{n\times d} :\quad \varphi^\sharp(x,u,F) = \inf_{v\in W_0^{1,p}(O;\mathbb{R}^n)} \int_O \frac{\varphi(x,u,F+\nabla v(\tilde{x}))}{\mathrm{meas}_d(O)}\,\mathrm{d}\tilde{x}\,;$$

this definition is independent of O but is only implicit and usually only some upper and lower estimates (namely, rank-one convex and polyconvex envelopes) are known explicitly or can numerically be evaluated.

To cope with both nonconvexity and with the unwanted phenomenon of nonexistence as in Example 17.2, one can consider *singular perturbations*, such as

$$\Phi_\varepsilon(u) = \int_\Omega \varphi(x,u,\nabla u) + \varepsilon\mathbb{H}\nabla^2 u \,\vdots\, \nabla^2 u\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \phi(x,u)\,\mathrm{d}S \qquad (17.98)$$

with a positive definite fourth-order tensor $\mathbb{H}$ and small $\varepsilon > 0$; cf. also $\mathbb{H}$ in Section 17.3.3.2. Under the growth/coercivity conditions on φ and ϕ induced by (17.53) with $1 < p < 2^*$, (17.98) possesses a (possibly nonunique) minimizer $u_\varepsilon \in W^{2,2}(\Omega;\mathbb{R}^d)$. The parameter ε determines an internal length scale of possible oscillations of ∇u_ε occurring if $\varphi(x,u,\cdot)$ is not convex, cf. also Figure 17.4. As ε is usually very small, it makes sense to investigate the asymptotics when it approaches 0. For $\varepsilon \to 0$, the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ possesses a subsequence converging weakly in $W^{1,p}(\Omega;\mathbb{R}^d)$ to some u and every such a limit u minimizes the relaxed functional

$$\bar{\Phi}(u) = \int_\Omega \varphi^\sharp(x,u,\nabla u)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \phi(x,u)\,\mathrm{d}S. \qquad (17.99)$$

The possible fast spatial oscillations of the gradient are smeared out in the limit.

To record some information about such oscillations in the limit, one should make a relaxation by continuous extension rather than only by weak lower semicontinuity. To ensure the existence of solutions, the extended space should support a compact topology which makes the extended functional continuous; such a relaxation is called a *compactification*. If the extended space also supports a convex structure (not necessarily coinciding with the linear structure of the original space), one can define variations, differentials, and the abstract Euler–Lagrange equality (17.13); then we speak about the *convex compactification* method, cf. [59].

A relatively simple example can be the relaxation of the micromagnetic problem (17.87)–(17.90) that, in general, does not have any solution if $\varepsilon = 0$ due to nonconvexity of the Heisenberg constraint $|u| = 1$. One can embed the set of admissible u's, namely $\{u \in L^\infty(\Omega;\mathbb{R}^3);\, u(x) \in S \text{ for a.a. } x\}$ with the sphere $S = \{|s| = 1\} \subset \mathbb{R}^3$ into a larger set $\mathscr{Y}(\Omega;S) = \{\nu = (\nu_x)_{x\in\Omega};\ \nu_x$ a probability[47] measure on S and $x \mapsto \nu_x$ weakly* measurable$\}$; the embedding is realized by the mapping $u \mapsto \nu = (\delta_{u(x)})_{x\in\Omega}$ where δ_s denotes here the Dirac measure supported at $s \in S$. The elements of $\mathscr{Y}(\Omega;S)$ are called *Young measures* [60][48] and this set is (considered as) a convex weakly* compact subset of $L^\infty_{w*}(\Omega;\mathrm{M}(S))$ where $\mathrm{M}(S) \cong C(S)^*$ denotes the set of Borel measures on S.[49] The problem (17.87)–(17.90) with $\varepsilon = 0$ then allows a continuous extension

$$\begin{aligned}
&\text{minimize } \bar{\Phi}(\nu,h) = \int_\Omega\int_S \omega(s) - h\cdot s\, \nu_x(\mathrm{d}s)\mathrm{d}x\\
&\text{subject to } \ \mathrm{div}(h-u) = 0, \ \ \mathrm{rot}\, h = j,\\
&\nu \in \mathscr{Y}(\Omega;S),\\
&\text{with } u(x) = \int_S s\, \nu_x(\mathrm{d}s) \text{ for } x \in \Omega. \qquad (17.100)
\end{aligned}$$

The functional $\bar{\Phi}$ is a continuous extension of $(u,h) \mapsto \Phi(u,h)$ from (17.87), which is even convex and smooth with respect to the geometry of $L^\infty_{w*}(\Omega;\mathrm{M}(S)) \times L^2(\Omega;\mathbb{R}^3)$. Existence of solutions to the relaxed problem (17.100) is then obtained by Theorem 17.1 modified for the constrained case. Taking into account the convexity of $\mathscr{Y}(\Omega;S)$, the necessary and sufficient optimality conditions of the type (17.13) for (17.100) lead, after a disintegration, to a pointwise condition

$$\int_S \mathfrak{H}_h(x,s)\,\nu_x(\mathrm{d}s) = \max_{s\in S} \mathfrak{H}_h(x,s),\quad \text{with } \ \mathfrak{H}_h(x,s) = h(x)\cdot s - \omega(s) \qquad (17.101)$$

to hold for a.a. $x \in \Omega$ with h satisfying the linear constraints in (17.100), that is, $\mathrm{div}(h-u) = 0$, $\mathrm{rot}\, h = j$, and $u = \int_S s\,\nu(\mathrm{d}s)$. The integrand of the type $\mathfrak{H}_h$ is sometimes called a *Hamiltonian* and conditions like (17.101), the *Weierstrass maximum principle*, formulated here for the relaxed problem and revealed as being a standard condition of the type (17.13) but with respect to a nonstandard geometry imposed by the space $L^\infty_{w*}(\Omega;\mathrm{M}(S))$. The solutions to (17.100) are typically nontrivial Young measures in the sense that ν_x is not a Dirac measure. From the maximum principle (17.101), one can often see that they are composed from a weighted sum of a finite number of Dirac measures supported only at such $s \in S$ that maximizes $\mathfrak{H}_h(x,\cdot)$. This implies that minimizing sequences for the original problem (17.87)–(17.90) with $\varepsilon = 0$ ultimately must exhibit finer and finer spatial oscillations of u's; this effect is experimentally observed in ferromagnetic materials, see Figure 17.4.[50] In fact, a small parameter $\varepsilon > 0$ in the original problem (17.87)–(17.90) determines the lengthscale of magnetic domains and also the typical width of the walls between the domains. For the Young measure relaxation in micromagnetism see, for example, [61, 63].

47) The adjective "probability" means here a positive measure with a unit mass but does not refer to any probabilistic concept.

48) In fact, L.C. Young had already introduced such measures in 1936 in a slightly different language even before the theory of measure had been invented. For modern mathematical theory see, for example, [30, Chapter 8], [61, Chapter 6–8], [62, Chapter 2], or [59, Chapter 3].

49) "L^∞_{w*}" denotes "weakly* measurable" essentially bounded" mappings, and $L^\infty_{w*}(\Omega;\mathrm{M}(S))$ is a dual space to $L^1(\Omega;C(S))$, which allows to introduce the weak* convergence that makes this set compact.

50) Actually, the minimization-energy principle governs magnetically soft materials where

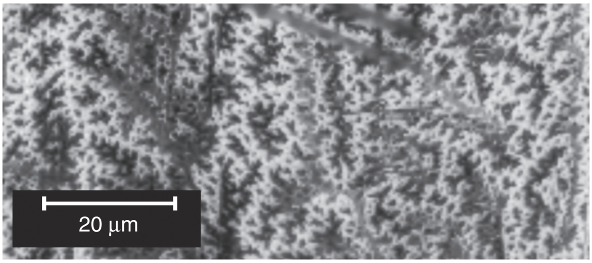

Figure 17.4 Fast spatial oscillations of the magnetization vector minimizing Φ from (17.87)–(17.90) with a double-well potential ω forming a fine microstructure in a ferromagnetic tetragonal single-crystal of NiMnGa with only one axis of easy magnetization normal to the observed surface. (Courtesy O. Heczko, Institute of Physics, ASCR.)

A relaxation by continuous extension of the originally discussed problem (17.52) is much more complicated because the variable exhibiting fast-oscillation tendencies (i.e., ∇u) is in fact subjected to some differential constraint (namely, $\mathrm{rot}\,(\nabla u) = 0$) and because, in contrast to the previous example, is valued on the whole $\mathbb{R}^{n\times d}$, which is not compact. We thus use only a subset of Young measures, namely,

$$\mathscr{G}^p(\Omega;\mathbb{R}^{n\times d}) = \Big\{ \nu\in\mathscr{Y}(\Omega;\mathbb{R}^{n\times d});\ \exists u\in W^{1,p}(\Omega;\mathbb{R}^n):\ \nabla u(x) = \int_{\mathbb{R}^{n\times d}} F\,\nu_x(\mathrm{d}F)\ \forall_{a.a.}x\in\Omega,\ \int_\Omega\int_{\mathbb{R}^{n\times d}} |F|^p\,\nu_x(\mathrm{d}F)\mathrm{d}x < \infty\Big\}.$$

The relaxed problem to (17.52) obtained by continuous extension then has the form

$$\begin{aligned} &\text{minimize } \bar{\Phi}(\nu,h) = \int_\Omega\int_{\mathbb{R}^{n\times d}} \varphi(u,F)\,\nu_x(\mathrm{d}F)\mathrm{d}x + \int_\Gamma \phi(u)\,\mathrm{d}S \\ &\text{subject to } \nabla u(x) = \int_{\mathbb{R}^{n\times d}} F\,\nu_x(\mathrm{d}F)\quad \forall_{a.a.}x, \\ &\qquad (u,\nu)\in W^{1,p}(\Omega;\mathbb{R}^n)\times\mathscr{G}^p(\Omega;\mathbb{R}^{n\times d}). \end{aligned} \tag{17.102}$$

Proving existence of solutions to (17.102) is possible although technically complicated[51] and, moreover, $\mathscr{G}^p(\Omega;\mathbb{R}^{n\times d})$ is unfortunately not a convex subset of $L^\infty_{\mathrm{w}*}(\Omega;\mathrm{M}(\mathbb{R}^{n\times d}))$ if $\min(n,d) > 1$. Only if $n = 1$ or $d = 1$, we can rely on its convexity and derive Karush—Kuhn–Tucker-type necessary optimality conditions of the type (17.13) with λ^* being the multiplier to the constraint $\nabla u(x) = \int_{\mathbb{R}^{n\times d}} F\,\nu_x(\mathrm{d}F)$; the adjoint operator $[R']^*$ in (17.16) turns "∇" to "div." The resulted system takes the form

$$\begin{aligned} &\int_{\mathbb{R}^{n\times d}} \mathfrak{H}_{u,\lambda^*}(x,F)\,\nu_x(\mathrm{d}F) = \max_{F\in\mathbb{R}^{n\times d}} \mathfrak{H}_{u,\lambda^*}(x,F), \\ &\quad\text{with } \mathfrak{H}_{u,\lambda^*}(x,F) = \lambda^*(x)\cdot F - \varphi(x,u(x),F) \\ &\quad\mathrm{div}\,\lambda^* = \int_{\mathbb{R}^{n\times d}} \varphi'_u(u,F)\,\nu_x(\mathrm{d}F) \text{ on } \Omega, \\ &\quad\lambda^*\cdot\vec{n} + \phi'_u(u) = 0 \qquad\qquad \text{on } \Gamma, \end{aligned} \tag{17.103}$$

cf. [59, Chapter 5]. If $\varphi(x,u,\cdot)$ is convex, then there exists a standard weak solution u, that is, $\nu_x = \delta_{\nabla u(x)}$, and (17.103) simplifies to

the hysteresis caused by pinning effects is not dominant.

51) Actually, using compactness and the direct method must be combined with proving and exploiting that minimizing sequences $\{u_k\}_{k\in\mathbb{N}}$ for (17.52) have $\{|\nabla u_k|^p;\ k\in\mathbb{N}\}$ uniformly integrable if the coercivity (17.53a) with $p > 1$ holds.

$$\mathfrak{H}_{u,\lambda^*}(x,\nabla u(x)) = \max_{F\in\mathbb{R}^{n\times d}} \mathfrak{H}_{u,\lambda^*}(x,F),$$
$$\operatorname{div}\lambda^* = \varphi'_u(u,\nabla u) \qquad \text{on } \Omega,$$
$$\lambda^*\cdot\vec{n} + \phi'_u(u) = 0 \qquad \text{on } \Gamma. \quad (17.104)$$

One can see that (17.104) combines the *Weierstrass maximum principle* with a half of the Euler–Lagrange equation (17.55). If $\varphi(x,u,\cdot)$ is not convex, the oscillatory character of ∇u for minimizing sequences can be seen from (17.103) similarly as in the previous example, leading to nontrivial Young measures.

We can illustrate it on Example 17.2, where (17.103) leads to the system $\mathrm{d}\lambda^*/\mathrm{d}x = 2u$ and $\mathrm{d}u/\mathrm{d}x = \int_{\mathbb{R}} F\,\nu_x(\mathrm{d}F)$ on $(0,6\pi)$ with the boundary conditions $\lambda^*(0) = 0 = \lambda^*(6\pi)$ and with the Young measure $\{\nu_x\}_{0\le x\le 6\pi} \in \mathscr{Y}^4((0,6\pi);\mathbb{R})$ such that ν_x is supported on the finite set $\{F\in\mathbb{R};\ \lambda^*(x)F - (|F|^2-1)^2 = \max_{\widetilde{F}\in\mathbb{R}} \lambda^*(x)\widetilde{F} - (|\widetilde{F}|^2-1)^2\}$. The (even unique) solution of this set of conditions is

$$u(x) = 0, \quad \lambda^*(x) = 0, \quad \nu_x = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1} \qquad (17.105)$$

for $x\in(0,6\pi)$. This (spatially constant) Young measure indicates the character of the oscillations of the gradient in (17.62).

Having in mind the elasticity interpretation from Section 17.3.3.2, this effect is experimentally observed in some special materials, see Figure 17.5;[52] for modeling of such microstructure by nonconvex problems see, for example, [20, 62, 64–66].

Models based on relaxation of continuous extensions such as (17.100) or (17.102), are sometimes called *mesoscopical*, in contrast to the original problems such as (17.87)–(17.90) or (17.98) with small $\varepsilon > 0$, which are called *microscopical*, while the models using original spaces but lower semicontinuous extensions such as (17.97), which forget any information about fine microstructures, are called *macroscopical*.

52) Actually, Figure 17.5 refers to a multidimensional vectorial case (i.e., $d > 1$ and $n > 1$) where (17.103) is not available.

17.4.3 Γ-Convergence

We saw above various situations where the functional itself depends on a parameter. It is then worth studying convergence of such functionals. In the context of minimization, a prominent role is played by Γ-*convergence* introduced by De Giorgi [67], sometimes also called *variational convergence* or *epigraph convergence*, cf. also [68–70]. We say that the functional Φ is the Γ-limit of a sequence $\{\Phi_k\}_{k\in\mathbb{N}}$ if

$$\forall u_k \to u:\ \liminf_{k\to\infty} \Phi_k(u_k) \ge \Phi(u), \qquad (17.106a)$$

$$\forall\, u\in V\ \exists\,\{u_k\}_{k\in\mathbb{N}} \text{ with } u_k \to u:\ \limsup_{k\to\infty} \Phi_k(u_k) \le \Phi(u). \qquad (17.106b)$$

One interesting property justifying this mode of convergence is the following:

Theorem 17.15 (Γ-convergence.) *If $\Phi_k \to \Phi$ in the sense* (17.106) *and if u_k minimizes Φ_k, then any converging subsequence of $\{u_k\}_{k\in\mathbb{N}}$ yields, as its limit, a minimizer of Φ.*[53]

Identifying the Γ-limit (if it exists) in concrete cases can be very difficult. Few relatively simple examples were, in fact, already stated above.

53) The proof is simply by a contradiction, assuming that $\Phi(u) > \Phi(v)$ for $u = \lim_{l\to\infty} u_{k_l}$ and some $v\in V$ and using (17.106) to have a recovery sequence $v_k \to v$ so that $\Phi(v) = \liminf_{k\to\infty}\Phi_k(v_k) \ge \liminf_{k\to\infty}\Phi_k(u_k) \ge \Phi(u)$.

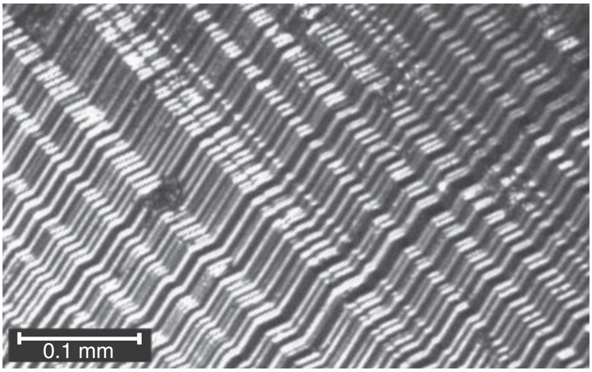

Figure 17.5 Oscillations of the deformation gradient minimizing Φ with a 6-well potential $\varphi(F)$: an orthorhombic martensite microstructure in a single-crystal of CuAlNi. (Courtesy H. Seiner, Inst. of Thermomechanics, ASCR.)

A simple example is the numerical approximation in Section 17.4.1 where we had the situation

$$V_k \subset V_{k+1} \subset V \quad \text{for } k\in\mathbb{N}, \text{ and} \tag{17.107a}$$

$$\Phi_k(v) = \begin{cases} \Phi(v) & \text{if } v\in V_k, \\ +\infty & \text{otherwise.} \end{cases} \tag{17.107b}$$

Let us further suppose that Φ is continuous with respect to the convergence used in (17.94). Then Φ_k Γ-converges to Φ.[54] Note that lower semicontinuity of Φ would not be sufficient for it, however.[55]

Another example of (17.106) with the weak topology we already saw is given by *singular perturbations*: the functionals $\Phi_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^d) \to \mathbb{R}\cup\{+\infty\}$ defined by (17.98) for $u \in W^{1,p}(\Omega;\mathbb{R}^d) \cap W^{2,2}(\Omega;\mathbb{R}^d)$ and by $+\infty$ for $u \in W^{1,p}(\Omega;\mathbb{R}^d) \setminus W^{2,2}(\Omega;\mathbb{R}^d)$ Γ-converge, for $\varepsilon \to 0$, to Φ from (17.52) if $\varphi(x,u,\cdot)$ is quasiconvex, otherwise one should use $\bar{\Phi}$ from (17.99). Alternatively, if Φ_ε is extended to the Young measures by $+\infty$ if the Young measure is not of the form $\{\delta_{\nabla u(x)}\}_{x\in\Omega}$ for some $u \in W^{1,p}(\Omega;\mathbb{R}^d) \cap W^{2,2}(\Omega;\mathbb{R}^d)$, one Γ-converges as $\varepsilon \to 0$ to the functional from (17.102).[56] Similarly, (17.87)–(17.90) Γ-converges to (17.100) if $\epsilon \to 0$.

Other prominent applications of Γ-convergence are dimensional reduction from three-dimensional problems to one-dimensional (springs, rods, beams) or two-dimensional (membranes, thin films, shells, plates), or homogenization of composite materials with periodic structure, cf. e.g. [48, 71].

Glossary

A lot of notions, definitions, and assertions are presented above. The following list tries to sort them according subjects or disciplines, giving the link to particular pages where the particular item is highlighted.

Topological notions:
Γ-convergence, p.581
continuous (weakly), p.552 (p.553)

54) Indeed, (17.106a) holds because $\Phi_k \geq \Phi_{k+1} \geq \Phi$ due to (17.107a). For any $\hat{v} \in V$, there is $\hat{v}_k \in V_k$ such that $\hat{v}_k \to \hat{v}$. Then $\lim_{k\to\infty} \Phi_k(\hat{v}_k) = \lim_{k\to\infty} \Phi(\hat{v}_k) = \Phi(\hat{v})$ and also $\lim_{k\to\infty} \hat{v}_k = v$ in V so that $\{\hat{v}_k\}_{k\in\mathbb{N}}$ is a recovery sequence for (17.106b).

55) A simple counterexample is $\Phi = +\infty$ everywhere except some $v \in V \setminus \cup_{k\in\mathbb{N}} V_k$; then $\Phi_k \equiv \infty$ obviously does not Γ-converge to Φ.

56) Again, (17.106a) is simply due to $\Phi_\varepsilon \geq \bar{\Phi}$ and $\bar{\Phi}$ is lower semicontinuous. The construction of particular recovery sequences for (17.106b) is more involved, smoothing the construction of a recovery sequence for $\varphi^\sharp$ or for the minimizing gradient Young measure as in [61, 2].

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compact set, p.553
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Smooth analysis:
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Fréchet subdifferential ∂_F, p.554
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Palais – Smale property, p.554
Plateau minimal-surface problem, p.567
Pontryagin maximum principle, p.585
relaxation, p.577
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Differential equations and inequalities:
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Carathéodory mapping, p.563
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Further Reading

The convex/smooth setting with one objective functional on which we primarily focused in Section 17.2 can be extensively generalized to nonconvex and nondifferentiable cases and/or to multi-objective situations, including dualization schemes, optimality conditions, sensitivity analysis, generalized equilibria, and many others, cf. e.g. [9, 12, 37, 72–74]. Many proof techniques are based on the remarkable *Ekeland variational principle* saying that, for a Gâteaux differentiable functional Φ bounded from below on a Banach space V, holds that

$$\begin{aligned} &\forall u \in V,\ \varepsilon > 0 : \quad \Phi(u) \le \inf \Phi + \varepsilon \\ &\Rightarrow \quad \exists v \in V : \quad \Phi(v) \le \Phi(u), \\ &\qquad \|v - u\| \le \sqrt{\varepsilon}, \quad \|\Phi'(v)\| \le \sqrt{\varepsilon}, \end{aligned}$$

See, for example, [11, 31, 37], in particular, also for a general formulation in metric spaces.

In concrete situations, solutions of variational problems often enjoy additional properties (typically, despite the counterexamples as [57, 58], some smoothness); there is an extensive literature in this direction of *regularity* of solutions, for example, [32, 50, 75].

There has been intensive effort leading to efficient and widely applicable methods to avoid the symmetry conditions (17.5), cf. also (17.57), based on the concept of monotonicity. Nonsymmetric nonlinear monotone-type operators (possibly generalized, for example, to pseudo-monotone operators or of the types (M) or (S), etc.) have been introduced on an abstract level in the work of Brézis [7], Minty [76], and others. Many monographs are available on this topic, also applied to concrete *nonsymmetric* quasilinear *equations* or inequalities, cf. e.g. [18, 50, 77, 78].

Even for situations conforming with the symmetry conditions of the type (17.57), Example 17.5 showed that sometimes variational methods even for linear boundary-value problem such as

$$-\mathrm{div}\nabla u = f \text{ on } \Omega, \quad \nabla u \cdot \vec{n} + u = g \text{ on } \Gamma,$$

are not compatible with natural physical demands that the right-hand sides f and g have an L^1-structure. This is why also *nonvariational methods* have been extensively developed. One method to handle general right-hand sides is Stampacchia's [79] transposition method, which has been analyzed for linear problems by Lions and Magenes [80]. Another general method is based on metric properties and contraction based on *accretivity* (instead of compactness and monotonicity) and, when applied to evolution problems, is connected with the theory of nonexpansive semigroups; from a very wide literature cf. e.g. the monographs by Showalter [78, Chapter 4], Vainberg [81, Chapter VII], or Zeidler [9, Chapter 57], or also [18, Chapter 3 and 9]. An estimation technique fitted with L^1-structure and applicable to thermal problems possibly coupled with mechanical or other physical systems, has been developed in [82], cf. also e.g. [18].

In fact, for $d = 1$ and $\Omega = [0, T]$, Section 17.3.2.1 dealt in particular with a very special *optimal control* problem of the *Bolza type*: minimize the objective $\int_0^T \varphi(t, u(t), v(t))\,\mathrm{d}t + \phi(T, u(T))$ for the initial-value problem for a simple (system of) ordinary differential equation(s) $\mathrm{d}u/\mathrm{d}t = v$, $u(0) = u_0$, with the *control* v being possibly subjected to a constraint $v(t) \in S$, with $t \in [0, T]$ playing the role of time. One can also think about generalization to (systems of) nonlinear ordinary differential equations of the type $\mathrm{d}u/\mathrm{d}t = f(t, u, v)$. If $\varphi(t, u, \cdot)$ is convex and $f(t, u, \cdot)$ is affine, one obtains existence of optimal control v and the corresponding response by the direct method as we did in Section 17.3.2.1. If fact, convexity of the so-called orientor field $Q(t, u) := \{(q_0, q_1);\ \exists\, s \in S(t):\ q_0 \geq \varphi(t, u, s),\ \ q_1 = f(t, u, s)\}$ is decisive for existence of optimal control. In the general case, the existence is not guaranteed and one can make a relaxation as we did in (17.100) obtaining the *relaxed optimal control* problem

$$\begin{aligned} &\text{minimize } = \int_0^T \int_S \varphi(t, u(t), s)\, v_t(\mathrm{d}s)\mathrm{d}t \\ &\text{subject to } \ \frac{\mathrm{d}u}{\mathrm{d}t} = \int_S f(t, u(t), s)\, v_t(\mathrm{d}s) \\ &\qquad\qquad v \in \mathscr{Y}([0, T]; S). \end{aligned} \tag{17.108}$$

The optimality conditions of the type (17.16) results in a modification of the Weierstrass maximum principle (17.103), namely,

$$\begin{aligned} &\int_S \mathfrak{H}_{u,\lambda^*}(t, s)\, v_t(\mathrm{d}s) = \max_{\widetilde{s} \in S} \mathfrak{H}_{u,\lambda^*}(t, \widetilde{s}) \quad \text{with} \\ &\mathfrak{H}_{u,\lambda^*}(t, s) = \lambda^*(t) \cdot f(t, u(t), s) - \varphi(t, u(t), s), \\ &\frac{\mathrm{d}\lambda^*}{\mathrm{d}x} + \int_S f_u'(t, u(t), s)^\top \lambda^*(t)\, v_t(\mathrm{d}s) \\ &\qquad = \int_S \varphi_u'(t, u(t), s)^\top\, v_t(\mathrm{d}s) \ \text{ on } [0, T], \\ &\lambda^*(T) = \phi_u'(T, u(T)). \end{aligned} \tag{17.109}$$

The linear terminal-value problem in (17.109) for λ^* is called the *adjoint system*, arising from the adjoint operator in (17.16). Of course, if (by chance) the optimal control v of the original problem exists, then the first condition in (17.109) reads as $\mathfrak{H}_{u,\lambda^*}(t, v(t)) = \max_{\widetilde{v} \in S} \mathfrak{H}_{u,\lambda^*}(t, \widetilde{v})$. Essentially, this has been formulated in [83, 84] and later become known as the *Pontryagin maximum principle*, here in terms of the so-called relaxed controls. We can see that it is a generalization of the Weierstrass principle and can be derived as a standard Karush–Kuhn–Tucker condition but with respect to the convex geometry induced from the space of relaxed controls.[57] One can also consider optimal control of partial differential equations instead of the ordinary ones, cf. also [59]. There is a huge literature about optimal control theory in all usual aspects of the calculus of variations as briefly presented above, cf. e.g. [73, 85, 86].

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57) The original and rather technical method was based on the so-called needle variations, however.

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