# 1
# The Fiber Bundle Model

Dear reader, if differential geometry is your field, please put this book back on the shelf. It is not for you. The fiber bundles that we deal with here are not spaces, but bundles of breakable fibers. Fibers that stretch and fail. They belong to the realm of engineers, physicists, and statisticians. They are models for how materials fail under duress.

Most materials do not consist of fibers. But materials are prone to failure under loading. Keeping material failure under control is one of the most important tasks of engineering. We need to be able to trust that our buildings will not collapse, our airplanes do not disintegrate in mid-air, our tankers do not rupture at sea …

Given the variety of materials and configurations they are used in, it seems a daunting task to attempt constructing a general theory of fracture and failure. Such a theory exists, however, and goes under the name of linear elastic fracture mechanics (*LEFM*) [1]. This has become a very refined theory over the years, and there is no doubt that it has been successful. Linear elastic fracture mechanics has as a starting point the theory of elasticity. This is a theory that treats materials as continuous, and as a result, linear elastic fracture mechanics is a top-down approach.

A completely different approach has come to life over the past couple of decades: atomistic modeling [2]. This approach hinges on the advent of the computer as a serious research tool. It is now possible to model materials with (fairly) realistic forces between the atoms in such quantities that it is possible to hook the results up with those approaches that start from a continuum description: top-down meets bottom-up.

Is there then any room for simplified fiber bundle models in the middle? Our answer is yes, and we will use the next couple of hundred pages or so to convince you, dear reader.

## 1.1
## Rivets Versus Welding

Here is a couple of examples of failures that seem to be opposite of each other in order to highlight the complexity of the central problem: how to ensure that structures do not fail.

*The Fiber Bundle Model: Modeling Failure in Materials,* First Edition.
Alex Hansen, Per C. Hemmer and Srutarshi Pradhan.

We zoom in on the failure of the hull of a Boeing 737 airplane during Aloha Airlines flight 243 on April 28, 1988, where a part of the fuselage of the airplane was ripped away mid-air; see Figure 1.1. Amazingly, the pilots were able to land the aircraft with 89 passengers and 6 crew members. The failure process had started long before as a small crack near a rivet due to metal fatigue initiated by crevice corrosion. The crack grew due to the cyclic pressure loading from flying and being on the ground. As the length of the crack grew, the stresses in front of it increased, and at some point, it became unstable, opening up the fuselage by moving from rivet to rivet in the way perforated paper fails. Clearly, understanding what happened and how it can be prevented from happening again belongs to the realm of engineering. However, the growth of the initial crack and how it went unstable are just as much a problem for fundamental science: what are the underlying mechanisms and how do they manifest themselves? In the AA flight 243 incident, the rivets played a crucial role.

The American *Liberty* cargo ships produced during World War II were the first ships that had hulls that were welded rather than riveted. Yet, 12 out of the 2710 ships built broke in half without warning. Cracks formed, grew slowly, went unnoticed, and at some point, they became unstable, breaking the ship apart, see Figure 1.2. By removing rivets, no mechanism was present that could lead to crack arrest. A growing crack in a car wind shield is effectively stopped by drilling a hole in front of it. The high stress at the crack tip, which drives the crack forward, is lowered as it is spread over the surface of the drilled hole when the crack reaches it. In the same way, rivets would stop growing cracks in the hull.

This contradicts the important role played by rivets in the AA flight 243 incident where rivets were the cause of the failure. Here, the lack of rivets was the reason for the failure.

Are there fundamental and general principles at work here that can explain the difference between the two incidents? The answer is yes. But, to be able to understand these principles, we need to simplify the problem. We need *models.*

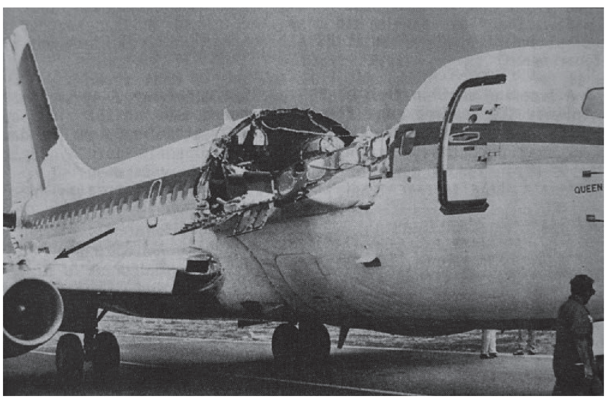


**Figure 1.1** The Boeing 737 after the explosive decompression that occurred during flight on April 28, 1988, in Hawaii. (Photo credit: National Transportation Safety Board)


**Figure 1.2** The Schenectady after it broke into two on January 16, 1943, in dock in Portland, Oregon. The ship had just been finished and was being outfitted. The failure was sudden and unexpected.

It is here that the fiber bundle models enter. They are models that simplify the problem of failure to the point where the very powerful methods of theoretical physics, statistics, and mathematics may be fully explored. They help us understand the subtle interplay between forces and strength that control the failure process. They help us understand what is generally present in all failure processes and what is specific for a given failure process.

### 1.1.1
### What Are Models Good For?

Since the use of models is sometimes viewed with some skepticism by the engineering community, we elaborate some more on what precisely is a model.

Fundamental sciences, and physics in particular, approach Nature in a *hierarchical way* [3]: more general questions are posed and answered before more specific questions. We may illustrate this by the following example: in the 1920s, general quantum mechanics was developed. In the 1930s, a general theory of metals was developed. This allowed for studying specific metals, but it also opened up for the search for a class of materials that were between metals and insulators: semiconductors. In the 1940s, this resulted in the construction of the first transistor – and the electronics age was born. One may only speculate how long it would have taken to construct the transistor if this path from the more general to the more specific had not been followed. How long would it take before someone accidentally stumbled across semiconductors?

This hierarchical approach lies behind the extensive use of models in theoretical physics. The fiber bundle model is a good example of the use of physical models to study the physical phenomena of interest with the minimum of ingredients needed: these models are stripped of any irrelevant contents. In fact, the models, and the approach of physics to science, are related to Occam's dictum: *Numquam ponenda est pluralitas sine necessitate* [plurality must never be posited without necessity] [4].[1)]

Still, the fiber bundle models have proved to be very effective in *practical* applications such as fiber-reinforced composites. In this context, the models have a history that goes back to the 1920s [6], and they constitute today an elaborate toolbox for studying such materials, rendering computer studies orders of magnitudes more efficient than brute force methods. Since the late 1980s [7], these models have received increasing attention in the physics community due to their deceivingly simple appearance coupled with an extraordinary richness of behaviors. As these models are just at the edge of what is possible analytically and typically not being very challenging from a numerical point of view so that extremely good statistics on large systems are available, they are perfect as model systems for studying failure processes as a part of theoretical physics.

## 1.2
## Fracture and Failure: A Short Summary

Fracture and material stability have for practical reasons interested humanity ever since we started using tools: our pottery should be able to withstand handling, our huts should be able to withstand normal weather. As science took on the form we know today during the Renaissance, Leonardo da Vinci studied 500 years ago experimentally the strength of wires – fiber bundles – as a function of their length [8]. Systematic strength studies, but on beams, were also pursued by Galileo Galilei 100 years later, as was done by Edme Mariotte (of gas law fame), who pressurized vessels until they burst in connection with the construction of a fountain at Versailles. For some reason, mainstream physics moved away from fracture and breakdown problems in the nineteenth century, and it is only during the last 20 years that fracture problems have been studied within physics proper. The reason for this is most probably the advent of computers as a research tool, rendering problems that were beyond the reach of systematic theoretical study now accessible.

If we were to single out the most important modern contribution from the physics community with respect to fracture phenomena, it must be the focus on *fluctuations* rather than averages. What good is the knowledge of the average behavior of a system when faced with a single sample and this being liable to breakdown given the right fluctuation? This book, being written by physicists, reflects this point of view, and hence, fluctuations play an important role throughout it.

1) Einstein is often quoted as having stated "Everything should be as simple as possible, but not simpler." This is of course the Occam razor [5].

## 1.3
## The Fiber Bundle Model in Statistics

Even though we may trace the study of fiber bundles to Leonardo da Vinci, their modern story starts with the already mentioned work by Peirce [6]. In 1945, Daniels published a seminal review-cum-research article on fiber bundles, which still today must be regarded as essential reading in the field [9]. In this paper, the fiber bundle model is treated as a problem of statistics and the analysis is performed within this framework, rather than treating it within materials science. The fiber bundle is viewed as a collection of elastic objects connected in parallel and clamped to a medium that transmits forces between the fibers. The elongation of a fiber is linearly related to the force it carries up to a maximum value. When this value is reached, the fiber fails by no longer being able to carry any force. The threshold value is assigned from some initially chosen probability distribution and does not change thereafter. When the fiber fails, the force it carried is redistributed. If the clamps deform under loading, fibers closer to the just-failed fiber will absorb more of the force compared to those further away. If the clamps, on the other hand, are rigid, the force is equally distributed to all the surviving fibers. Daniels discussed this latter case. A typical question posed and answered in this paper would be the average strength of a bundle of $N$ fibers, but also the variance of the average strength of the entire bundle. This book takes the same point of view, discussing the fiber bundle model as a *statistical model.*

**Fredrick Thomas Peirce**



Fredrick Thomas Peirce was born in 1896 in Southport, Australia, where his father was a minister. He was awarded the B.Sc. degree by the University of Sidney at the age of just 19. During World War I, he served in the Australian Army Signal Service and was severely wounded in Palestine.

After the war, Peirce went to England and studied X-ray crystallography under the Nobel laureate W. H. Bragg, at University College, London, as well as other topics in physics and chemistry.

In 1921, he joined the Physics Department of the British Cotton Industry Research Association in Manchester. His background enabled him to apply the principles of physics to the study of textile structures. When a new Testing Department was created in 1924, Peirce was chosen as its head.

In the beginning, his research was centered on fundamental physical properties of cotton fibers and yarns: rigidity, plasticity, and tensile behavior. His article *Theorems of the strength of long and of composite specimens* was probably the first scientific treatment of fiber bundles [6]. Here he developed a *weak link* theory, which dealt with the variation in bundle strength due to the probability of random weak spots.

After 1924, his research focused on the influence of humidity on textiles: how to prevent water from entering from the outside while allowing it to escape from the inside. It has been suggested that this was triggered by his change of environment from the dry Australia to the damp Manchester.

During World War II, he and his staff of 60 were devoted to meeting military requirements, as textiles for parachutes, and for arctic and tropic clothing. In 1944, his expertise was requested in the United States, and he accepted an invitation to become Director of Textile Research in the School of Textiles at North Carolina State College. However, after a stroke, he resigned and moved back to Australia, where he died at an early age of 53.

## 1.4 The Fiber Bundle Model in Physics

Theoretical physics has changed quite profoundly over the last three decades. This coincides with the computer coming of age. We are not there yet, but computational physics is rapidly establishing itself as a third way of doing physics on equal footing with experimental and theoretical physics. The power of the modern computer, being in the form of a huge machine such as the Japanese K-computer consisting of 800 racks or in the form of GPUs – Graphic Processing Units – which allows enormous power on the desktop thanks to the computer game industry, allows for handling problems that would be forbidding even to think of in earlier times.

When looking back at the history of fracture and failure, one realizes that the early giants, for example, Galilei, struggled with such problems. However, in the eighteenth century physics moved away from such problems. They were, we suspect, too dirty (i.e., practical) to touch for pure physics. Perhaps substituting the term "dirty" by "difficult" is closer to the truth, however. The divorce between physics and the science of fracture and failure lasted until the 1980s when the statistical physics community – still quite elated after the tremendous successes in connection with critical phenomena – started considering such problems. In these early days of the renewed interest in fracture, it was the fuse model that got all the attention. This numerical model, which we will consider in Chapter 6, simulates a network of electrical fuses, each having a current threshold drawn from some probability distribution. As the current is increased through the network, how do fuses burn out? In the beginning of the failure process, fuses burn out because they have small thresholds. However, as the failure process proceeds, the current distribution in the network evolves and fuses carrying high currents appear. Hence, there are now fuses that fail not because they are weak, but because the currents they are carrying are high. A competition between these two effects starts and a rich variety of different effects may be studied. The problem with the fuse model is that it is very difficult to get any hard – mathematically derived – results. This model is a numerical model only.

The fiber bundle model in its more sophisticated version – the local-load-sharing fiber bundle model (see Chapter 4) – shows the same competition between stress enhancement and weak fibers as the fuse model does [10]. The distribution of fiber strengths makes the cracks – missing fibers – repulsive with respect to each other. This is easy to understand. If we sit on a failed fiber and look for weak fibers around it, the further away we look, the weaker the weakest fiber we find will be. On the other hand, the stress enhancement due to failure will be the highest near the failed fiber. Hence, if fibers fail for this reason, they will be near the already-failed fiber. Hence, stress enhancement makes cracks – failed fibers – attractive.

Referring back to Section 1.1, we discussed two accidents: the explosive compression of an airplane due to fractures evolving from the rivets holding the hull together and the breaking up of a tanker as a result of there being no rivets in the hull. These two accidents seem in some sense opposite of each other. However, in light of the competition between weak thresholds and stress enhancement, we may understand the principle behind the two accidents. In the case of the rivets in the airplane hull, they were weak spots where failure would occur. Cracks grew from these weak spots and at some point (April 28, 1988) the stress enhancement took control with the result that the crack growth took off. The same principle happened in fact in connection with the tanker. Even though there were no rivets, the hull would not be completely uniform and microcracks would appear. However, the competition would soon be won by the stress enhancement at the crack tips.

To our knowledge, the first use of the *fiber bundle model* in physics came in 1989 with Sornette as author [7]. In contrast to the fuse model, the fiber bundle model offers a fine balance between being analytically tractable and numerically amenable. In its simplest form, the equal-load-sharing fiber bundle model, essentially everything may be calculated. In more advanced models, for example, the local-load-sharing fiber bundle model (see Chapter 4), some quantities may be calculated analytically in these models, but others may not.

The authors of this book are physicists. This book reflects this fact. However, one of the principal tools that statistical physicists use is statistics. We will use statistics extensively in this book. It is only in the choice of which subjects to emphasize that our background will shine through.

## 1.5
## The Fiber Bundle Model in Materials Science

The fiber bundle model has so far been presented here as a general model rather than as a tool that can be used in engineering. The fiber bundle model does have a place in engineering. Perhaps not surprisingly, it has carved out a fairly sizable field in fiber reinforced composites [11]. We will not go into detail in this field, but just point out that this endeavor began in 1952 with the Cox shear-lag model [12]. It then passed through different stages of development, for example, the Hedgepeth model [13], until now being a mature model that is used for strength calculations in an engineering setting; see, for example, [14].

In the fiber bundle model, the material failure occurs when the fibers are stretched and some may possibly fail. A similar model, in which, however, failures occur under compression, is the pillar model [15, 16]. In this, two solid horizontal planes are supported by a set of pillars. The pillars have statistically distributed thresholds for failure under compression. In their simplest versions, there is a one-to-one correspondence between the pillar and the fiber bundle models. In both models, one is free to specify how stress is redistributed around a failure. The pillar model has been used for describing several aspects of rock failure under compression.

In Chapter 7, we will say a few more words about fiber-reinforced composites and revisit the pillar model.

## 1.6
## Structure of the Book

This monograph summarizes the authors' current knowledge of the fiber bundle model. It is written from the perspective of theoretical physics. Chapter 2 introduces the equal-load-sharing fiber bundle model and takes the reader through the "classical" results concerning its average properties. Chapter 3 focuses on fluctuations and here important concepts such as avalanches are introduced. Chapter 4 introduces the local-load-sharing model as the opposite limit of the equal-load-sharing model. Whereas the latter distributes the forces carried by the failed fibers equally among all remaining fibers, the local – load-sharing model distributes the forces to the nearest surviving fibers. Also, more sophisticated models such as the soft clamp model are discussed here. Chapter 5 returns to the average properties of the equal-load-sharing fiber bundle, but now as an iterative system. The theory of iterative maps is used to demonstrate the relation between the equal-load-sharing fiber bundle model and critical phenomena. Chapter 6 discusses the possibility to predict the point at which the fiber bundle model collapses under load, based on the theory developed in Chapter 3 on avalanches and on the iterative theory in Chapter 5. In Chapter 7, we discuss the use of fiber bundle models in connection with important phenomena such as creep, fatigue and crushing in addition to considering the influence of thermal noise on the fiber bundle. Finally, in Chapter 8, we discuss briefly the use of fiber bundle models in geophysics, particularly in connection with snow and landslide avalanches.