1 The MHD Equations

1.1 Derivation of the MHD Equations

In this book, we will treat the description of equilibrium and stability properties of magnetically confined fusion plasmas in the framework of a fluid theory, the so-called Magnetohydrodynamic (MHD) theory. In this chapter, we are going to derive the MHD equations and discuss some of their basic properties and the limitations for application of MHD to the description of fusion plasmas. The derivation follows the treatment given in [1]. For a more in-depth discussion of the MHD equations, the reader is referred to [2]. Non-linear aspects of MHD are treated in [3]. A good overview of general tokamak physics can be found in [4].

1.1.1 Multispecies MHD Equations

As a magnetized plasma is a many-body system, its description cannot be done by solving individual equations of motion that would typically be a set of, say, 10^{20} equations[1] that are all coupled through the electromagnetic interaction. Hence, some kind of mean field theory is needed.

Starting point of our derivation is the kinetic equation known from statistical physics. It describes the many-body system in terms of a distribution function f_α in six-dimensional space $d^3x d^3v$, where

$$f_\alpha(\mathbf{x}, \mathbf{v}, t)\ d^3x\ d^3v \tag{1.1}$$

is the probability to find a particle of species α at $\mathbf{x}$ with velocity $\mathbf{v}$ at time t. Here, $\mathbf{x}$ and $\mathbf{v}$ are independent variables that, in the sense of classical mechanics, fully describe the system.

The basic assumption of kinetic theory is that fields and forces are macroscopic in the sense that they have already been averaged over a volume containing many particles (say, a Debye-sphere[2]) and the microscopic fields and forces at the exact

1) Here, we think of a typical fusion plasma of density 10^{20} particles per cubic metre.
2) The Debye length λ_D is the typical distance on which the electric field in a plasma is shielded so that its action is limited to a sphere of radius λ_D.

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particle location can be expressed through a collision term giving rise to a change of f_α along the particle trajectories in six-dimensional space. We note that this has reduced the microscopic problem of the 10^{20} interactions to the proper choice of the collision term.

Evaluating the total change of f_α along the trajectories and keeping in mind that along these, $d\mathbf{x}/dt = \mathbf{v}$ and $d\mathbf{v}/dt = \mathbf{F}_\alpha/m_\alpha$, where $\mathbf{F}_\alpha$ is the force acting on the particle and m_α its mass, the kinetic equation can be expressed as

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\mathbf{\nabla} f_\alpha + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\text{coll}} \tag{1.2}$$

where we have assumed that the only relevant force is the Lorentz force and hence explicitly neglected gravity (which is a good approximation for magnetically confined fusion plasmas, but generally not true in Astrophysical applications). According to the above-mentioned description of mean field theory, the fields $\mathbf{E}$ and $\mathbf{B}$ will have to be calculated from Maxwells equations using the charge density and current resulting from appropriate averaging over the distribution function in velocity space as will be described in the following.

The kinetic equation is used to describe phenomena that arise from f_α not being a Maxwellian, which is the particle distribution in thermodynamic equilibrium to which the system will relax through the action of collisions. In fusion plasmas, this frequently occurs as the mean free path is often large compared to the system length as is for example the case for turbulence dynamics in a tokamak along field lines. Another important example is when the relevant timescales are short compared to the collision time, such as in RF (radio frequency) wave heating and current drive that can occur by Landau damping rather than collisional dissipation. Here, a description using the Vlasov or Fokker–Planck equation is needed.

However, in situations where f_α is close to Maxwellian, one can average the kinetic equation over velocity space to obtain hydrodynamic equations in configuration space. When doing so, one encounters so-called moments of f_α. The kth moment, which is related to the velocity average of $\mathbf{v}^k$, is given by

$$\int \mathbf{v}^k f_\alpha d^3v = n_\alpha\langle\mathbf{v}^k\rangle \tag{1.3}$$

These moments are related to the hydrodynamic quantities used to describe the plasma in configuration space. For the zeroth moment, we obtain

$$n_\alpha(\mathbf{x}, t) = \int f_\alpha(\mathbf{x}, \mathbf{v}, t)d^3v \tag{1.4}$$

which is the number density in real space. The first moment of f_α is related to the fluid velocity in the centre of mass frame by

$$\mathbf{u}_\alpha(\mathbf{x}, t) = \frac{1}{n_\alpha}\int \mathbf{v}f_\alpha(\mathbf{x}, \mathbf{v}, t)d^3v \tag{1.5}$$

For the second moment, it is of advantage to separate the particle velocity into the fluid velocity and the random thermal motion $\mathbf{w}$ according to

$$\mathbf{v} = \mathbf{u}_\alpha + \mathbf{w} \tag{1.6}$$

It is easy to show that $\langle \mathbf{w} \rangle = 0$, as expected for thermal motion, as

$$\langle \mathbf{w} \rangle = \langle \mathbf{v} \rangle - \langle \mathbf{u}_\alpha \rangle = \mathbf{u}_\alpha - \mathbf{u}_\alpha = 0 \tag{1.7}$$

However, the quadratic average is non-zero, representing the thermal energy via

$$\frac{1}{2} m_\alpha \int \mathbf{w}^2 f_\alpha d^3 v = \frac{3}{2} n_\alpha k_B T_\alpha = \frac{3}{2} p_\alpha \tag{1.8}$$

where k_B is the Boltzmann constant and we have used the definition of the thermal energy density and its relation to the pressure p_α for an ideal plasma. We note that this definition relies on the previous assumption that f_α is close to Maxwellian. More generally, the second moment is defined as a tensor of rank 2, the pressure tensor

$$\mathbf{P}_\alpha = m_\alpha \int \mathbf{w} \otimes \mathbf{w} f_\alpha d^3 v, \tag{1.9}$$

where $\otimes$ denotes the dyadic product. The non-diagonal terms of this tensor are related to viscosity, whereas from Eq. (1.8), it is clear that the trace of $\mathbf{P}_\alpha$ is equal to $3p_\alpha$, that is for an isotropic system, the diagonal elements of $\mathbf{P}_\alpha$ are just equal to the scalar pressure. Therefore, the pressure tensor is also often written as

$$\mathbf{P}_\alpha = p_\alpha 1 + \Pi_\alpha, \tag{1.10}$$

where 1 is the unit tensor and Π_α the anisotropic part of $\mathbf{P}_\alpha$.

We now integrate the kinetic equation (Eq. 1.2) over velocity space[3] to obtain

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0 \tag{1.11}$$

which is the equation of continuity for species α. Here, we have assumed that the velocity space average of the collision term is zero, meaning that the total number of particles is conserved for each species. Should this not be the case (e.g. by ionization or fusion), the right-hand side would consist of a source term $S(\mathbf{x}, t)$.

The next moment is obtained by multiplying the kinetic equation by $\mathbf{v}$ and integrating over velocity space. This yields the momentum balance

$$\begin{aligned} m_\alpha \frac{\partial (n_\alpha \mathbf{u}_\alpha)}{\partial t} &= -\nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha + \mathbf{P}_\alpha) \\ &\quad + n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \mathbf{R}_{\alpha\beta}, \end{aligned} \tag{1.12}$$

where the friction force $\mathbf{R}_{\alpha\beta}$ is the first moment of the collision term for collisions with species β. We note that only collision with unlike particles lead to a net friction force while collisions within one species, which are important for thermalization, do not transfer net momentum to that species. This form is also called the *conservative* form as, like the equation of continuity, it relates the temporal derivative of a quantity (in this case, the momentum) to the divergence of a flux.

3) When integrating over velocity space, it is useful to remember that t, $\mathbf{x}$ and $\mathbf{v}$ are independent so that the derivative with respect to t and x can be taken out of the integral. In addition, terms containing a v derivative are integrated partially and the surface term vanishes as $f_\alpha \to 0$ faster than any power of v for $v \to \infty$.

However, this equation can be rearranged using the continuity equation into a form in which the dyadic product of the velocity can be absorbed in the derivative on the left-hand side:

$$m_\alpha n_\alpha \left(\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right) = -\nabla \cdot \mathbf{P}_\alpha + n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \mathbf{R}_{\alpha\beta}, \tag{1.13}$$

which is usually called the *force balance*. Here, the operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \tag{1.14}$$

is called the *substantial* or *convective derivative* and measures the change along the trajectory of a fluid element in the laboratory frame. In ordinary hydrodynamics, Eq. (1.13) is called the *Euler equation* while equations in the co-moving frame are referred to as the *Lagrange description*.

The system of equations so far is not closed as a second moment appears in the first moment equation, just as the velocity as first-order moment occurs in the zeroth-order continuity equation. It is clear that this problem cannot be solved by adding the second moment of the kinetic equation as a third moment will appear. This is the closure problem of MHD, where at each step, an additional relation will be required to close the system. If we want to stop here, we obviously need a relation for the pressure, that is an equation of state. This could be the adiabatic equation

$$\frac{d}{dt} \left(\frac{p_\alpha}{{\rho_\alpha}^{\gamma_\alpha}} \right) = 0, \tag{1.15}$$

where γ_α is the adiabatic coefficient and we have assumed that we only deal with the scalar pressure in Eq. (1.13). Together with Maxwell's equations for the fields **E** and **B**, we now have indeed a closed system. However, we will still simplify this system for a two-component plasma in Section 1.1.2.

1.1.2
One-Fluid Model of Magnetohydrodynamics

For the case of a two-component plasma consisting of one ion species and electrons, the system of two-fluid equations can be combined to give a set of one-fluid equations. Here, owing to the large mass difference between the two species, the mass and momentum are more or less contained in the ions, whereas the electrons guarantee quasineutrality and lead to an electrical current if their velocity is different from that of the ions. In the following, we will assume a hydrogen plasma, that is charge number $Z = 1$. Specifically, the one-fluid variables are the mass density

$$\rho = n_i m_i + n_e m_e \approx n m_i \tag{1.16}$$

where we have used charge neutrality ($n_e = n_i = n$), the centre of mass fluid velocity

$$\mathbf{v} = \frac{1}{\rho}(m_i n_i \mathbf{u}_i + n_e m_e \mathbf{u}_e) \approx \mathbf{u}_i \tag{1.17}$$

and the electrical current density

$$\mathbf{j} = en_i\mathbf{u}_i - en_e\mathbf{u}_e = en(\mathbf{u}_i - \mathbf{u}_e). \tag{1.18}$$

The one-fluid equations are obtained by adding or subtracting the continuity and force balance equations for the individual species and expressing them in the one-fluid variables, neglecting terms of the order m_e/m_i. In this process, addition will give a one-fluid equation for the velocity, whereas the subtraction will yield one for the current density.

Adding the continuity equations yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1.19}$$

that is a one-fluid continuity equation while subtracting them leads to

$$\frac{\partial \rho_{\mathrm{el}}}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{1.20}$$

which is the continuity equation for the electrical current. As we assume the plasma to be quasi-neutral, the electrical charge density $\rho_{\mathrm{el}} = en_i - en_e$ vanishes and the equation just reads $\nabla \cdot \mathbf{j} = 0$.

Adding the force equations leads to

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla \cdot \mathbf{P} + \mathbf{j} \times \mathbf{B} \tag{1.21}$$

the Euler equation, where $\mathbf{P} = \mathbf{P_i} + \mathbf{P_e}$ as in an ideal plasma, the total pressure is the sum of the partial pressures of the individual species. As pointed out earlier, the fluid velocity is mainly the ion velocity. To determine the role that the electrons play, we can re-write the electron equation of motion in terms of the one-fluid velocity $\mathbf{v}$ to obtain

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{\sigma}\mathbf{j} + \frac{1}{en_e}(\mathbf{j} \times \mathbf{B} - \nabla p_e) - \frac{m_e}{e}\frac{d\mathbf{u}_e}{dt} \tag{1.22}$$

which is Ohm's law for a plasma. One can see that here, not all two-fluid variables could be eliminated from the equation through $m_e \ll m_i$. However, we will argue in the following that the last two terms are usually small for our applications and can be neglected so that this problem will not appear in what follows.

Assuming that we will deal with the scalar pressure only, we can use the adiabaticity equation

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0. \tag{1.23}$$

In ordinary hydrodynamics, neglecting the viscous part of the pressure tensor corresponds to infinite Reynolds number and hence the use of the Euler instead of the Navier Stokes equation that rules out a proper description of fluid turbulence. However, if we keep finite conductivity in Ohm's law, there is still dissipation in the system and the relevant dimensionless number becomes the magnetic Reynolds number (Chapter 8).

Finally, we use Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.24}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{1.25}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.26}$$

and we have a closed system of equations to describe a plasma as a single fluid. We note that we have neglected the polarization current in Ampere's law, eliminating phenomena with (phase) velocity close to that of the speed of light, such as electromagnetic waves that arise from this term. In addition, we do not need to solve explicitly an equation for $\nabla \cdot \mathbf{E}$ in the quasi-neutral plasma. Finally, counting the number of variables and equations reveals that one scalar equation seems obsolete; this is related to the fact that any solution that satisfies $\nabla \cdot \mathbf{B}$ in the beginning will always do so and hence this is rather a boundary condition than a separate equation.

1.1.3
Validity of the One-Fluid Model of Magnetohydrodynamics

The system of equations derived earlier relies on a number of assumptions that have been made during the derivation. Here, we briefly review them and point out the restrictions arising.

- By assuming that we can use a continuum description, we think of the plasma described as fluid elements that are infinitesimally small such that individual particles are not distinguished. This means that the typical 'extension' of the particle orbit, that is the Larmor radius r_L, is small compared to a typical system length L:

  $$r_{Li} = \frac{\sqrt{m_i k T_i}}{eB} \ll L. \tag{1.27}$$

  This is also known as the condition for a *magnetized plasma* and is usually very well fulfilled in the fusion plasmas under study here where typical ion Larmor radii are of the order of millimetres and the electron Larmor radius is even smaller by a factor $\sqrt{m_e/m_i}$, which is the reason why we have used the ion Larmor radius earlier. For the MHD instabilities treated in this book, it is important to remember that the validity of our results will break down for very small scales, and finite Larmor radius (FLR) effects set the limit to the applicability in the limit $L \to 0$.
- Defining a local temperature requires that f_α is close to a Maxwellian. This relies on considering timescales that are long compared to the collision time

  $$\tau_{coll} \sim T^{3/2}/n \ll \tau$$

  or, in terms of spatial scales, the mean free path λ_{mfp} being small compared to the system length:

  $$\lambda_{\mathrm{mfp}} \sim T^2/n \ll L.$$

These conditions are usually not well fulfilled in typical hot fusion plasmas, at least parallel to the magnetic field, as typical values for λ_{mfp} can easily reach the order of kilometres while L is usually of the order of metres such that the particles pass through the system many times before equilibrating. Hence, a kinetic description is often needed to describe the dynamics along field lines, as is the case for example in gyrokinetic description of turbulence. On the other hand, perpendicular to the field, the typical mean free path is the Larmor radius and the validity condition is fulfilled if Eq. (1.27) is fulfilled.

- In Ohm's law (Eq. (1.22)), the ratio of the 'Hall term' $(\mathbf{j} \times \mathbf{B} - \nabla p_e)/(en_e)$ to the term $\mathbf{v} \times \mathbf{B}$ can be estimated using the force balance $\mathbf{j} \times \mathbf{B} - \nabla p_e \approx \nabla p_i \approx p_i/L$ to be small if the typical velocity v fulfills the condition $v \gg r_{Li}/Lv_{\text{th},i}$. As we have already assumed $r_{Li}/L \ll 1$ (Eq. (1.27)), this condition is usually well fulfilled for the fast (v of the order of $v_{\text{th},i}$) MHD phenomena treated in our context. However, it breaks down for phenomena that are slow enough that the difference between ion and electron fluid matters, such as drift waves. Then, two-fluid theory will have to be used and diamagnetic effects will become important.
- Applying a similar argument to the term $(m_e/e)(d\mathbf{u}_e/dt)$ shows that it can be neglected whenever the typical length scale L is long compared to the electron Larmor radius $r_{L,e}$, which, as pointed out earlier, is fulfilled whenever Eq. (1.27) is fulfilled.
- Finally, while the remaining terms in Ohm's law are already true one-fluid terms, another important simplification can often be made considering that a hot plasma is a very good electrical conductor, meaning that the typical timescale for current diffusion

  $$\tau_R = L^2 \mu_0 \sigma \sim L^2 T^{3/2} \gg \tau \tag{1.28}$$

  can be of the order of seconds while typical MHD instabilities grow much faster. Hence, one can often also neglect the finite conductivity effects in Eq. (1.22), which formally corresponds to the limit $\sigma \to \infty$. The resulting Ohm's law

  $$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \tag{1.29}$$

  is the fundamental ingredient of what is called *ideal MHD*, and its consequences will be discussed in Section 1.1.2. We will come back to finite resistivity in Chapter 8.

We mention here that for toroidal confinement systems, dimensionless variables are often used for scaling arguments in a way comparable to the wind tunnel approach in ordinary hydrodynamics. It can be shown that, if the plasma can be assumed to be quasi-neutral[4)], a consistent set of dimensionless variables is given by normalizing energy, length and time scales as follows: the kinetic plasma energy can be normalized by the magnetic field energy, a parameter known as β (Eq. (1.46)). For a typical length scale, we define the ratio between Larmor radius and system length as $\rho^* = r_L/L$. Finally, if we want to relate the time to the time between collisions, a convenient definition for a toroidal system is the so-called

4) This corresponds to the assumption of small Debye length, $\lambda_D/L \to 0$.

collisionality ν^*, which is the inverse ratio of the above-mentioned collision time to the time a particle takes to complete a characteristic orbit in a toroidal confinement system, the so-called banana transit time[5]. According to the discussion earlier, MHD corresponds to the limits $\rho^* \to 0$ and $\nu^* \to \infty$, with the caveat that in ideal MHD, the electrical conductivity is still high enough that the condition described by Eq. (1.28) holds. In this case, the typical time scale is not given by the collision time, but rather by the inertia of the plasma. This so-called Alfvén timescale is discussed in Section 1.2.

1.2 Consequences of the MHD Equations

The system of equations derived earlier describes a magnetized plasma as a one-component fluid. Before applying the equations to specific magnetic confinement schemes, we will point out some general consequences arising from the equations.

1.2.1 Magnetic Flux Conservation

An important consequence of the ideal Ohm's law is the fact that magnetic flux is conserved when moving with the plasma. To prove this statement, we consider a contour C moving through the plasma with velocity $\mathbf{u_C}$. The geometry is shown in Figure 1.1a. The change of magnetic flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$ through this contour is given by

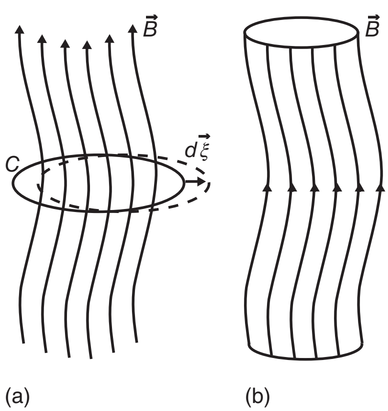

Figure 1.1 Geometry used for the derivation of flux conservation (a) and visualization of the concept of flux tubes (b).

5) In a toroidal system, there is a population of particles that bounce back and forth in the magnetic mirror created by the inhomogeneous B-field.

$$\frac{d\Psi}{dt} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint \mathbf{u_C} \times \mathbf{B} \mathbf{d}\ell, \tag{1.30}$$

where the first term accounts for explicit change of $\mathbf{B}$ with time and the second term comes from the fact that the change of surface of C can be calculated by integrating the vector product of the curve's tangent $\mathbf{d}\ell$ and the displacement $d\boldsymbol{\xi}$ (where $d\boldsymbol{\xi}/dt = \mathbf{u_C}$) along the contour C.

Using Faraday's law (Eq. (1.26)) in combination with Ohm's law (Eq. (1.22)), in the first integral of Eq. (1.30) and applying Stokes' theorem, leads to

$$\frac{d\Psi}{dt} = \oint \left(\mathbf{v} - \frac{\mathbf{j}}{en_e} - \mathbf{u_C} \right) \times \mathbf{B} \cdot d\ell \tag{1.31}$$

where we have assumed the plasma to be an ideal conductor, $\sigma \to \infty$. In deriving Eq. (1.31), we have made use of the fact that the curl of a gradient vanishes, eliminating the $\nabla p_e/(n_e e)$ term which is true as long as $\nabla p_e/n_e = \nabla(p_e/n_e)$, that is for polytropic equations of state[6]. This means that the magnetic flux is constant if the contour is moved with the electron velocity[7]

$$\mathbf{u}_C = \mathbf{v} - \frac{\mathbf{j}}{en_e} = \mathbf{v}_e \tag{1.32}$$

Consequently, one can imagine the plasma consisting of 'flux tubes' that are small cylinders with their mantle defined by the magnetic field lines as depicted in Figure 1.1b. As the flux is constant in these cylinders, the flux tubes are convected with v_e. In some sense, magnetic field lines thus become real objects in an ideal plasma. Sometimes, the field lines are also said to be 'frozen into the plasma'[8]. An important consequence for the motion of flux tubes in ideal MHD is that they cannot intersect as this would change the flux within the individual tubes. Hence, the motion is constrained to not change the topology of the flux tubes. Such topological changes are only possible invoking additional terms in Ohms law outside of the ideal MHD model, such as finite resistivity or finite electron inertia. Resistive effects are treated in Chapter 8.

For the MHD instabilities treated in this book, it means that they are expected to move with v_e, which has to be determined from the force balance equation of the electrons and will in general consist of a combination of fluid velocity and diamagnetic velocity as we have

$$\mathbf{v}_{e,\perp} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\nabla p_e \times \mathbf{B}}{en_e B^2} = \mathbf{v}_{E\times B} + \mathbf{v}_{e,\mathrm{dia}} \tag{1.33}$$

6) Note that this argument does in general not hold for the $\mathbf{j} \times \mathbf{B}$ term as it is only equal to ∇p in stationary force equilibrium while here, we are concerned with dynamic changes of the equilibrium.
7) We note that using the ideal Ohm's law (Eq. 1.29), the field lines are found to be frozen into the fluid velocity rather than the electron velocity, since then, no current, that is no difference between electron and ion velocity is considered.
8) This terminology is slightly misleading as the effect is due to the high electrical conductivity that comes from the plasma being quite hot!

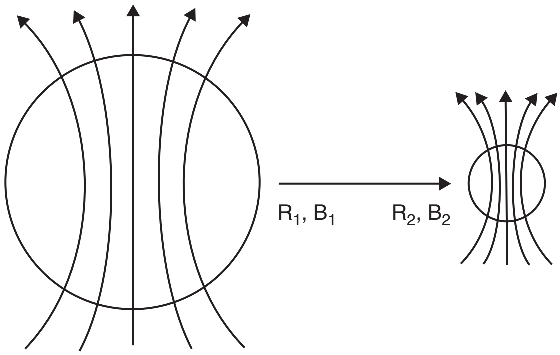

Figure 1.2 Schematic visualization of the consequence of flux conservation for the collapse of a neutron star: owing to the decrease of the equatorial surface, the magnetic field has to increase accordingly (remember that the density of field lines is representative for the field strength).

which follows from the vector product of Eq. (1.13) for the electrons with $\mathbf{B}$. Note that $\mathbf{v}_{E\times B}$ is the same for each species α and hence, the frame where $\mathbf{E} = 0$ is often referred to as the *rest frame of the plasma*.

An interesting illustration of the consequences of flux conservation is the collapse of a star of mass exceeding the critical mass for formation of a neutron star: during this process, the radius changes from $R_1 \approx 10^6$ to $R_2 \approx 10$ km while conserving magnetic flux, schematically shown in Figure 1.2. If we assume that we start with a dipole field of 10^{-5} T (roughly the Earth's magnetic field), we arrive at a final field of 10^5 T as flux conservation yields $B_2 = B_1(R_1/R_2)^2$. Such enormous magnetic field strengths can indeed be inferred from measuring the electromagnetic radiation coming from neutron stars, giving proof of the applicability of ideal MHD even under these extreme conditions.

1.2.2
MHD Equilibrium

A common application is the calculation of an equilibrium configuration using the MHD force balance (Eq. (1.21)). An equilibrium state is characterized by stationarity, that is $\partial/\partial t \to 0$. If we are interested in equilibria where the dynamic pressure coming from the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ can be neglected, the force balance reads

$$\nabla p = \mathbf{j} \times \mathbf{B} \tag{1.34}$$

stating that a pressure gradient can be sustained by currents possessing a component perpendicular to magnetic fields. Using Ampère's law for the current density, we can re-write the equilibrium force balance:

$$\nabla p = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = -\nabla\frac{B^2}{2\mu_0} + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = -\nabla\frac{B^2}{2\mu_0} + \frac{B^2}{\mu_0}\boldsymbol{\kappa}, \tag{1.35}$$

where the curvature $\boldsymbol{\kappa}$ has been introduced according to

$$\boldsymbol{\kappa} = \frac{\mathbf{B}}{B} \cdot \nabla \left(\frac{\mathbf{B}}{B} \right), \quad \text{with} \quad |\boldsymbol{\kappa}| = \frac{1}{R_c}, \tag{1.36}$$

where R_c is the local radius of curvature of the field line. Hence, there are two contributions by which the magnetic field can exert a force on the plasma. The magnetic pressure $B^2/(2\mu_0)$ produces a restoring force when the field lines are compressed, whereas the field line tension $B^2/\mu_0\boldsymbol{\kappa}$ exerts a force in order to straighten out a field line once it is bent.

Using Eq. (1.35), we can also evaluate the condition under which the dynamic pressure can be neglected in the force balance. Obviously, the magnetic field mainly balances the kinetic pressure as long as

$$\nabla p \gg \rho \mathbf{v} \cdot \nabla \mathbf{v} \;\rightarrow\; \frac{p}{L} \gg \rho \frac{v^2}{L} \;\rightarrow\; \sqrt{\frac{p}{\rho}} \gg v \tag{1.37}$$

which states that the flow velocity should be much smaller than roughly the speed of sound.

1.2.3
Magnetohydrodynamic Waves

In this section, we address a particular point of the dynamics of ideal MHD. When deriving the set of MHD equations, we neglected the displacement current in Ampère's law, thus eliminating electromagnetic waves from the solutions. However, in the preceding section, we saw that MHD provides two kinds of restoring forces to displacement of a field line, magnetic pressure and field line tension. These give rise to MHD waves, the so-called Alfvén waves, which exist within the system of equations derived earlier. Figure 1.3 shows these two situations.

As a starting point, we linearize the force equation, assuming that all quantities can be written as a zeroth-order term that is constant in time and space and a small first-order perturbation that may vary in time and space. In addition, we set $\mathbf{v}_0 = 0$, assuming that flow does not play a role in the zeroth-order force balance.

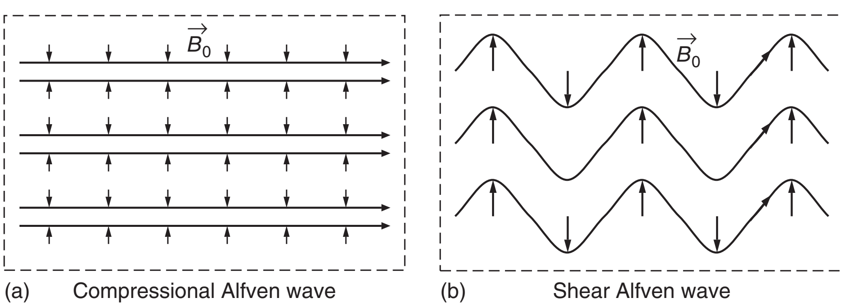

Figure 1.3 (a,b) Geometry for the derivation of compressional (a) and shear (b) Alfvén waves. The small arrows represent the perturbation of the system giving rise to an oscillation.

Linearizing (1.21), writing the right-hand side in the form (1.35) and differentiating with respect to time, we obtain

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = -\nabla \frac{\partial p_1}{\partial t} + \frac{1}{\mu_0}\left(\mathbf{B}_0 \cdot \nabla \frac{\partial \mathbf{B}_1}{\partial t} - \nabla\left(\mathbf{B}_0 \cdot \frac{\partial \mathbf{B}_1}{\partial t}\right)\right), \tag{1.38}$$

where the last two terms represent field line tension and pressure, respectively. In the following, we will hence distinguish two cases, namely pure compression (where the restoring force is given by magnetic pressure) and incompressible displacement (where the restoring force is due to field line tension). In reality, there will be waves that are a mixture of these two limiting cases.

1.2.3.1 Compressional Alfvén Waves

In this section, we assume that the plasma is compressed homogeneously in the direction perpendicular to the equilibrium magnetic field lines that are straight as we assumed $\mathbf{B}_0 = \text{const.}$. This corresponds to Figure 1.3a and $\mathbf{B}_0 \cdot \nabla \rightarrow 0$ in Eq. (1.38). In ordinary hydrodynamics, this wave is the sound wave, propagating in the longitudinal direction due to the restoring force of kinetic pressure.

In order to re-write Eq. (1.38) in terms of $\mathbf{v}_1$, we express the first term on the right-hand side using the adiabatic law:

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \;\rightarrow\; \frac{\partial}{\partial t}(\rho_0 p_1 - \gamma p_0 \rho_1) = 0 \tag{1.39}$$

Using the linearized continuity equation, we obtain

$$\frac{\partial p_1}{\partial t} = -\gamma p_0 \nabla \cdot \mathbf{v}_1 \tag{1.40}$$

stating that the change of pressure comes from the compression of a fluid element.

Next, we obtain an equation for $\mathbf{B}_1$ by combining Faraday's law and Ohm's law.

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E}_1 = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0). \tag{1.41}$$

This can be rewritten using a vector theorem

$$\nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) = (\mathbf{B}_0 \cdot \nabla)\mathbf{v}_1 - (\mathbf{v}_1 \cdot \nabla)\mathbf{B}_0 + \mathbf{v}_1(\nabla \cdot \mathbf{B}_1) - \mathbf{B}_0(\nabla \cdot \mathbf{v}_1). \tag{1.42}$$

In the case of pure compression, the first term on the right-hand side vanishes due to geometry (no change along the equilibrium field). The second term vanishes due to $\mathbf{B}_0 = \text{const.}$ As $\nabla \cdot \mathbf{B} = 0$ to each order, the third term generally vanishes. Hence, we arrive at

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\mathbf{B}_0(\nabla \cdot \mathbf{v}_1). \tag{1.43}$$

Now, we can express (Eq. (1.38)) as follows:

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \left(\gamma \frac{p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0}\right) \Delta \mathbf{v}_1, \tag{1.44}$$

where we have used the fact that the perturbation does not introduce any vortices, that is $\nabla \times \mathbf{v}_1 = 0$, which leads to $\nabla(\nabla \cdot \mathbf{v}_1) = \Delta \mathbf{v}_1$. Equation (1.44) is a wave equation with phase velocity

$$v_{\mathrm{ph}} = \sqrt{\frac{\gamma p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0}}. \tag{1.45}$$

There are two contributions, namely the contribution of kinetic pressure and that of magnetic pressure. Depending on their ratio, which in plasma physics is known as the *plasma beta*

$$\beta = \frac{2\mu_0 p}{B^2} \tag{1.46}$$

the wave will propagate at the speed of sound $c_s = \sqrt{\gamma p_0 / \rho_0}$ for $\beta \gg 1$ or, at $\beta \ll 1$, with the Alfvén speed

$$v_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}. \tag{1.47}$$

More generally, the wave speed will be a combination of both and we call the wave a magneto-aoustic wave.

1.2.3.2 **Shear Alfvén Waves**

Now, we turn to the second case, depicted in Figure 1.3b, where we assume that the only restoring force is due to field line tension, that is the plasma is perturbed in an incompressible way, meaning $\nabla \cdot \mathbf{v} = 0$. In this case, the pressure perturbation vanishes (Eq. (1.40)) and in Eq. (1.42), the last term is zero while now, the first term on the right-hand side is non-zero as the plasma is perturbed along the field lines.

Hence, Eq. (1.38) becomes

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho_0} \nabla_{\|}{}^2 \mathbf{v}_1 \tag{1.48}$$

which again is a wave equation, this time for waves travelling along the equilibrium field lines, that is a transverse wave, analogous to the oscillation of for example a guitar string. The phase velocity is the Alfvén velocity v_A, this time without any contribution of the kinetic pressure as we have assumed the motion to be incompressible.

The Alfvén velocity is quite important for the dynamics of ideal MHD as it sets the 'natural' timescale limited by inertia. Estimating the Alfvén timescale as

$$\tau_A = L / v_A \tag{1.49}$$

and inserting typical parameters of magnetically confined fusion plasmas, it is of the order of $1-10\ \mu$s, that is quite fast, because of the small mass of the very low density plasma. Consequently, ideal MHD instabilities in tokamaks often grow so fast that they have to be slowed down by passive structures such as conducting wall elements in order to be accessible for magnetic feedback control. An example for this is the vertical displacement event (VDE) treated in Section 4.4 or the Resistive Wall Mode (RWM) treated in Section 7.4.

In ideal MHD, Alfvén waves will normally appear as a damping term for instabilites as their excitation is a sink of free energy. In a toroidal confinement system, the Alfvén spectrum usually is a continuum, that is there are no resonances at discrete frequencies that extend in radial direction and hence they lead to strong damping. However, owing to toroidicity, there also exists a kind of discrete Alfvén waves that can be excited under certain circumstances, for example by a population of fast particles. These can be quite important for future reactor grade fusion plasmas in which a large population of fast particles is expected, but their detailed treatment is beyond the scope of this book.