# 1

# Hydromagnetic Oscillations in Homogeneous Plasma

In this section, we will address the basic properties of MHD oscillations in a homogeneous plasma, without going into a detailed research regarding various plasma states. Such examinations of MHD oscillations in a homogeneous plasma can be found in monographs [23–25].

We will generally use the ideal magneto-hydrodynamics approximation to describe hydromagnetic oscillations. According to this approximation, the system of MHD equations has the following form:

$$\bar{\rho}\frac{d\bar{\mathbf{v}}}{dt} = -\nabla\bar{P} + \frac{1}{4\pi}[\mathrm{rot}\,\bar{\mathbf{B}}\times\bar{\mathbf{B}}], \tag{1.1}$$

$$\frac{\partial\bar{\mathbf{B}}}{\partial t} = \mathrm{rot}\,[\bar{\mathbf{v}}\times\bar{\mathbf{B}}], \tag{1.2}$$

$$\frac{\partial\bar{\rho}}{\partial t} + \nabla(\bar{\rho}\bar{\mathbf{v}}) = 0, \tag{1.3}$$

$$\frac{d}{dt}\frac{\bar{P}}{\bar{\rho}^{\bar{\gamma}}} = 0, \tag{1.4}$$

where $\bar{\mathbf{v}}$ and $\bar{\mathbf{B}}$ are the plasma motion velocity and magnetic field vectors, $\bar{\rho}$ and $\bar{P}$ are plasma density and pressure and $\bar{\gamma}$ is the adiabatic index, $d/dt = \partial/\partial t + \bar{\mathbf{v}}\nabla$.

Let us linearise this system with respect to small perturbations. We will subscribe the parameters of an unperturbed background plasma with $_0$, while leaving the perturbation parameters with no subscript. Let us examine small-amplitude oscillations. In the linear approximation, the plasma and magnetic field parameters can then be written as $\bar{\mathbf{B}} = \mathbf{B}_0 + \mathbf{B}$, $\bar{\mathbf{v}} = \mathbf{v}_0 + \mathbf{v}$, $\bar{\rho} = \rho_0 + \rho$, $\bar{P} = P_0 + P$. The system of Eqs. (1.1)–(1.4) linearised with respect to small perturbations reduces to the following equations:

$$\rho_0\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}_0\nabla\mathbf{v} + \mathbf{v}\nabla\mathbf{v}_0\right) = -\nabla P + \frac{1}{4\pi}\left\{\left[\mathrm{rot}\,\mathbf{B}\times\mathbf{B}_0\right] + \left[\mathrm{rot}\,\mathbf{B}_0\times\mathbf{B}\right]\right\}, \tag{1.5}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \mathrm{rot}\,[\mathbf{v}\times\mathbf{B}_0] + \mathrm{rot}\,[\mathbf{v}_0\times\mathbf{B}], \tag{1.6}$$

$$\frac{\partial P}{\partial t} = -\bar{\gamma}P_0\,\mathrm{div}\mathbf{v}. \tag{1.7}$$

*Magnetospheric MHD Oscillations: A Linear Theory*, First Edition. Anatoly Leonovich, Vitalii Mazur, and Dmitri Klimushkin.

Here, $\mathbf{B}, \mathbf{v}$ are the vectors of perturbed magnetic field and plasma velocity and $P$ is the perturbed pressure. The perturbed electric field is determined by the frozen-in condition:

$$\mathbf{E} = -\frac{1}{\bar{c}}\left([\mathbf{v} \times \mathbf{B}_0] + [\mathbf{v}_0 \times \mathbf{B}]\right), \tag{1.8}$$

where $\bar{c}$ is the speed of light in vacuum, and the perturbed electric current is given by

$$\mathbf{j} = \frac{\bar{c}}{4\pi}\,\mathrm{rot}\,\mathbf{B}. \tag{1.9}$$

For a homogeneous immobile plasma ($\mathbf{v}_0 = 0$) inside a homogeneous magnetic field, an arbitrary oscillation can be represented as a superposition of Fourier harmonics of the form $\exp(i\mathbf{kx} - i\omega t)$, where $\mathbf{k}$ is the wave vector, $\mathbf{x}$ is the coordinate vector, $\omega$ is the wave frequency and $t$ is the time. Substituting the expressions for the wave field components into (1.5)–(1.7), for each such harmonic, yields

$$\omega^2 = k_\parallel^2 v_A^2, \quad \omega^4 - \omega^2 k^2 (v_A^2 + v_s^2) + k_\parallel^2 k^2 v_A^2 v_s^2 = 0, \tag{1.10}$$

– dispersion equations for Alfvén and magnetosonic waves. Here, $v_A = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén speed, $v_s = \sqrt{\bar{\gamma} p_0/\rho_0}$ is the sound speed in plasma, $k = \sqrt{k_\perp^2 + k_\parallel^2}$ is the modulus of the wave vector, and $k_\parallel, k_\perp$ are its components along and across the magnetic field.

The solutions of the second equation (1.10)

$$\omega^2 = \frac{k^2}{2}(v_A^2 + v_s^2) \pm \sqrt{\frac{k^4}{4}(v_A^2 + v_s^2)^2 - k^2 k_\parallel^2 v_A^2 v_s^2}$$

describe two branches of magnetosonic waves: fast magnetosonic (FMS, '+' before the radical) and slow magnetosonic (SMS, '–' before the radical). These equations have an especially simple form if one of the following conditions holds: $v_s \ll v_A$ (equivalent to $\beta \ll 1$, where $\beta = 8\pi p_0/B_0^2$ is the plasma gas-kinetic to magnetic pressure ratio), $v_s \gg v_A$ ($\beta \gg 1$), or $|k_\parallel| \ll |k_\perp|$, found in most real natural plasma formations. In this case, the approximate dispersion equation for FMS waves can be written as

$$\omega^2 \approx k^2 v_f^2, \tag{1.11}$$

where $v_f^2 = v_A^2 + v_s^2$, while the equation for SMS waves can be written as

$$\omega^2 \approx k_\parallel^2 c_s^2, \tag{1.12}$$

where $c_s = v_A v_s/\sqrt{v_A^2 + v_s^2}$. We will hereafter restrict ourselves to these approximations.

It can be seen from the first equation in (1.10) that the Alfvén wave group velocity $\mathbf{v}_{gA} = \partial\omega/\partial\mathbf{k} = v_A(\mathbf{B}_0/B_0)$ is along the magnetic field lines. The FMS group velocity $\mathbf{v}_{gf} = v_f(\mathbf{k}/k)$ is along the wave vector, and SMS group velocity $\mathbf{v}_{gs} = c_s(\mathbf{B}_0/B_0)$ is along the magnetic field, same as for Alfvén waves. As follows from (1.10), the latter statement is only approximate (even in the ideal MHD approximation) and no longer valid when small corrections are taken into account in order to satisfy the above conditions. Given these properties of the MHD-wave group velocity, the FMS is called the 'isotropic' mode and the Alfvén and SMS waves 'guided' modes of MHD oscillations. Useful representation of the phase and group velocities of MHD modes is given by the Friedrichs (polar) diagrams, which show the relation between the parallel ($v_{ph\parallel}$ and $v_{g\parallel}$) and transverse ($v_{ph\perp}$ and $v_{g\perp}$) components of these velocities for each of the three modes (Figure 1.1).

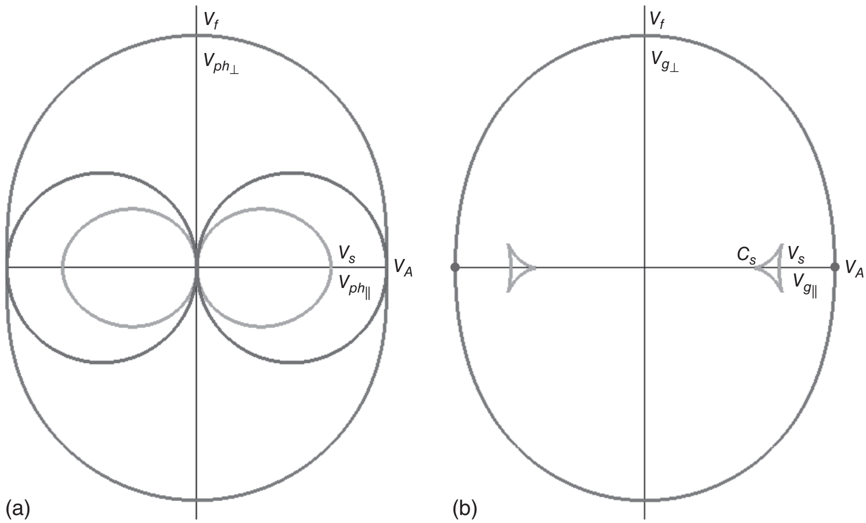


**Figure 1.1** Friedrichs diagrams for the Alfvén (black), slow (light gray) and fast (dark gray) MHD modes for case when the Alfvén speed $v_A$ is greater than the speed of sound $v_s$: (a) for the phase velocity $v_{ph}$ and (b) for the group velocity $v_g$. Here indices ∥ and ⊥ mean projections on the direction parallel and perpendicular to the ambient magnetic field, respectively.

The character of perturbed field oscillations in various MHD modes is presented in Table 1.1. The system of coordinates used here is given by unit vectors $\mathbf{e}_\parallel = \mathbf{B}_0/B_0$, $\mathbf{e}_\perp = \mathbf{k}_\perp/k_\perp$, while the third unit vector $\mathbf{e}_b$ is chosen such that the three unit vectors $(\mathbf{e}_\perp, \mathbf{e}_b, \mathbf{e}_\parallel)$ are righthanded. The table lists the (phase-shift corrected) amplitudes of various components of the wave field as expressed through the full amplitudes of the field components $\tilde{B}, \tilde{E}, \tilde{j}, \tilde{v}, \tilde{P}$ ($\tilde{B} \equiv |\mathbf{B}| \ldots$). The relations between the amplitudes of these components are different for different modes. For the Alfvén (A) wave,

$$\tilde{E} = \frac{v_A}{\bar{c}}\tilde{B}, \quad \tilde{v} = v_A\frac{\tilde{B}}{B_0}, \quad \tilde{j} = \frac{\bar{c}}{4\pi}k\tilde{B}. \tag{1.13}$$

For the FMS, the relations between $\tilde{B}, \tilde{E}, \tilde{j}$ and $\tilde{v}$ have the same form, while for the perturbed pressure we have

$$\tilde{P} = \bar{\gamma}P_0\frac{k_\perp}{k}\frac{\tilde{B}}{B_0} = \bar{\gamma}P_0\frac{k_\perp}{k}\frac{\tilde{v}}{v_A}. \tag{1.14}$$

While the energy is distributed equally between magnetic field and plasma oscillations for the Alfvén and FMS waves, the SMS wave is dominated by plasma oscillation energy. In the latter case, it is expedient to express all the values in terms of $\tilde{v}$:

$$\tilde{B} = B_0\frac{k_\perp}{k}\frac{v_s}{v_A}\frac{\tilde{v}}{v_A}, \quad \tilde{E} = B_0\frac{k_\perp k_\parallel}{k^2}\frac{v_s^2}{v_A^2}\frac{\tilde{v}}{\bar{c}}, \quad \tilde{j} = \frac{\bar{c}}{4\pi}k_\perp B_0\frac{v_s}{v_A}\frac{\tilde{v}}{v_A}, \quad \tilde{P} = \gamma P_0\frac{\tilde{v}}{v_s}. \tag{1.15}$$

One particular consequence of this is the fact that electric and magnetic fields in Alfvén waves are in-phase, while plasma motion velocity is antiphase (phase shift is $\pi$), and the current oscillations are $\pi/2$ phase-shifted relative to them. The longitudinal and transverse components of the current are either in-phase or antiphase to each other, depending on the sign of $k_\parallel$. In magnetosonic waves, plasma motion velocity and pressure oscillate in-phase.

**Table 1.1** Amplitudes of wave field components for various MHD modes.

| **Modes\components** | $B_\perp$ | $B_b$ | $B_\parallel$ | $E_\perp$ | $E_b$ | $E_\parallel$ |
|---|---|---|---|---|---|---|
| A | 0 | $\tilde{B}$ | 0 | $\tilde{E}$ | 0 | 0 |
| FMS | $-\frac{k_\parallel}{k}\tilde{B}$ | 0 | $\frac{k_\perp}{k}\tilde{B}$ | 0 | $\tilde{E}$ | 0 |
| SMS | $\frac{k_\parallel}{k}\tilde{B}$ | 0 | $-\frac{k_\perp}{k}\tilde{B}$ | 0 | $-\tilde{E}$ | 0 |

| **Modes\components** | $j_\perp$ | $j_b$ | $j_\parallel$ | $v_\perp$ | $v_b$ | $v_\parallel$ | $P$ |
|---|---|---|---|---|---|---|---|
| A | $-i\frac{k_\parallel}{k}\tilde{j}$ | 0 | $i\frac{k_\perp}{k}\tilde{j}$ | 0 | $-\tilde{v}$ | 0 | 0 |
| FMS | 0 | $-i\tilde{j}$ | 0 | $\tilde{v}$ | 0 | 0 | $\tilde{P}$ |
| SMS | 0 | $i\tilde{j}$ | 0 | 0 | 0 | $\tilde{v}$ | $\tilde{P}$ |

In FMS waves, the electric field and the parallel component of the magnetic field are also in-phase to plasma motion velocity and pressure, while $B_\perp$ is either in-phase or antiphase, depending on the sign of $k_\parallel$. In SMS waves, the electric field and the parallel component of the magnetic field oscillate antiphase to plasma motion velocity and pressure, while $B_\perp$ is either in-phase or antiphase, depending on the sign of $k_\parallel$. Current oscillations in magnetosonic waves are $\pi/2$ phase-shifted relative to the oscillations of the other components.

In Alfvén and FMS waves, the plasma oscillations are directed across the magnetic field lines, while being directed along them in SMS waves. The pressure in Alfvén waves is not perturbed. This means that the field lines move at the same velocity as the plasma is displaced. A certain analogy may be drawn between Alfvén wave propagation and oscillations propagating along a 1D string. In an FMS wave, transverse displacement of the plasma is accompanied by magnetic and plasma pressure perturbations. The propagation of these oscillations is similar to how common sound waves propagate in gas. In SMS waves, the plasma displacement is along the magnetic field. Such oscillations resemble the propagation of sound waves in a straight 1D channel. The magnetic field lines serve as walls determining the propagation direction. A qualitative picture of magnetic field and plasma oscillations in various MHD modes is shown in Figure 1.2.

Let us examine some effects that are beyond the ideal MHD framework. The first is hydromagnetic wave decay. Two effects cause this decay: plasma particle collisions resulting in a viscous plasma with finite conductivity and collisionless – Cherenkov and cyclotron – interaction between the waves and particles. The collision-induced decay of hydromagnetic waves in the magnetosphere is negligibly small. The decay decrement can be represented in the same form for the three modes:

$$\frac{\gamma}{\omega} \sim \frac{\omega \nu_m}{v_A^2} = \frac{\bar{c}^2}{v_A^2}\frac{\omega \nu_e}{\omega_{pe}^2}, \tag{1.16}$$

where $\gamma$ is the oscillation decrement, $\nu_m = \bar{c}^2/4\pi\sigma = \bar{c}^2\nu_e/\omega_{pe}^2$ is the magnetic viscosity, $\sigma$ is the longitudinal conductivity of plasma, $\nu_e$ is the electron collision frequency, $\omega_{pe} = \sqrt{4\pi n_e e^2/m_e}$ is the electron Langmuir frequency. The parameters in (1.16)

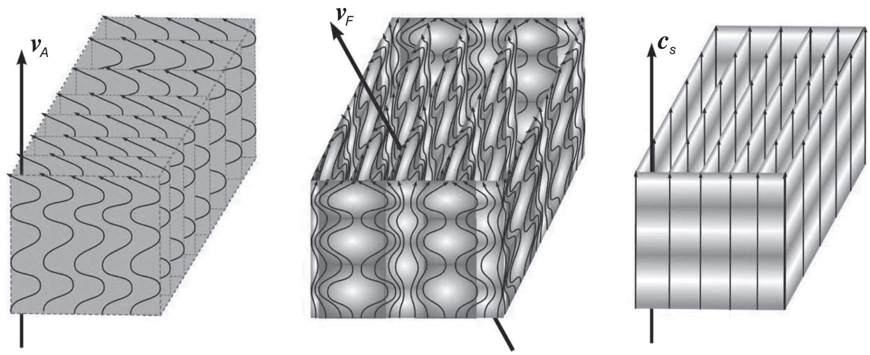


**Figure 1.2** Schematic of magnetic field oscillations (field lines), plasma pressure (shades of gray) and group velocity directions for Alfvén (a), fast magnetosonic (b) and slow magnetosonic (c) waves propagating in a homogeneous plasma.

vary very widely in the magnetosphere – from $v_A \sim 3 \cdot 10^3$ km/s, $\omega_{pe} \sim 10^6$ s$^{-1}$, $\nu_e \sim 10^{-2}$ s$^{-1}$ on inner magnetic shells to $v_A \sim 3 \cdot 10^2$ km/s, $\omega_{pe} \sim 3 \cdot 10^4$ s$^{-1}$, $\nu_e \sim 10^{-5}$ s$^{-1}$ in the outer magnetosphere. Given the characteristic frequencies of the observable magnetospheric MHD oscillations $\omega = (10^{-2} \div 10)$ s$^{-1}$, we arrive at an estimated $\gamma/\omega \sim 10^{-7} \div 10^{-14}$. The estimates are somewhat larger if we take into account particle collisions with small-scale wave turbulence of the magnetospheric plasma. However, nearer to the Earth ionosphere, where the collision-induced decay becomes significant, the collision frequency rises dramatically.

The collisionless decay of Alfvén and FMS waves is also small. Their phase velocities ($\sim v_A$) are much higher than the thermal velocities of ions ($v_i \sim v_s$). Their decrement due to ions is therefore proportional to exponentially small factor $\exp(-v_A^2/v_i^2)$ [26]. The thermal velocity of electrons, $v_e$, can be above, below or of order $v_A$, in the magnetosphere. The decay due to electrons being small results from the big difference between ion and electron masses. Half of the Alfvén wave and FMS wave energy is contained in the time-averaged kinetic energy of ions and the light electrons cannot slow them down effectively. Therefore, the decrements of the collisionless decay of these waves due to electrons contain a small factor, $m_e/m_i$ (see [24]):

$$\frac{\gamma_{eA}}{\omega} \sim \frac{m_e}{m_i}\frac{v_e}{v_A}\frac{k^2 v_A^2}{\omega_i}, \quad \frac{\gamma_{ef}}{\omega} \sim \frac{m_e}{m_i}\frac{v_e}{v_A}.$$

If $v_A \gg v_e$ these formulae also contain the exponentially small factor $\exp(-v_A^2/v_e^2)$.

The picture is completely different for a collisionless Landau decay in SMS waves. The dispersion equation for low-frequency oscillations of plasma (with Maxwell distribution of ions and electrons over velocities) has the following form (see [24]):

$$1 + \sum_{\alpha=i,e} \frac{\omega_{p\alpha}^2}{k^2 v_\alpha^2}\left[1 + i\sqrt{\pi} z_0^\alpha e^{x_\alpha} \sum_{n=-\infty}^{\infty} I_n(x_\alpha) w(z_n^\alpha)\right] = 0, \quad (1.17)$$

where the summing is according to particle types (the $\alpha$ index denotes plasma ions, $\alpha = i$ or electrons, $\alpha = e$) and cyclotron harmonics (the $n$ index). The notations are $k = \sqrt{k_\parallel^2 + k_\perp^2}$ is the wave vector module, $x_\alpha = k_\perp^2 \rho_\alpha^2$, where $\rho_\alpha = v_\alpha/\omega_\alpha$ is the

Larmor radius, $\omega_\alpha = eB_0/m_\alpha \bar{c}$ is the cyclotron frequency, $\omega_{p\alpha} = \sqrt{4\pi n_\alpha e^2/m_\alpha}$ is the plasma frequency and $v_\alpha = \sqrt{T_\alpha/m_\alpha}$ is the thermal velocity of the $\alpha$ type particles, $z_n^\alpha = (\omega - n\omega_\alpha)/\sqrt{2}k_\parallel v_\alpha$. For small values of the argument (the condition $|k_\perp \rho_\alpha| \ll 1$ is assumed to hold), the modified Bessel function $I_n(x_\alpha)$ is approximately represented as $I_n(x_\alpha) \approx (x_\alpha/2)^n/n!$. The function $w(z)$ is the probability integral having the following asymptotic representations (see [27]):

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2/2}dt\right) \approx \begin{cases} 1 - z^2 + 2iz/\sqrt{\pi}, & |z| \ll 1, \\ \exp(-z^2) + i/\sqrt{\pi}z, & |z| \gg 1. \end{cases}$$

In the sum over $n$ in (1.17), in the known limiting case $v_i \ll |\omega/k_\parallel| \ll v_e$, we can restrict ourselves to the zero harmonics and write the dispersion equation approximately as

$$\frac{\omega_{pi}^2}{k^2}\left(\frac{1}{v_{es}^2}(1 + i\sqrt{\pi}z_e^0) - \frac{k_\parallel^2}{\omega^2}\right) \approx 0,$$

where $\omega_{pi}^2/v_{es}^2 = \omega_{pe}^2/v_e^2$, $v_{es} = v_i\sqrt{T_e/T_i} = \sqrt{\bar{\gamma}T_e/m_i}$ is the ion thermal velocity determined from the electron temperature. In the zero order of perturbation theory, the solution of this equation gives the dispersion equation for SMS waves in the limit $T_e \gg T_i$: $\omega^2 = k_\parallel^2 v_{es}^2$. Given the next order of perturbation theory, we obtain the dispersion equation taking into account oscillation energy absorption:

$$\omega^2 = k_\parallel^2 v_{es}^2\left(1 - i\sqrt{\frac{\pi m_e}{2m_i}}\right).$$

In this limiting case, the value of the relative SMS decrement we are interested in $\bar{\varepsilon}_s = \gamma_s/\mathrm{Re}(\omega) \equiv \bar{\varepsilon}_{s\infty} = -\sqrt{\pi m_e/2m_i}/2 \approx -0{,}015$, where $\gamma_s \equiv \mathrm{Im}\omega$. The full numerical solution of (1.17) for $\bar{\varepsilon}_s$ as a function of $(T_e/T_i)$ is shown in Figure 1.3. The calculated curve $\bar{\varepsilon}_s(T_e/T_i)$ has a universal form for a large enough variation range of the plasma parameters including the parameter variation range in the Earth magnetosphere (1 nT $\leq B_0 \leq$ 10 T; 1 km/s$\leq v_A \leq 10^4$ km/s; $10^{-2} \leq \beta \leq 1$).

Note that, unlike Alfvén and FMS waves, the SMS decrement is rather large ($|\bar{\varepsilon}_s| \sim 1$) when $T_e/T_i \lesssim 1$ (and $\beta \lesssim 1$). This results from the SMS phase velocity being close to the plasma ion thermal velocity, in this case,

$$c_s \sim v_s = \sqrt{\bar{\gamma}\frac{P_0}{\rho_0}} = \sqrt{\bar{\gamma}\frac{T_e + T_i}{m_i}} \sim v_i,$$

making the decay due to ions very large ($\gamma \sim \omega$), thus preventing SMS waves from propagating.

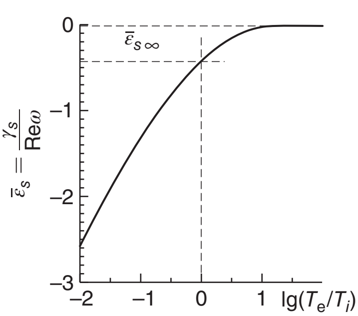


**Figure 1.3** Relative SMS decrement $\bar{\varepsilon}_s = \gamma_s/\mathrm{Re}\omega$ vs. non-isothermality, lg $T_e/T_i$, of homogeneous plasma.

Let us examine another effect unaccounted for within the ideal MHD framework, namely the transverse dispersion of Alfvén waves, absent from (1.10). We will use the following equations for the perturbed electric and magnetic fields:

$$\operatorname{rot}\mathbf{E} = i\frac{\omega}{\bar{c}}\,\mathbf{B}, \quad \operatorname{rot}\mathbf{B} = -i\frac{\omega}{\bar{c}}\hat{\varepsilon}\,\mathbf{E}, \tag{1.18}$$

where $\hat{\varepsilon}$ is the dielectric permeability tensor of plasma. Let us employ the following representation of the tensor $\hat{\varepsilon}$ for MHD waves (see [24]):

$$\hat{\varepsilon} = \frac{\bar{c}^2}{v_A^2}\begin{pmatrix} 1-\frac{3}{4}k_\perp^2\rho_i^2 & iu & 0 \\ -iu & 1-2k_\perp^2\rho_i^2 & 0 \\ 0 & 0 & \tilde{G}(\omega/k_\parallel v_e)/k_\parallel^2\rho_s^2 \end{pmatrix}, \tag{1.19}$$

where the notations are $u = \omega/\omega_i$, $\rho_s = v_{es}/\omega_i$ (hereafter we assume $u, |k_\perp\rho_i|, |k_\perp\rho_s/\tilde{G}| \ll 1$), and the function $\tilde{G}(z)$ is expressed in terms of the familiar Kramp function $\tilde{W}$,

$$\tilde{G}(z) = 1 + i\sqrt{\frac{\pi}{2}}z\tilde{W}\left(\frac{z}{\sqrt{2}}\right) \equiv 1 - ze^{-z^2/2}\int_0^z e^{t^2}dt + i\sqrt{\frac{\pi}{2}}ze^{-z^2/2},$$

and has the following asymptotic representations:

$$\tilde{G}(z) \approx \begin{cases} 1 - z^2 + \cdots + i\sqrt{\pi/2}\,z, & |z| \ll 1, \\ -z^{-2} - \frac{3}{4}z^{-4} + \cdots + \sqrt{\pi/2}\,ze^{-z^2/2}, & |z| \gg 1. \end{cases}$$

For Alfvén waves – even if their transverse dispersion is taken into account – the approximate equality $\omega \approx k_\parallel v_A$ holds true. Hence, $\omega/k_\parallel v_e \approx s_e/\rho_s$, where $s_e = \bar{c}/\omega_{pe}$ is a characteristic electron skin depth in plasma, and we have, in the limiting cases,

$$\frac{\rho_s^2}{\tilde{G}(s_e/\rho_s)} \approx \begin{cases} \rho_s^2, & s_e \ll \rho_s, \\ -s_e^2, & s_e \gg \rho_s. \end{cases} \tag{1.20}$$

Based on (1.18), (1.19), we obtain the following dispersion equation

$$\nu q^4 - (\alpha-1)(1+\nu)q^2 + (\alpha-1)^2 - u^2\alpha = 0, \tag{1.21}$$

with these notations: $q = k_\perp/k_\parallel$ is dimensionless transverse wave number, $\alpha = \omega^2/k_\parallel^2 v_A^2$, $\nu = k_\parallel^2\Lambda^2$,

$$\Lambda^2 \equiv \frac{\rho_s^2}{\tilde{G}(s_e/\rho_s)} + \frac{3}{4}\rho_i^2 = \begin{cases} -s_e^2, & s_e \gg \rho_s, \quad (\beta \ll m_e/m_i), \\ \rho_{si}^2 = \rho_s^2 + \frac{3}{4}\rho_i^2, & s_e \ll \rho_s, \quad (\beta \gg m_e/m_i). \end{cases} \tag{1.22}$$

Accordingly, for these limiting cases,

$$\nu = \begin{cases} -\mu^4 \equiv -(m_e/m_i)u^2, & (\beta \ll m_e/m_i), \\ \varkappa^4 \equiv (k_\parallel^2/m_i\omega_i^2)(T_i + 3T_e/4), & (\beta \gg m_e/m_i). \end{cases}$$

When $s_e \sim \rho_s$ ($\beta \sim m_e/m_i$), the value of $\Lambda^2$ is complex. Its variation in the complex $\Lambda^2$ plane for $s_e \gg \rho_s$ to $s_e \ll \rho_s$ is shown in Figure 1.4. Equation (1.21) describes two MHD oscillation branches – the Alfvén and FMS waves. The behaviour of these branches in the plane $(q^2, \alpha)$ for the limiting cases of 'cold' ($\beta \ll m_e/m_i$) and 'hot' ($\beta \gg m_e/m_i$) dispersion of Alfvén waves is examined in Appendix A.

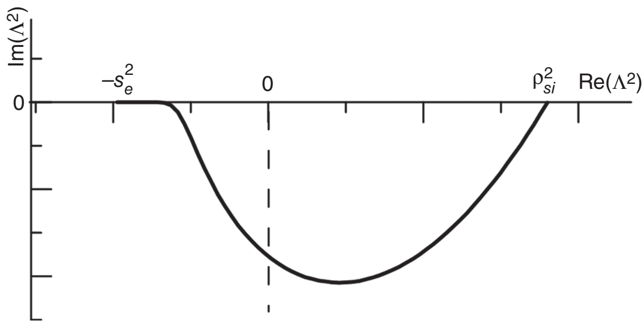


**Figure 1.4** Qualitative behaviour of the function $\Lambda^2(s_e/\rho_s)$ in the complex $\Lambda^2$ plane as the argument varies from $s_e/\rho_s \gg 1$ to $s_e/\rho_s \ll 1$.

Small corrections for effects unaccounted for within the ideal MHD framework practically do not affect FMS dispersion, so that the dispersion Eq. (1.11) can be used for these waves. Conversely, it is these small corrections that determine the transverse dispersion (dependence of frequency $\omega$ on the wave-vector transverse component $k_\perp$) of Alfvén waves. Presence of this dispersion means that Alfvén waves can propagate, not only along magnetic field lines, but also in a transverse direction to them. For quasi-parallel Alfvén waves, $k_\perp \lesssim \sqrt{u}k_\parallel$, the transverse dispersion is determined by ion inertia, and the solution of the dispersion equation has the form:

$$\omega = k_\parallel v_A \left[ 1 - \frac{u}{2} \left( \frac{k_\perp^2}{k_0^2} + \sqrt{1 + \frac{k_\perp^4}{k_0^4}} \right)^{-1} \right], \tag{1.23}$$

where $k_0^2 = 2uk_\parallel^2$. The characteristic group velocity of the transverse propagation of quasi-parallel Alfvén waves is

$$v_{g\perp} = \frac{\partial \omega}{\partial k_\perp} = v_A \frac{k_\perp}{2k_\parallel} \ll v_A.$$

For large $k_\perp \gg \sqrt{u}k_\parallel$ the Alfvén wave dispersion is called 'kinetic', and the solution of their dispersion equation has the form

$$\omega = k_\parallel v_A \left[ 1 + k_\perp^2 \Lambda^2 \right], \tag{1.24}$$

where $\Lambda^2$ is determined from (1.22). When $\beta \sim m_e/m_i$ the value of $\Lambda^2$ is complex ($|\Lambda^2| \sim s_e^2 \sim \rho_{si}^2$), and the imaginary part of (1.24) is the decrement of the Cherenkov decay due to background plasma electrons, their thermal velocity being close to the Alfvén wave propagation velocity $v_e \sim v_A$.

The values of $s_e$ and $\rho_i$ characteristic of the magnetosphere lie within the 0.1–10 km range. This means that dispersion (1.24) is only significant for waves that are extremely small scale in the transverse direction. The characteristic transverse group velocity of propagating kinetic Alfvén waves is

$$v_{g\perp} = \mathrm{Re}\, \frac{\partial \omega}{\partial k_\perp} = 2 v_A k_\parallel k_\perp \mathrm{Re}\,(\Lambda^2) \ll v_A.$$

Another important feature of Alfvén waves is the fact that, unlike magnetosonic waves, they have a non-zero longitudinal component of the perturbed electric field. As follows from (1.18), (1.19),

$$E_{\parallel} = \frac{k_{\parallel} k_{\perp}}{k_{\perp}^2 - \tilde{G}(s_e/\rho_s)/\rho_s^2} E_{\perp} \approx -k_{\parallel} k_{\perp} \frac{\rho_s^2}{\tilde{G}(s_e/\rho_s)} E_{\perp}. \tag{1.25}$$

The above can be summarised as follows. Taking into account effects determined by the thermal movement of particles results in a strong decay of SMS waves in a $T_i \gtrsim T_e$ plasma. Under typical conditions, $T_i \gg T_e$ in nearly all magnetospheric regions. This makes it impossible for the SMS oscillation eigenmodes to exist, in nearly the entire magnetosphere. The only exception is the inner plasmasphere, where $T_e > T_i$ and the SMS decay decrement is small (see [28]). For Alfvén and FMS waves, the decay effects related to their interaction with plasma particles are small. As for the transverse dispersion of magnetosonic waves, it is large enough even in the ideal MHD approximation so that taking into account small corrections unaccounted for within that approximation fails to result in any significant effects. Situations are possible for Alfvén waves when it is necessary to take into account their transverse dispersion (1.23) or (1.24) determining their slow movement across magnetic field lines or the presence of a parallel component of the electric field (1.25) in them.